Solutions to Math 42 First midterm exam

1. (10 points) Evaluate each of the following indefinite integrals.

   (a) $\int xe^{2x}dx$

Integration by parts, with $u = x, v = e^{2x}/2$. We have $\int xe^{2x}dx = \int udv = uv - \int v \cdot du$, so

$$\int xe^{2x} = xe^{2x}/2 - \int \frac{e^{2x}}{2} = \frac{1}{2}xe^{2x} - \frac{e^{2x}}{4} + C$$

   (b) $\int x\ln(x-1)dx$

We put $x = u + 1$ to make the integral $\int(u+1)\ln(u)du$. Now we use integration by parts:

$$\int (u+1)\ln(u)du = (u^2/2 + u)\ln(u) - \int (u^2/2 + u) \cdot \frac{1}{u}du$$

$$= (u^2/2 + u)\ln(u) - \int (u/2 + 1)du = (u^2/2 + u)\ln(u) - (u^2/4 + u) + C.$$

Finally, substituting $u = x - 1$, we get

$$(x^2/2 - 1/2)\ln(x-1) + (-x^2/4 - x/2) + C.$$

Comment: this problem was a bit tricky, because many people tried to integrate by parts directly (rather than first making the substitution). This led to integrate $\frac{x^2}{x-1}$, which can be done by polynomial division, but we hadn't discussed that.

2. (10 points) Determine (with justification) whether each of the following improper integrals is convergent or divergent.

(a) $\int_0^1 \frac{\cos^2(x)}{\sqrt{x}} dx$.

We compare to the integral of $\frac{1}{\sqrt{x}}$: because $\cos^2(x) \leq 1$, we have

$$\frac{\cos^2(x)}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}$$

Now

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{t \to 0} (2\sqrt{x})|_t^1 = 2,$$

and therefore the integral is convergent.

Common errors: citing p-series with $p < 1$ to conclude that $\int_0^1 \frac{1}{\sqrt{x}} dx$ diverges. (The p-series refers to $\sum_{n=1}^{\infty} 1/n^p$; this converges if $p > 1$ and diverges for $p \leq 1$. The corresponding integral would be $\int_{x=1}^{\infty} \frac{1}{x^p}$; this converges if $p > 1$ and diverges for $p \leq 1$. Note the limits of integration.)

(b) $\int_0^{\infty} \frac{e^{-x}}{x} dx$.

The integral is improper both because the function behaves badly at $x = 0$, and because the limit of integration involves ∞. Accordingly we split up:

$$\int_0^1 \frac{e^{-x}}{x} + \int_1^{\infty} \frac{e^{-x}}{x} dx$$

To handle the first, we compare to $\frac{1}{x}$: We note that $e^{-x} \geq e^{-1}$ if $x \leq 1$, because e^{-x} is a decreasing function. Therefore,

$$\frac{e^{-x}}{x} \geq e^{-1} \frac{1}{x}$$

Now $\int_0^1 e^{-1} \frac{1}{x} dx$ diverges (because its indefinite integral $e^{-1} \ln |x|$ approaches infinity as $x \to 0$). By the comparison test, $\int_0^1 \frac{e^{-x}}{x} dx$ diverges too.

Therefore, the complete integral $\int_0^{\infty} \frac{e^{-x}}{x}$ diverges, because one of the part diverges.

3. (10 points) Let $I = \int_1^2 \frac{dx}{x}$.

(a) Write an expression that estimates I using the trapezoid rule with $n = 6$ intervals. **You do not need to explicitly evaluate the result.**

Explain, by means of a picture, whether this is an overestimate or an underestimate.

$I \approx \frac{1}{12}\left(1 + 2\frac{6}{7} + 2\frac{6}{8} + 2\frac{6}{9} + 2\frac{6}{10} + 2\frac{6}{11} + \frac{1}{2}\right)$. Looking at the graph (see graph in separate file) this is an overestimate, i.e. I is less than the right-hand side.

Common errors: Leaving the expression for the approximation in terms of f, i.e. just copying down the formula for the trapezoid rule without saying what $f(x)$ is.

(b) Compute an error bound for both the trapezoid rule and Simpson's rule (applied to $I = \int_1^2 \frac{dx}{x}$) with $n = 10$ intervals.

We have (with $f(x) = 1/x$) that $f'(x) = -1/x^2$, $f'' = 2/x^3$, $f^{(3)} = -6/x^4$ and $f^{(4)} = 24/x^5$. Therefore, we can take $K_2 = 2, K_4 = 24$. The error in the trapezoid rule is therefore at most

$$\frac{K_2}{12n^2} = \frac{2}{1200} = \frac{1}{600}$$

and the error in Simpson's rule is

$$\frac{K_4}{180n^4} = \frac{24}{180 \cdot 10^4} = \frac{2}{15 \cdot 10^4}.$$

Common errors: Using equality signs (=) instead of inequalities when discussing error bounds. Choosing a K that is less than zero, or not justifying why a given choice of K works.

