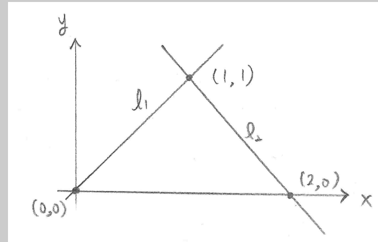
Solutions to Math 42 Final Exam — March 17, 2014

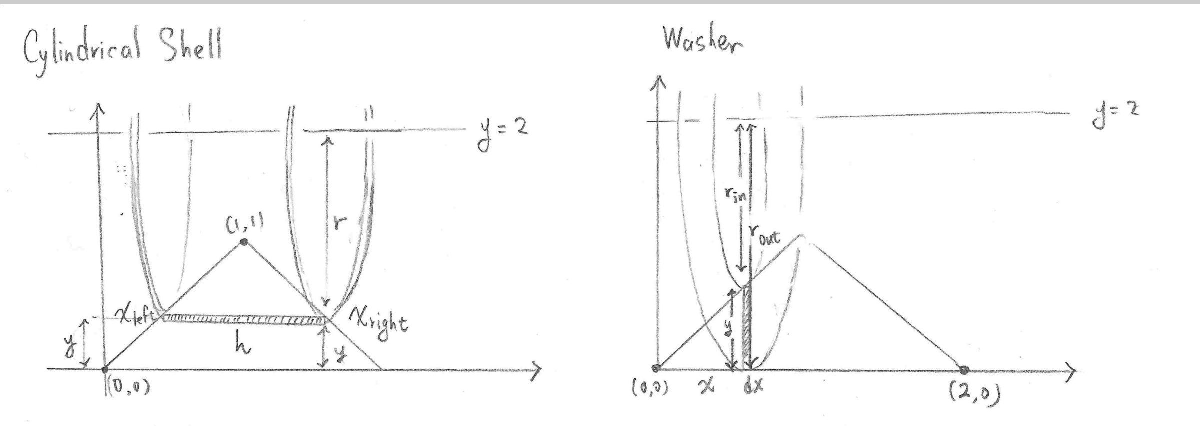
1. (10 points) Let T be the triangular region in the xy-plane with vertices $(0,0)$, $(2,0)$, and $(1,1)$.

To find the equation for the line l_1, note that the slope is $\frac{1-0}{1-0} = 1$, and l_1 passes through the origin. Hence the equation is $y = x$; to find the equation for the line l_1, note that the slope is $\frac{1-0}{1-2} = -1$, and l_2 passes through $(2,0)$. Hence the equation is $(y-0) = -(x-2)$, yielding $y = -x+2$.



(a) Set up, but do not evaluate, an expression for the volume of the solid formed by rotating T about the line $y = 2$. Justify your answer by drawing a picture, labeling sample slices, and citing the method used.

(5 points)



<u>Cylindrical shell</u>: $V = \int 2\pi r h dr$. From the picture, we know $r = 2-y$, $dr = dy$, y ranges from 0 to 1, and $h = x_{\text{right}} - x_{\text{left}}$. From the equations of l_1 and l_2, we find $x_{\text{left}} = y$ and $x_{\text{right}} = 2-y$. Therefore,

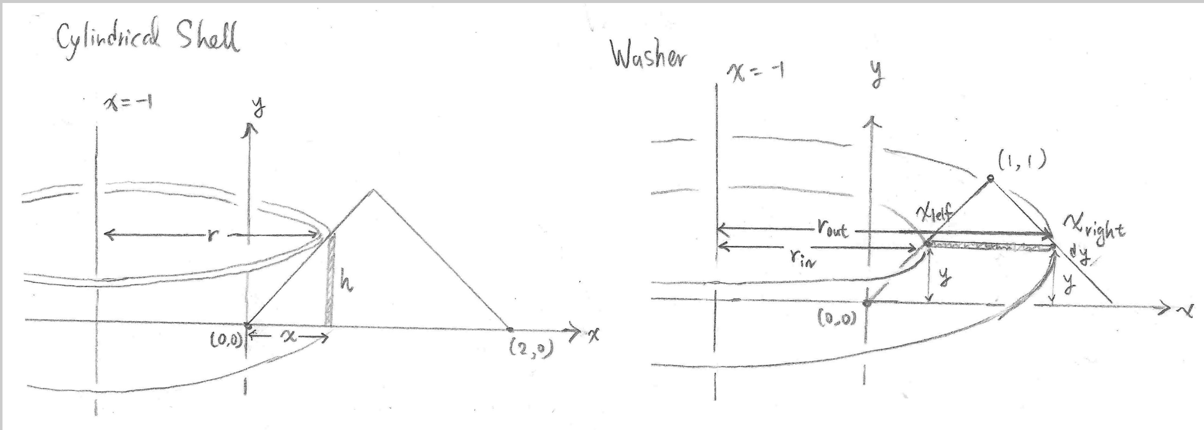
$$V = \int_0^1 2\pi(2-y)(2-y-y)dy.$$

<u>Washer</u>: $V = \int \pi(r_{\text{out}}^2 - r_{\text{in}}^2)dh$. From the picture, we know $dh = dx$, $r_{\text{out}} = 2$, and $r_{\text{in}} = 2-y$. For x between 0 and 1, using the equation for l_1 we have $y = x$, yielding $r_{\text{in}} = 2-x$; for x between 1 and 2, using the equation for l_2 we have $y = 2-x$, yielding $r_{\text{in}} = 2-(2-x)$. Therefore,

$$V = \int_0^1 \pi\left(2^2 - (x-2)^2\right)dx + \int_1^2 \pi\left(2^2 - (2-(x-2))^2\right)dx.$$

(b) Set up, but do not evaluate, an expression for the volume of the solid formed by rotating T about the line $x = -1$. Justify your answer by drawing a picture, labeling sample slices, and citing the method used.

