Solutions to Math 42 Second midterm exam

1. (10 points) (a) Compute the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ (i.e., not just the radius of convergence, but what happens at the endpoints.)

   Use the ratio test: $|a_{n+1}/a_n| = |x\frac{n^2}{(n+1)^2}| \to |x|$, so the radius of convergence is 1. At $x = 1$ the series is $\sum 1/n^2$, which converges because it's a p-series with $p > 2$; and $x = -1$ it converges by the absolute convergence test.
   So it converges exactly for $-1 \leq x \leq 1$.

   (b) Write down the Taylor series for e^{2x} around $a = 0$ (no justification is needed). Compute the radius of convergence of this power series, with full justification.

   By substitution, it equals $\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$. Now use the ratio test: $|\frac{a_{n+1}}{a_n}| = |2x/(n+1)|$, with limit zero. It converges for all x by the ratio test, so $R = \infty$.

2. (10 points) (a) The power series for arcsin starts as

$$\arcsin(2x) = 2x + \frac{4}{3}x^3 + \frac{12}{5}x^5 + \frac{40}{7}x^7 + \dots$$

Given this, write down the first four terms of the power series for $\frac{1}{\sqrt{1-4x^2}}$. (Recall that $\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$.)

We differentiate:

$$\frac{2}{\sqrt{1-4x^2}} = 2 + 4x^2 + 12x^4 + 40x^6 + \dots$$

and dividing by 2 we get

$$\frac{1}{\sqrt{1-4x^2}} = 1 + 2x^2 + 6x^4 + 20x^6 + \dots$$

(b) Write down an infinite series that will give $\int_0^1 \sin(2x^2)dx$. Your answer should be expressed in summation notation.

We start with the series for $\sin(x)$ and do a substitution:

$$\sin(2x^2) = \sum_{n=0}^{\infty}(-1)^n \frac{(2x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n 2^{2n+1}\frac{(x)^{4n+2}}{(2n+1)!}$$

Integrating from 0 to 1 gives

$$\sum_{n=0}^{\infty}(-1)^n \frac{2^{2n+1}}{(2n+1)!\cdot(4n+3)}.$$

3. (10 points) Let $f(x)$ be a function. You are given the following values of f and its derivatives:

$$f(0) = 1, f'(0) = 2, f''(0) = 1, f^{(3)}(0) = -4.$$

$$f(2) = 2, f'(2) = -1, f''(2) = 0, f^{(3)}(2) = 3.$$

(a) Using a Taylor series, write down an approximation to $f(1.9)$. You do not need to evaluate the expression. (You will need to decide whether to use a Taylor series about $a = 0$ or $a = 2$. Decide which one will give a better approximation and briefly explain your choice.)

Use 2 because 1.9 is much closer to 2 than 0. We get

$$2 + 0.1 + \frac{3}{3!}(0.1)^3$$

(b) Now suppose you are given that $|f^{(4)}(x)| \leq 10$ for all values of x. Use Taylor's inequality to estimate the error in your approximation from (a). Again you do not need to evaluate the resulting expression.

Taylor's inequality gives that the error is at most $\frac{10 \cdot (0.1)^4}{4!}$.

4. (10 points) (a) Compute a Taylor polynomial for $f(x) = x^{1/3}$ around $a = 1$, up to and including the term involving x^3.

The derivatives are as follows:

$$f(x) = x^{1/3}, f'(x) = \frac{1}{3}x^{-2/3}, f''(x) = \frac{-2}{9}x^{-5/3}, f'(x) = \frac{10}{27}x^{-8/3}.$$

Substituting $x = 1$, we get

$$f(1) = 1, f'(1) = 1/3, f''(1) = -2/9, f'''(1) = 10/27$$

and Taylor polynomial

$$1 + \frac{1}{3}(x-1) - \frac{1}{9}(x-1)^2 + \frac{5}{81}(x-1)^3$$

(b) Give an upper bound for the error $|f(0.8) - T_3(0.8)|$, where T_3 is your series from (a). You do not need to numerically evaluate the expression you give.

We use Taylor's inequality, noting that $f^{(4)}(x) = \frac{-80}{81}x^{-11/3}$. This is bounded in absolute value by $x^{-11/3}$, which is at most $(0.8)^{-11/3}$. So Taylor's inequality gives

$$|f(0.8) - T_3(0.8)| \leq (0.8)^{-11/3} \cdot \frac{(0.2)^4}{4!}$$

5. (10 points) (a) Consider a sphere of radius $r = 3$. Slice it with a plane parallel to the equator, but at height 1 above the equatorial plane. What is the volume of the resulting "spherical cap," i.e. the portion of the sphere above this plane?