4. (10 points) We consider the numerical approximation of the integral $\int_0^1 e^{-\frac{x^2}{2}}\,dx$.

(a) Show that the function $f(x) = e^{-\frac{x^2}{2}}$ satisfies $|f''(x)| \leq 1$ for $0 \leq x \leq 1$.

We have $f'(x) = -xe^{-x^2/2}$ and then

$$f''(x) = (x^2 - 1)e^{-\frac{x^2}{2}}.$$

Therefore we get

$$|f''(x)| = |(x^2 - 1)||e^{-\frac{x^2}{2}}|.$$

The statement now follows from the observation that for $0 \leq x \leq 1$ it holds

$$|(x^2 - 1)| \leq 1 \qquad \text{and} \qquad |e^{-\frac{x^2}{2}}| \leq 1.$$

Common errors: just checking $f''(0)$ and $f''(1)$. Missing absolute values: Working with f'' and not $|f''|$. Saying a function was decreasing without justification.

(b) How many intervals are needed for the trapezoid rule, applied to $\int_0^1 e^{-\frac{x^2}{2}}\,dx$, to ensure that the error is at most $\frac{1}{48}$?

Hint: Use the statement of part (a). You may use this even if you did not answer part (a).

The Error bound of the trapezoid Rule together with part (a) yields the estimate

$$|E_T| \leq \frac{1}{12n^2}$$

Therefore we want $\frac{1}{12n^2} \leq \frac{1}{48}$. Therefore we have to choose n so that

$$n^2 \geq \frac{48}{12} = 4, \qquad \text{which means} \quad n \geq 2.$$

Common error: setting E_T *equal to* $\frac{1}{48}$, plugging the bound in to say that $\frac{1}{48} \leq \frac{1}{12n^2}$, and concluding $n \leq 2$. Make sure why you understand why this is an error.

5. (10 points) (a) Determine whether the following series is convergent or divergent, giving full justification: $\sum_{n=2}^{\infty} \frac{1}{1+n\ln(n)}$.

We use the limit comparison test for $a_n = \frac{1}{1+n\ln(n)}$ with the series $b_n = \frac{1}{n\ln(n)}$.
First of all $\lim_{n\to\infty} \frac{b_n}{a_n} = 1 + \frac{1}{n\ln(n)} = 1$. So the series either both converge or both diverge.
To check convergence for b_n, use the integral test applied to $f(x) = \frac{1}{x\ln(x)}$ and use the u-substitution $x = e^u$:

$$\int_2^{\infty} \frac{dx}{x\ln(x)} = \int_2^{\infty} \frac{du}{u}$$

which is divergent. Therefore $\sum b_n$ diverges, and so does $\sum a_n$.
Common error: misuse of the comparison test.

(b) For what values of $r > 0$ does the series

$$\frac{1}{1+r} + \frac{1}{2+r^2} + \frac{1}{3+r^3} + \dots$$

converge?

It converges for $r > 1$ and diverges for $0 \leq r \leq 1$.

(a) If $r > 1$, we can compare our series $a_n = \frac{1}{n+r^n}$ to the geometric series $b_n = \frac{1}{r^n} = (1/r)^n$. This geometric series is convergent because $1/r < 1$.

(b) If $r = 1$ we have a harmonic series, with is divergent. If $r < 1$, we have

$$n + r^n \leq n + 1 \implies \frac{1}{n+r^n} \geq \frac{1}{n+1}$$

and again, by comparison with the harmonic series, the series is divergent.

6. (10 points) (a) Determine the radius of convergence of

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n$$

We use the ratio test. Write $a_n = \frac{(n!)^2}{(2n)!} x^n$. Now

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!^2}{(n!)^2} \cdot \frac{(2n)!}{(2n+2)!} \cdot \frac{x^{n+1}}{x^n} = (n+1)^2 \cdot \frac{1}{(2n)(2n+1)} x.$$

As $n \to \infty$, this limit approaches $x/4$, and the limit of $|\frac{a_{n+1}}{a_n}|$ approaches $|x/4|$. Therefore, the series converges for $|x/4| < 1$ and diverges for $|x/4| > 1$, in other words, converges for $|x| < 4$ and diverges for $|x| > 4$. The radius of convergence is 4.

Note that we don't know whether or not the series converges for $x = \pm 4$, but the question doesn't ask for that either.

(b) Determine all x for which

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}$$

converges. (In other words, you should determine not only the radius of convergence, but what happens at the endpoints of the interval.)