(5 points)



Cylindrical shell: $V = \int 2\pi r h dr$. From the picture, we know $dr = dx$, and $r = x + 1$. For x between 0 and 1, using the equation for l_1 we have $h = x$; for x between 1 and 2, using the equation for l_2 we have $h = 2 - x$. Therefore,

$$V = \int_0^1 2\pi(x+1)x dx + \int_1^2 2\pi(x+1)(2-x)dx.$$

Washer: $V = \int \pi(r_{\text{out}}^2 - r_{\text{in}}^2)dh$. From the picture, we know $dh = dy$, y ranges from 0 to 1, $r_{\text{out}} = x_{\text{left}} + 1 = 2 - y + 1$, and $r_{\text{in}} = x_{\text{left}} + 1 = y + 1$. Therefore,

$$V = \int_0^1 \pi\Big((2-y+1)^2 - (y+1)^2\Big)dy.$$

2. (12 points) For this problem, use the following information:

- If g is a normal ("bell-shaped" or "Gaussian") probability density function, then g has the general form

$$g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

- A partial list of approximate values of the function

$$P(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt \quad \text{is given at right:}$$

$P(0.5) \approx 0.69$	$P(1.3) \approx 0.90$
$P(0.6) \approx 0.72$	$P(1.4) \approx 0.92$
$P(0.7) \approx 0.76$	$P(1.5) \approx 0.93$
$P(0.8) \approx 0.79$	$P(1.6) \approx 0.94$
$P(0.9) \approx 0.82$	$P(1.7) \approx 0.95$
$P(1.0) \approx 0.84$	$P(1.8) \approx 0.96$
$P(1.1) \approx 0.86$	$P(1.9) \approx 0.97$
$P(1.2) \approx 0.88$	$P(2.0) \approx 0.98$

Suppose that a manufacturer of pressure gauges is testing the accuracy of its products before placing them on the market; this is accomplished by using each gauge to measure a sample of compressed air with a known pressure value of exactly 50 psi. Suppose that the reading found by each gauge is a random variable having a normal distribution.

(a) Given that the mean reading is 50 psi, and the standard deviation is 1.2 psi, find the approximate probability that a randomly selected gauge will make a reading on the sample that is less than 50.6 psi. (Your final answer should be a number, but first express it as a probability integral and show how it can be evaluated using the information given.)

(4 points)

$$\text{Prob(pressure} < 50.6) = \int_{-\infty}^{50.6} \frac{1}{1.2\sqrt{2\pi}} e^{-\frac{(x-50)^2}{2(1.2)^2}}\, dx.$$

Let $t = \frac{x-50}{1.2}$, $dt = \frac{dx}{1.2}$. Continuing, we have

$$= \int_{-\infty}^{\frac{50.6-50}{1.2}} g(t)dt = \int_{-\infty}^{0.5} g(t)dt = P(0.5) \approx 0.69 = 69\%.$$

Note that the new lower bound for t in the integral is $-\infty$ because $\lim_{x\to-\infty} \frac{x-50}{1.2} = -\infty$.

(b) With the same values of mean and standard deviation as in part (a), find the approximate probability that a randomly selected gauge will make a reading that is less than 48.8 psi. (Again give a number as your final answer, but use an appropriate integral expression as part of your justification.)

(4 points)

$$\text{Prob}(\text{pressure} < 48.8) = \int_{-\infty}^{48.8} \frac{1}{1.2\sqrt{2\pi}} e^{-\frac{(x-50)^2}{2(1.2)^2}} dx.$$

Again, by the substitution $t = \frac{x-50}{1.2}$, this integral becomes

$$\int_{-\infty}^{\frac{48.8-50}{1.2}} g(t)dt = \int_{-\infty}^{-1} g(t)dt.$$

Next, by the symmetry $g(t) = g(-t)$ of the standard normal distribution, this integral is equal to

$$\int_{1}^{\infty} g(t)dt.$$

Finally, the last integral can be further written as

$$1 - \int_{-\infty}^{1} g(t)dt = 1 - P(1) \approx 1 - 0.84 = 16\%.$$

(c) Now suppose that after a change in the manufacturing process, it is determined that the mean reading of pressure is still 50 psi, but the standard deviation has changed. In addition, it is now found that approximately 14 percent of readings are outside the range from 49 to 51 psi. What is the new standard deviation? (Again use an appropriate integral expression as part of your justification.)