Slicing by discs, the volume is

$$\int_1^3 \pi(9 - y^2)dy = \pi(9y - y^3/3)|_1^3 = \pi(18 - 9 + 1/3) = \pi \cdot (9 + \frac{1}{3}).$$

(b) Consider a triangle with vertices $(2, 1), (1, 0), (3, 0)$. Rotate this triangle around the y-axis. What is the volume of the resulting solid?

Use the washer method:

$$\int_0^1 \pi((3 - y)^2 - (y + 1)^2)dy = \pi \int_0^1 (8 - 8y)dy = 8\pi(y - y^2/2)|_0^1 = 4\pi.$$

6. (10 points) (a) A log that is 6 feet long is cut at one-foot intervals. Each of the resulting cross-sections is circular, with diameter as given in the table below. Write down a numerical approximation to the volume of the log. You do not need to evaluate the resulting expression.

Distance to one end of log (in feet)	0	1	2	3	4	5	6
Diameter of cross-section	3	3.6	4	4.2	3.8	3.6	3.2

WARNING: question uses diameter not radius! Use Simpson's rule, starting from $V = \int A(y)dy$. Get

$$V = \frac{1}{3}\pi \left(1.5^2 + 4(1.8)^2 + 2(2^2) + 4(2.1)^2 + 2(1.9)^2 + 4(1.8)^2 + (1.6)^2\right)$$

in cubic feet.

(b) Set up an integral that computes the volume of the solid obtained when the area between the curves $y = x$ and $y = x^4$ is rotated around $x = 2$. You do not need to evaluate the resulting integral.

By the method of circular shells we get

$$\int_0^1 \underbrace{(x - x^4)}_{\text{height}} \cdot \underbrace{(2\pi)(2 - x)}_{\text{circum}} \, dx$$

7. (10 points) (a) Compute $\int_1^2 \frac{1}{x(x+3)}dx$.

We substitute $\frac{1}{x(x+3)} = \frac{A}{x} + \frac{B}{x+3}$. Now this becomes $1 = A(x+3) + Bx$. Substitute $x = 0$ we get $A = 1/3$ and $B = -1/3$ and we get

$$\frac{1}{x(x+3)} = \frac{1}{3}\left(\frac{1}{x} - \frac{1}{x+3}\right)$$

The integral becomes

$$\begin{aligned}\int_1^2 \frac{1}{x(x+3)}dx &= \int_1^2 \frac{1}{3}\left(\frac{1}{x} - \frac{1}{x+3}\right)dx \\ &= \frac{1}{3}\ln|x| - \frac{1}{3}\ln|x+3|\Big|_1^2 \\ &= \frac{1}{3}\ln 2 - \frac{1}{3}\ln 5 + \frac{1}{3}\ln 4 \\ &= \frac{1}{3}(\ln 2 - \ln 5 + \ln 4) \\ &= \frac{1}{3}\ln\frac{8}{5}.\end{aligned}$$

(b) Compute $\int_0^{1/\sqrt{2}} \sqrt{1-x^2}dx$.

Substitute $x = \sin(\theta)$. The θ-integral goes from 0 to $\theta = \pi/4$. We get

$$\int_0^{\pi/4} \cos^2(\theta)d\theta = \int_0^{\pi/4} \frac{\cos(2\theta)+1}{2}d\theta = \left(\frac{\sin(2\theta)}{4} + \theta/2\right)|_0^{\pi/4} = \frac{1}{4} + \pi/8.$$

8. (10 points) Match the function with the power series. **NO JUSTIFICATION IS NECESSARY!**

(a) e^{-x}

(b) $\frac{1}{1+x^2}$

(c) $\ln(1-x)$

(d) $\cos(2x)$

(e) $\frac{1}{10+x}$

(i) $1 - 2x^2 + 16\frac{x^4}{4!} - \dots$

(ii) $1 + x^2 + x^4 + x^6 + x^8 + \dots$

(iii) $\frac{1}{10} - \frac{x}{100} + \frac{x^2}{1000} - \dots$

(iv) $1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \dots$

(v) $1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$

(vi) $1 - x^2 + x^4 - x^6 + x^8 \dots$

(vii) $-\frac{1}{10} + \frac{x}{100} - \frac{x^2}{1000} + \dots$

(viii) $-x - \frac{x^2}{2} - \frac{x^3}{3} - \dots$

$(a) \to (v)$
$(b) \to (vi)$
$(c) \to (viii)$
$(d) \to (i)$
$(e) \to (iii)$

Common power series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

Midpoint rule:

$$\int_a^b f(x)dx \approx (\Delta x) \cdot (f(\bar{x}_1) + \cdots + f(\bar{x}_n))$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = \frac{1}{2}(x_{i-1} + x_i)$.

Trapezoid rule:

$$\int_a^b f(x)dx \approx \frac{(\Delta x)}{2} \cdot (f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n))$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

Simpson rule:

$$\int_a^b f(x)dx \approx \frac{\Delta x}{3} \cdot (f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n))$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$ and n is even.