At first we use the ratio test. Write $a_n = \frac{(x-2)^n}{n^2}$. We have $\frac{a_{n+1}}{a_n} = (x-2) \cdot \frac{n^2}{(n+1)^2}$. As $n \to \infty$ we have $\lim \left| \frac{a_{n+1}}{a_n} \right| = |(x-2)|$, so the series converges if $|x-2| < 1$ and diverges if $|x-2| > 1$.

This does not tell us what happens when $x - 2 = \pm 1$, i.e. when $x = 1$ or $x = 3$. When $x = 3$ we get the harmonic series which diverges. When $x = 1$ we get the series $1 - \frac{1}{2} + \frac{1}{3} - \dots$ which converges by the alternating series test.

In conclusion, the series converges when $1 \le x < 3$.

7. (10 points) (a) Consider the series $s = \sum_{n=1}^{\infty} \frac{1}{n^2}$. How large must k be in order that the k-th partial sum s_k approximates s to an error of less than 0.0005? In other words, how large must k be in order that $s - s_k = \sum_{n=k+1}^{\infty} \frac{1}{n^2} \leq 0.0005$?

We are going to use the remainder estimate for the integral test. For this observe that

$$|s - s_k| \leq \int_k^{\infty} \frac{1}{x^2} dx = \frac{1}{k}.$$

To guarantee that $|s - s_k| \leq 0.0005$, we want $\frac{1}{k} \leq 0.0005$, that is to say, $k \geq 2000$.
Common error: using $\int_{k+1}^{\infty}$ instead.

(b) Explain why

$$\sum_{n=101}^{200} \frac{1}{n} \leq \ln(2).$$

(Use the same ideas as the integral test.)

We get from a similar picture to that used in the integral test (see picture in separate file) that

$$\sum_{n=101}^{200} \frac{1}{n} \leq \int_{100}^{200} \frac{dx}{x} = \ln(2).$$

Common error: many people wrote that $\int_{100}^{200} = \ln(2)$, but didn't explain at all why that was relevant. Many people also said "by the integral test," but that's not enough – in the integral test, one of the limits is ∞, why can you plug in 200 instead? (A quick picture to explain would suffice.) Also, many people used instead $\int_{101}^{200}$, which gives an inequality in the wrong direction.

8. (10 points) Suppose that a_n is a sequence of *positive* numbers and $\lim_{n\to\infty} a_n = 0$. Suppose also that a_n is decreasing, in other words $a_1 \geq a_2 \geq \ldots$. **Please be careful to read these hypotheses carefully.**

For each of the following series circle either CONVERGE (the series must converge), DIVERGE (the series must diverge), or UNKNOWN (it is not possible to determine from the given information whether the series converges or diverges).

NO JUSTIFICATION IS NECESSARY!

(a) $\sum (-1)^n a_n$.

CONVERGE DIVERGE UNKNOWN

CONVERGE by alternating series test (no justification was necessary but it has been included for solutions).

(b) $\sum e^{a_n}$.

CONVERGE DIVERGE UNKNOWN

DIVERGE, note that $e^{a_n} \to 1$, use test for divergence.

(c) $\sum \frac{a_n}{n^2}$.

CONVERGE DIVERGE UNKNOWN

CONVERGE, note that a_n goes to zero, so we must have $a_n \leq C$ for some constant C. Now compare to C/n^2.

(d) $\sum a_n$.

CONVERGE DIVERGE UNKNOWN

UNKNOWN, consider $a_n = 1/n$ and $a_n = 1/n^2$ for examples with different answers.

(e) $\sum \frac{a_n}{n}$.

CONVERGE DIVERGE UNKNOWN

UNKNOWN, consider $a_n = 1/\ln(n)$ or $a_n = 1/n$ for examples with different answers.

Formulas for numerical integration.

Midpoint rule:

$$\int_a^b f(x)dx \approx (\Delta x)\cdot(f(\bar{x}_1)+\cdots+f(\bar{x}_n))$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = \frac{1}{2}(x_{i-1}+x_i)$.

Trapezoid rule:

$$\int_a^b f(x)dx \approx \frac{(\Delta x)}{2}\cdot(f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n))$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

Simpson rule:

$$\int_a^b f(x)dx \approx \frac{\Delta x}{3}\cdot(f(x_0)+4f(x_1)+2f(x_2)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n))$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$ and n is even.

Error-Bound Formulas for Approximate Integration: If $|f^{(2)}| \leq K_2$ and $|f^{(4)}| \leq K_4$ on $[a,b]$,

$$|E_T| \leq \frac{K_2(b-a)^3}{12n^2} \qquad |E_M| \leq \frac{K_2(b-a)^3}{24n^2} \qquad |E_S| \leq \frac{K_4(b-a)^5}{180n^4},$$

where E_T, E_M, E_S are respectively the errors in the trapezoidal rule, the midpoint rule, and Simpson's rule, with n steps.