(4 points) We are given that

$$\text{Prob}(\text{pressure} > 51) + \text{Prob}(\text{pressure} < 49) = 14\% = 0.14.$$

After the substitution $t = \frac{x-50}{1.2}$ this equation becomes

$$0.14 = \int_{1/\sigma}^{\infty} g(t)dt + \int_{-\infty}^{-1/\sigma} g(t)dt.$$

By the symmetry $g(t) = g(-t)$ of the standard normal distribution, the first and second integrals are the same. Hence we have

$$0.7 = \int_{1/\sigma}^{\infty} g(t)dt = 1 - \int_{-\infty}^{1/\sigma} g(t)dt = 1 - P(1/\sigma).$$

Solving this equation, we get $P(1/\sigma) = 0.93 \approx P(1.5)$. Hence $\sigma \approx (1.5)^{-1} = \frac{2}{3}$ psi.

3. (12 points) A Silicon Valley venture capitalist models the lifespan of an Internet start-up company as a random variable, with probability density function

$$f(t) = \begin{cases} \dfrac{Ct}{(t^2+1)^2} & \text{if } t \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where t is measured in years, and C is a positive constant.

(a) Find C, using the fact that f is a probability density function.

(6 points) For f to be a probability density function, we need $\displaystyle\int_{-\infty}^{\infty} f(t)dt = 1$. Since f vanishes for all $t < 0$, we need only to integrate over $[0, \infty)$:

$$\int_0^\infty \frac{Ct}{(t^2+1)^2} dt = 1.$$

Use the u-substitution $u = t^2 + 1$, $du = 2tdt$. The lower bound for u is $1^2 + 0 = 1$, and the upper bound is $\lim\limits_{t\to\infty}(t^2+1) = \infty$. The integral then becomes

$$\begin{aligned} 1 &= C \int_1^\infty \frac{1/2}{u^2} du \\ &= \frac{C}{2} \lim_{U\to\infty} \Big(-u^{-1} \Big)\Big|_1^U = \frac{C}{2}\Big(-0 - (-1) \Big) = \frac{C}{2}. \end{aligned}$$

Hence, $C = 2$.

(b) Find the mean lifespan of a company according to this venture capitalist's model.

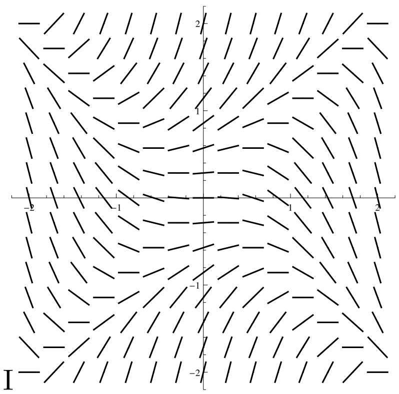
(6 points) By definition, the mean μ is equal to $\int_{-\infty}^{\infty} tf(t)dt$. Again, since f vanishes when $t < 0$, we need only to integrate over $[0, \infty)$:

$$\mu = \int_0^\infty \frac{2t}{(t^2+1)^2} tdt = \int_0^\infty \frac{2t^2}{(t^2+1)^2} dt.$$

Consider the trigonometric substitution $t = \tan\theta$, $dt = \sec^2\theta d\theta$. The lower bound for θ is $\arctan(0) = 0$; the upper bound for θ is $\lim\limits_{t\to\infty} \arctan(t) = \frac{\pi}{2}$. Continuing, we have

$$\begin{aligned} \mu &= \int_0^{\frac{\pi}{2}} \frac{2\tan^2\theta}{(1+\tan^2\theta)^2} \sec^2\theta d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{2\tan^2\theta}{(\sec^2\theta)^2} \sec^2\theta d\theta \\ &= \int_0^{\frac{\pi}{2}} 2\sin^2\theta d\theta \\ &= \int_0^{\frac{\pi}{2}} 2\, \frac{1-\cos(2\theta)}{2} d\theta \\ &= \left(\theta - \frac{\sin(2\theta)}{2}\right)\Big|_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{2} - 0 - (0 - 0) = \boxed{\frac{\pi}{2}}. \end{aligned}$$

4. (12 points) Each of the equations given below corresponds to exactly one of the eight direction fields displayed; the scale on each is $-2 \le x \le 2,\ -2 \le y \le 2$. Determine the direction field that corresponds to each equation. No justification is necessary. (Note that two fields do not have a matching equation.)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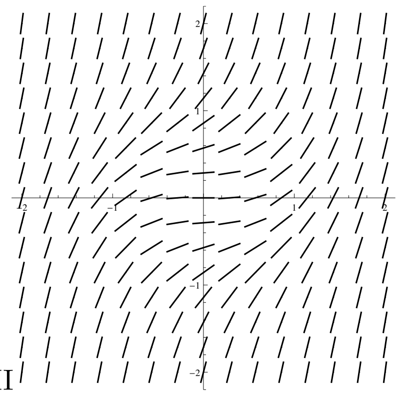
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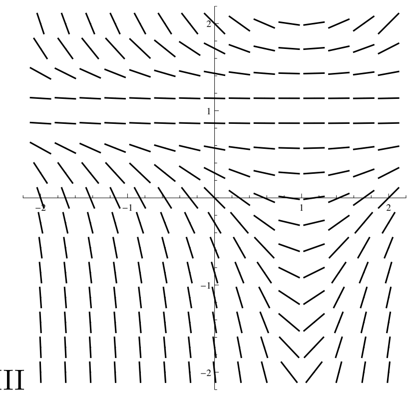
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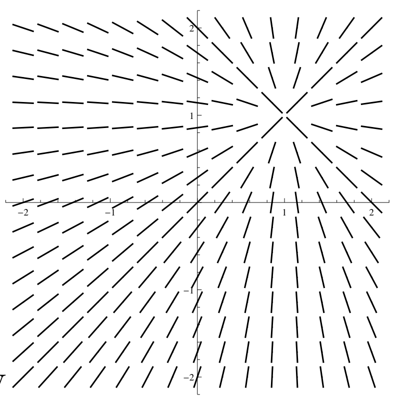
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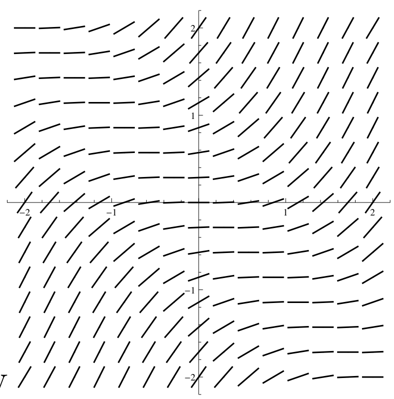
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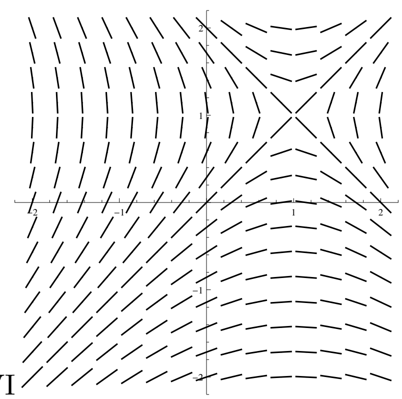
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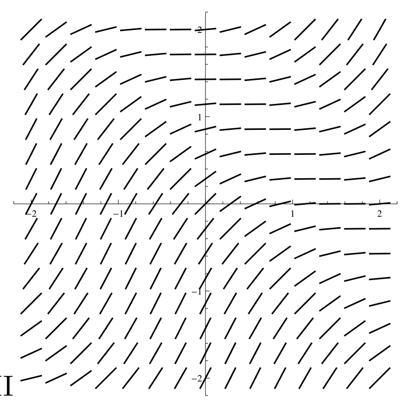
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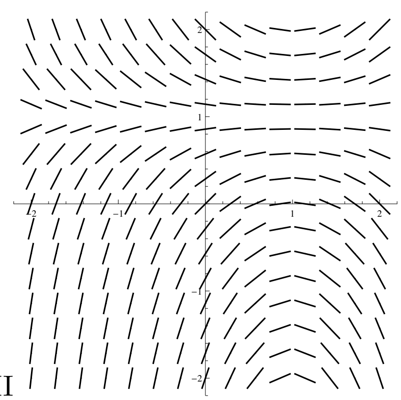

VIII

Equation	I, II, III, IV, V, VI, VII, or VIII	Equation	I, II, III, IV, V, VI, VII, or VIII
$y' = (y-1)^2(x-1)$	**III**	$y' = y^2 - x^2$	**I**
$y' = \dfrac{y-1}{x-1}$	**IV**	$y' = 1 - \sin(x+y)$	**VII**
$y' = \dfrac{x-1}{y-1}$	**VI**	$y' = 1 - \cos(x+y)$	**V**

5. (12 points)

(a) Show all steps in solving the initial value problem

$$\frac{dx}{dt} = x\cos t - \cos t, \quad x(0) = 2$$

(6 points) Taking out the common factor,

$$\frac{dx}{dt} = \cos t(x-1).$$

For this equation, $x = 1$ is an equilibrium solution, which doesn't satisfy $x(0) = 2$, so we may assume $x \neq 1$. Then the equation is separable. Dividing both side by $x - 1$ and integrating, we have

$$\ln|x-1| = \int \frac{1}{x-1}dx = \int \cos t dt = \sin t + C.$$

Using initial condition $x(0) = 2$,

$$\ln(2-1) = \sin 0 + C \qquad \Longrightarrow C = 0$$

Solving in x,

$$|x-1| = e^{\sin t} \qquad x - 1 = \pm e^{\sin t}.$$

However, the sign has to be positive from the initial condition. Therefore

$$x = 1 + e^{\sin t}$$

is the desired solution.

(b) Show all steps in solving the initial value problem

$$\frac{dy}{dx} = -xe^{y+x^2}, \quad y(0) = 0$$

(6 points) Using the exponential rule,

$$\frac{dy}{dx} = -xe^y e^{x^2}.$$

Then the equation is separable. Dividing by e^y and integrating, we have

$$-e^y = \int e^{-y}dy = -\int xe^{x^2}dx = -\frac{1}{2}e^{x^2} + C$$

where the second integral can be integrated using the substitution $u = x^2$. From the initial condition,

$$-e^0 = -\frac{1}{2}e^0 + C, \qquad \Longrightarrow C = -\frac{1}{2}.$$

Simplifying the equation, we have

$$y = -\ln\left(\frac{1}{2}e^{x^2} + \frac{1}{2}\right),$$

which is the desired solution.

6. (8 points) In a certain country the population grows according to natural growth with relative growth rate $k = \frac{1}{20}$ per year, but internal strife also encourages *emigration* (that is, departures) at a constant rate of 2 million people per year.

(a) Set up (but do not solve) a differential equation for $P(t)$, the population of the country in millions of people at time t, measured in years.

(3 points)

$$\frac{dP}{dt} = \frac{1}{20}P - 2.$$

(b) Find the equilibrium values of population based on your equation in part (a).

(2 points) The right hand side is a function of P, and the equilibrium solution is where $dP/dt = 0$. Thus $P = 40$ (million people).

Remark: the model doesn't make sense if $P < 0$, thus if the population reaches $P = 0$, then it will be likely to stay at $P = 0$, but the problem simply asks the equilibrium based on the differential equation in part (a). It is preferred to include $P = 40$ only.

(c) Suppose the population at time $t = 0$ is 75 million; solve the differential equation to find an expression for the population (in millions of people) after t years.

(3 points) The population is modeled by the initial value problem

$$\begin{cases} \frac{dP}{dt} = \frac{1}{20}P - 2 = \frac{1}{20}(P - 40) \\ P(0) = 75 \end{cases}$$

75 is not an equilibrium value, so $P \neq 40$. The equation is separable, and dividing both side by $P - 40$,

$$\ln|P - 40| = \int \frac{1}{P - 40} dP = \int \frac{1}{20} dt = \frac{t}{20} + C$$

Using $P(0) = 75$,

$$\ln(75 - 40) = 0 + C, \qquad \Longrightarrow C = \ln 35.$$

Simplifying the equation,

$$|P - 40| = e^{t/20 + \ln 35} = 35e^{t/20}$$

and thus $P - 40 = \pm 35e^{t/20}$. However, the sign has to be positive from the initial condition. Thus, the population after t years is

$$P(t) = 40 + 35e^{t/20} \text{ (million people)}.$$

7. (14 points) A charged particle in an idealized physics experiment is confined by electromagnetic forces to move along a (theoretically) infinite line so that at any time t, in seconds, its position $x(t)$, in meters, to the right of a fixed origin is governed by the differential equation

$$\frac{dx}{dt} = x - \frac{x^3}{4}.$$

(a) Find the equilibrium solutions of the differential equation.

(2 points) The right hand side is a function of x. Taking $dx/dt = 0$, we have

$$0 = x - \frac{x^3}{4} = x\left(1 - \frac{x}{2}\right)\left(1 + \frac{x}{2}\right).$$

Thus $x = 0$, $x = 2$, $x = -2$ are three equilibrium solutions.

(b) At what positions is the particle moving to the right? To the left? Standing still? Justify your answers.

(3 points) The particle is moving to the right at the position where $dx/dt > 0$, to the left where $dx/dt < 0$, and standing still if $dx/dt = 0$. Solving $dx/dt > 0$,

$$\frac{dx}{dt} = -x\left(\frac{x}{2} + 1\right)\left(\frac{x}{2} - 1\right) > 0 \iff x < -2 \text{ or } 0 < x < 2$$

using the graph of the cubic function. Similarly $dx/dt < 0$ if and only if $-2 < x < 0$ or $x > 2$, and $dx/dt = 0$ if and only if $x = 0, \pm 2$.

Answer. The particle moves to the left where $x < -2$ or $0 < x < 2$, to the right where $-2 < x < 0$ or $x > 2$, and stands still when $x = -2$, 0 or 2.

(c) Suppose $x(0) = 5$. What is the behavior of the particle as $t \to \infty$? Justify your answer. (*Hint:* you don't have to solve the differential equation.)

(3 points) From part (b), the particle moves to the left when $P > 2$, so initially the particle moves to the left and get close to 2. Then as x approaches to 2, dx/dt approaches to 0 and the particle slows down indefinitely. The particle never reaches the position at $x = 2$, because if we had $x(t_0) = 2$ at some time $t = t_0$, then solving initial condition problem with $x(t_0) = 2$ will guarantee $x(t) = 2$ everywhere, because $x(t) = 2$ is the equilibrium solution, but this is inconsistent with the initial condition $x(0) = 5$.

Remark. You may use the fact that the solution curves of a first-order differential equation (with various initial values) doesn't have a self-intersection point.

For easy reference, the differential equation is: $\dfrac{dx}{dt} = x - \dfrac{x^3}{4}$.

(d) Now suppose instead $x(0) = 1$. Use Euler's method with step size $\frac{1}{2}$ to estimate the value $x(1)$. Show your steps, but you don't need to simplify your answer.

(3 points) Let $F(t, x) = x^2 - x^3/4$. Using the step size $h = 1/2$, we may create the following table.

t	x	dx/dt
$t_0 = 0$	$x_0 = 1$	$F(t_0, x_0) = 1 - \frac{1}{4} = \frac{3}{4}$
$t_0 = \frac{1}{2}$	$x_1 = x_0 + hF(t_0, x_0) = 1 + \frac{1}{2} \cdot \frac{3}{4} = \frac{11}{8}$	$F(t_1, x_1) = \frac{11}{8} - \frac{1}{4}\left(\frac{11}{8}\right)^3$
$t_0 = 1$	$x_2 = x_1 + hF(t_1, x_1) = \frac{11}{8} + \frac{1}{2}\left(\frac{11}{8} - \frac{1}{4}\left(\frac{11}{8}\right)^3\right)$	

Thus $x(1) \approx \frac{11}{8} + \frac{1}{2}\left(\frac{11}{8} - \frac{1}{4}\left(\frac{11}{8}\right)^3\right)$.

(e) In reality, the experiment takes place only between $x = -2$ and $x = 2$ meters, not $x = \pm\infty$. Determine those positions located between $x = -2$ and $x = 2$ when the particle is moving the fastest. (*Hint:* first use the differential equation to find an expression for $\dfrac{d^2x}{dt^2}$.)

(3 points) The problem asks when the speed of particle $\left|\dfrac{dx}{dt}\right|$ attains the maximum when $-2 \le x \le 2$. Then the maximum is attained where either dx/dt attains its maximum or minimum, and the extrema of dx/dt, defined on the closed interval $[-2, 2]$, is attained either at the endpoint or at the critical point, i.e., where

$$\frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} = 0.$$

By the chain rule,

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(x - \frac{x^3}{4}\right) = \frac{d}{dx}\left(x - \frac{x^3}{4}\right) \cdot \frac{dx}{dt} = \left(1 - \frac{3x^2}{4}\right)\left(x - \frac{x^3}{4}\right) = 0.$$

Thus the critical values are when $x = 0, \pm 2, \pm 2\sqrt{3}/3$. At $x = 0, \pm 2$, $dx/dt = 0$. At $x = 2\sqrt{3}/3$, $dx/dt = 4\sqrt{3}/9$. At $x = -2\sqrt{3}/3$, $dx/dt = -4\sqrt{3}/9$. Therefore the particle is at the same speed (with the opposite direction) when $x = \pm 2\sqrt{3}/3$, which is the maximum of $|dx/dt|$, i.e., where the particle moves fastest.

8. (15 points) In a certain closed ecosystem, let functions $x(t)$ and $y(t)$ represent the population sizes (in thousands of beings) of two species, X and Y, respectively; here the time t is measured in months. Suppose further that the population sizes are modeled by the equations

$$\frac{dx}{dt} = \frac{x}{8} - \frac{xy}{400}$$
$$\frac{dy}{dt} = \frac{y}{8} - \frac{y^2}{800} - \frac{xy}{400}$$

(a) Describe the nature of the relationship between the species X and Y: is it one of competition, cooperation, or predation? If the relationship is one of predation, indicate which species is the predator and which is the prey. Explain every part of your answer.

(3 points) From the term $-\frac{xy}{400}$ that occurs in both dx/xt and dy/dt, one see that the growth rate of each species suffers from the presence of the other. That is, the species are in competition.

(b) Find all equilibrium solutions of the above system. (You may omit negative x, y here.)

(3 points) Suppose that $x(t)$ and $y(t)$ are constant function (independent of t), so their derivative are zero:

$$\frac{x}{8} - \frac{xy}{400} = 0, \qquad \frac{y}{8} - \frac{y^2}{800} - \frac{xy}{400} = 0$$

We first factor the first equation to get

$$x\left(\frac{1}{8} - \frac{y}{400}\right) = 0$$

That is,

$$x = 0 \text{ or } y = 50$$

If $x = 0$, then the second equation becomes

$$0 = \frac{y}{8} - \frac{y^2}{800} = y\left(\frac{1}{8} - \frac{y}{800}\right)$$

That is,

$$y = 0 \text{ or } y = 100$$

Thus, we have two equilibrium points $(0, 0)$ and $(0, 100)$ from the first case. In the second case, we have $y = 50$, so the second equation becomes

$$0 = \frac{50}{8} - \frac{50^2}{800} - \frac{50x}{400} = \frac{50 - 25 - x}{8} = \frac{25 - x}{8}$$

That is, $x = 25$, so for the second case, we have an equilibrium point $(25, 50)$. To sum up, we have three equilibrium points $(0, 0), (0, 100)$ and $(25, 50)$.

For easy reference, here again is the system: $\begin{cases} \dfrac{dx}{dt} = \dfrac{x}{8} - \dfrac{xy}{400} \\ \dfrac{dy}{dt} = \dfrac{y}{8} - \dfrac{y^2}{800} - \dfrac{xy}{400} \end{cases}$

(c) Suppose that at time $t = 0$ months, we have $x(0) = 0$ and $y(0) = 20$. Solve for an explicit formula that gives the population size $y(t)$ in terms of t. What happens to x and y as t approaches infinity?

(4 points) Since $x(0) = 0$, we can use the equation for dx/dt and find that $dx/dt(0) = 0$. The population of species x is zero and it is staying that way. So $x(t) = 0$ for all t. Substitute that into the equation for dy/dt, one gets

$$\frac{dy}{dt} = \frac{y}{8} - \frac{y^2}{800} = \frac{1}{8}y\left(1 - \frac{y}{100}\right)$$

which is a logistic equation with $k = \frac{1}{8}, M = 100$. Therefore, the solution is given by

$$y(t) = \frac{M}{1 + Ae^{-kt}} = \frac{100}{1 + Ae^{-t/8}}$$

where A is a constant. We substitute $y(0) = 20$ to find A:

$$20 = y(0) = \frac{100}{1 + A} \qquad \Rightarrow \qquad A = 4$$

So

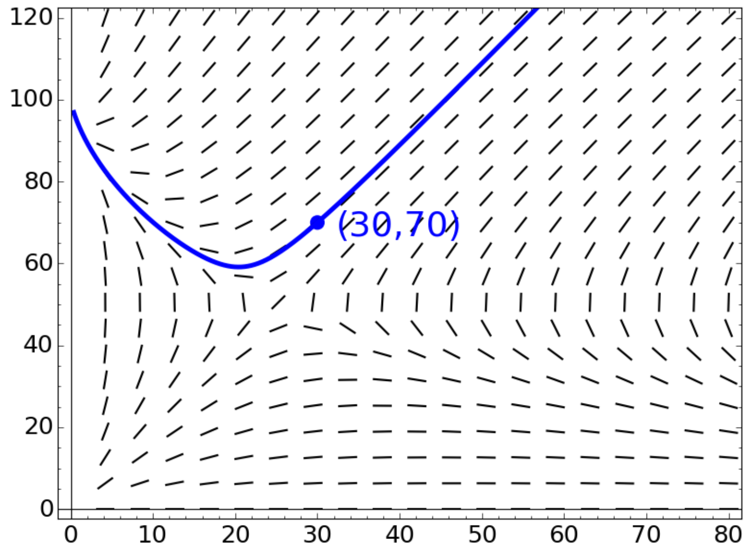
$$y(t) = \frac{100}{1 + 4e^{-t/8}}$$

Taking limit as t approaches infinity, one get $y(t)$ approaches 100 (its carrying capacity), while $x(t) = 0$ for all time t, so the limit is simply zero. That is,

$$\lim_{t\to\infty} x(t) = 0 \qquad \lim_{t\to\infty} y(t) = 100$$

For easy reference, here again is the system: $\begin{cases} \dfrac{dx}{dt} = \dfrac{x}{8} - \dfrac{xy}{400} \\ \dfrac{dy}{dt} = \dfrac{y}{8} - \dfrac{y^2}{800} - \dfrac{xy}{400} \end{cases}$

(d) Now suppose instead that the species populations are measured to be $x(0) = 30$ and $y(0) = 70$ at time $t = 0$ months. Below is a picture of a direction field for the system of differential equations satisfied by species X and Y, and on it is drawn the phase trajectory corresponding to the initial condition $x(0) = 30$, $y(0) = 70$:



How should an arrow be drawn on the above trajectory so that it represents how the species' populations change as the time t increases? Explain your reasoning precisely. (*Hint*: for each of the two populations, determine conditions that predict if it is increasing, decreasing, or not changing size at a given moment.)

(2 points) Note that

$$\frac{dx}{dt} = \frac{x}{8} - \frac{xy}{400} = \frac{x}{400}(50 - y)$$

Thus, whenever $y > 50$ and $x > 0$, we get that $dx/dt < 0$, so $x(t)$ is decreasing. One see that the given trajectory always satisfies these two conditions, so $x(t)$ is decreasing along this trajectory, i.e. the arrow is pointing to the left.

(e) Use the information provided above to describe the eventual fate of the species X and Y, starting from the initial condition $x(0) = 30$, $y(0) = 70$.

(3 points) From the given trajectory and its direction, one see that both species will first decrease for awhile until a point where species x is sufficiently low that the growth of species y outpaces the effect of the competition from species x. As a result, species x approaches 0, while y approaches its carrying capacity of 100.

9. (12 points) Suppose the power series $\sum_{n=0}^{\infty} c_n x^n$ converges for $x = -4$, but diverges for $x = 6$; no other information about the values of c_n is given. Decide whether each of the following statements is either *always true* ("T"), or *always false* ("F"), or *sometimes true and sometimes false, depending on the situation* ("MAYBE"). Circle your answer. You do not need to provide justification.

(2 points each) Let R be the radius of convergence of the power series. By assumption, we observe that $|-4| \leq R \leq |6|$.

(a) $\sum_{n=0}^{\infty} c_n 3^n$ converges. **[T]** F MAYBE

$x = 3 < 4$ is inside the radius.

(b) $\sum_{n=0}^{\infty} c_n (-5)^n$ converges. T F **[MAYBE]**

$x = -5$ is 5 units from the center, may or may not be outside the radius.

(c) $\sum_{n=1}^{\infty} \frac{c_n}{n}$ converges absolutely. **[T]** F MAYBE

One can compute the ratio of the series $\sum_{n=1}^{\infty} |\frac{c_n}{n}|$, which is less or equal to $1/R \leq 1/4$. Then it is absolutely convergent by ratio test.

(d) $\lim_{n\to\infty} c_n = 0$. **[T]** F MAYBE

By the Test for Divergence, this is true because $\sum_{n=1}^{\infty} c_n$ is convergent ($x = 1$ is inside the radius).

(e) $\lim_{n\to\infty} c_n 6^n = \infty$. T F **[MAYBE]**

Since $\sum_{n=1}^{\infty} c_n 6^n$ is divergent, the Test for Divergence fails; $\lim_{n\to\infty} c_n 6^n$ may or may not be infinity.

(f) $\lim_{n\to\infty} c_{2n} 9^n = \infty$. T **[F]** MAYBE

By (a) and the Test for Divergence, $\lim_{n\to\infty} c_n 3^n = 0$. This implies $\lim_{n\to\infty} c_{2n} 3^{2n} = \lim_{n\to\infty} c_{2n} 9^n = 0$.

10. (10 points) Show all steps in completing the parts below; if you wish, you may freely use without proof the fact that

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \text{for all } x.$$

(a) Compute the limit, or show that it does not exist: $\lim_{x\to 0} \frac{x^2\cos x + 2\cos x - 2}{x^2 + 2\cos x - 2}$

(5 points) Using the power series for $\cos x$, we get

$$\begin{aligned}
\lim_{x\to 0} \frac{x^2\cos x + 2\cos x - 2}{x^2 + 2\cos x - 2} &= \lim_{x\to 0} \frac{x^2\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right) + 2\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right) - 2}{x^2 + 2\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right) - 2} \\
&= \lim_{x\to 0} \frac{\left(-\frac{x^4}{2!} + \frac{x^6}{4!} - \frac{x^8}{6!} + \dots\right) + \left(\frac{2x^4}{4!} - \frac{2x^6}{6!} + \dots\right)}{2\left(\frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots\right)} \\
&= \lim_{x\to 0} \frac{x^4\left[\left(-\frac{1}{2!} + \frac{x^2}{4!} - \frac{x^4}{6!} + \dots\right) + \left(\frac{2}{4!} - \frac{2x^2}{6!} + \dots\right)\right]}{x^4\left[2\left(\frac{1}{4!} - \frac{x^2}{6!} + \frac{x^8}{8!} - \dots\right)\right]} \\
&= \lim_{x\to 0} \frac{\left(-\frac{1}{2!} + \frac{x^2}{4!} - \frac{x^4}{6!} + \dots\right) + \left(\frac{2}{4!} - \frac{2x^2}{6!} + \dots\right)}{2\left(\frac{1}{4!} - \frac{x^2}{6!} + \frac{x^8}{8!} - \dots\right)} = \frac{-\frac{1}{2!} + \frac{2}{4!}}{\frac{2}{4!}} = \boxed{-5}
\end{aligned}$$

Alternatively, one can use L'Hospital's Rule, but we need to apply it four times to reach the answer.

(b) Determine with justification whether the series $\sum_{k=1}^{\infty}\left(1 - \cos\left(\frac{1}{k}\right)\right)$ converges or diverges.

(5 points) **Solution 1 (Limit Comparison Test):** The series given yields the approximation $\cos x \approx T_2(x) = 1 - \frac{x^2}{2!}$, and thus that $1 - \cos x \approx \frac{x^2}{2}$, for x near 0. Thinking of x as $\frac{1}{k}$ in our case, and noting $\sum_{k=1}^{\infty} \frac{1}{2k^2} = \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges by the p-Series Rule with $p = 2$, this suggests we should try using the Limit Comparison Test (it applies, as $1 - \cos(\frac{1}{k}) > 0$ for all $k \geq 1$). Indeed,

$$\lim_{k\to\infty} \frac{1 - \cos(\frac{1}{k})}{\frac{1}{2k^2}} = \lim_{x\to 0^+} \frac{1 - \cos x}{\frac{x^2}{2}} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1 > 0$$

via l'Hospital's Rule, so the series $\sum_{k=1}^{\infty}\left(1 - \cos\left(\frac{1}{k}\right)\right)$ converges since $\sum_{k=1}^{\infty}\frac{1}{2k^2}$ does.

Solution 2 (Comparison Test): We will show that $1 - \cos x < \frac{x^2}{2}$ for $x > 0$. Note that the Taylor series for $\cos x$ for $x > 0$ is an alternating series. Using the Alternating Series Remainder Estimate from the approximation by the first term, we get

$$|\cos x - 1| = \left|\left(\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\right) - 1\right| \leq \frac{x^2}{2!} \qquad \Longrightarrow \qquad 1 - \cos x \leq |\cos x - 1| \leq \frac{x^2}{2!} = \frac{x^2}{2}$$

as we wanted. (Alternatively, we can show this directly by differentiating the function $f(x) = \cos x + \frac{x^2}{2}$ to get $f'(x) = \sin x + x$ which is positive for all $x > 0$. Thus, the function is increasing so $f(x) > f(0) = 1$ for all $x > 0$, that is, $\cos x + \frac{x^2}{2} > 1$ or equivalently $1 - \cos x < \frac{x^2}{2}$.) Using this fact for $x = 1/k$, we get

$$0 < 1 - \cos\left(\frac{1}{k}\right) < \frac{1}{2k^2}$$

Thus, by Comparison Test, $\sum_{k=1}^{\infty}\left(1 - \cos\left(\frac{1}{k}\right)\right)$ converges since (as in solution 1) $\sum_{k=1}^{\infty}\frac{1}{2k^2}$ does.