Math 42 Final Exam — March 17, 2014

Name: ______________________________ **SUID#:** ______________

Circle your section:			
Junsoo Ha 03 (10-10:50am) 08 (1:15-2:05pm)	Nick Ronchetti 05 (11-11:50am) 07 (9-9:50am)	Li-Cheng Tsai 02 (10-10:50am) 04 (11-11:50am)	Sunny Ahuja ACE

- Complete the following problems. In order to receive full credit, please show all of your work and justify your answers. You do not need to simplify your answers unless specifically instructed to do so. You may use any result proved in class or the text, but be sure to clearly state the result before using it, and to verify that all hypotheses are satisfied.
- Please check that your copy of this exam contains 16 numbered pages and is correctly stapled.
- This is a closed-book, closed-notes exam. No electronic devices, including cellphones, headphones, or calculation aids, will be permitted for any reason.
- **You have 3 hours.** Your organizer will signal the times between which you are permitted to be writing, including anything on this cover sheet, and to have the exam booklet open. During these times, the exam and all papers must remain in the testing room. When you are finished, you must hand your exam paper to a member of teaching staff.
- Paper not provided by teaching staff is prohibited. If you need extra room for your answers, use the back side of the question page or other extra space provided at the front of this packet, and clearly indicate that your answer continues there. Do not unstaple or detach pages from this exam.
- Please sign the following:

> "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ______________________________

1. (10 points) Let T be the triangular region in the xy-plane with vertices $(0, 0)$, $(2, 0)$, and $(1, 1)$.

   (a) Set up, but do not evaluate, an expression for the volume of the solid formed by rotating T about the line $y = 2$. Justify your answer by drawing a picture, labeling sample slices, and citing the method used.

   (b) Set up, but do not evaluate, an expression for the volume of the solid formed by rotating T about the line $x = -1$. Justify your answer by drawing a picture, labeling sample slices, and citing the method used.

2. (12 points) For this problem, use the following information:

- If g is a normal ("bell-shaped" or "Gaussian") probability density function, then g has the general form

$$g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

- A partial list of approximate values of the function

$$P(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt \quad \text{is given at right:}$$

$P(0.5) \approx 0.69$	$P(1.3) \approx 0.90$
$P(0.6) \approx 0.72$	$P(1.4) \approx 0.92$
$P(0.7) \approx 0.76$	$P(1.5) \approx 0.93$
$P(0.8) \approx 0.79$	$P(1.6) \approx 0.94$
$P(0.9) \approx 0.82$	$P(1.7) \approx 0.95$
$P(1.0) \approx 0.84$	$P(1.8) \approx 0.96$
$P(1.1) \approx 0.86$	$P(1.9) \approx 0.97$
$P(1.2) \approx 0.88$	$P(2.0) \approx 0.98$

Suppose that a manufacturer of pressure gauges is testing the accuracy of its products before placing them on the market; this is accomplished by using each gauge to measure a sample of compressed air with a known pressure value of exactly 50 psi. Suppose that the reading found by each gauge is a random variable having a normal distribution.

(a) Given that the mean reading is 50 psi, and the standard deviation is 1.2 psi, find the approximate probability that a randomly selected gauge will make a reading on the sample that is less than 50.6 psi. (Your final answer should be a number, but first express it as a probability integral and show how it can be evaluated using the information given.)

(b) With the same values of mean and standard deviation as in part (a), find the approximate probability that a randomly selected gauge will make a reading that is less than 48.8 psi. (Again give a number as your final answer, but use an appropriate integral expression as part of your justification.)

For easy reference, here is all of the information provided on the previous page:

- If g is a normal ("bell-shaped" or "Gaussian") probability density function, then g has the general form

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(c) Now suppose that after a change in the manufacturing process, it is determined that the mean reading of pressure is still 50 psi, but the standard deviation has changed. In addition, it is now found that approximately 14 percent of readings are outside the range from 49 to 51 psi. What is the new standard deviation? (Again use an appropriate integral expression as part of your justification.)

3. (12 points) A Silicon Valley venture capitalist models the lifespan of an Internet start-up company as a random variable, with probability density function

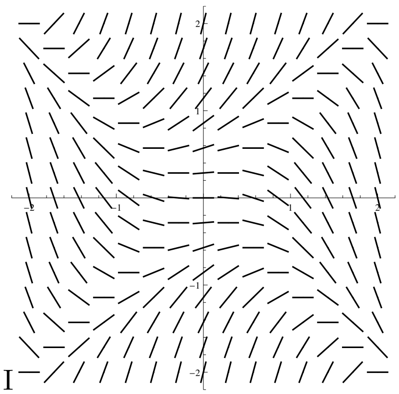
$$f(t) = \begin{cases} \dfrac{Ct}{(t^2+1)^2} & \text{if } t \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

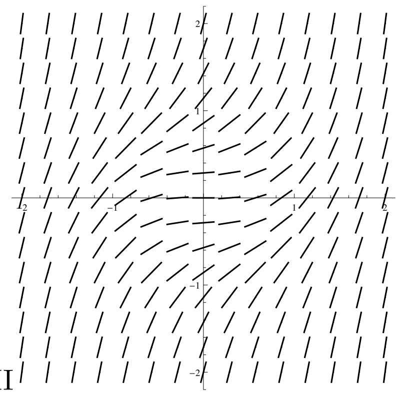
where t is measured in years, and C is a positive constant.

(a) Find C, using the fact that f is a probability density function.

(b) Find the mean lifespan of a company according to this venture capitalist's model.

4. (12 points) Each of the equations given below corresponds to exactly one of the eight direction fields displayed; the scale on each is $-2 \le x \le 2$, $-2 \le y \le 2$. Determine the direction field that corresponds to each equation. No justification is necessary. (Note that two fields do not have a matching equation.)


I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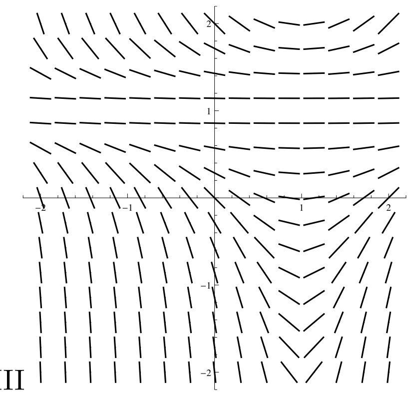
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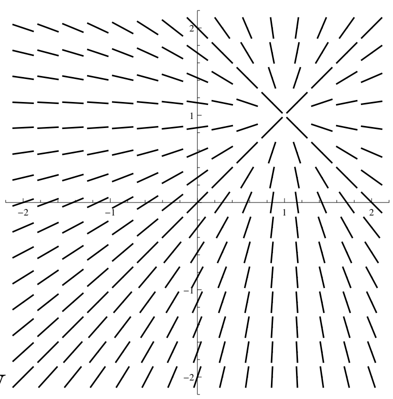
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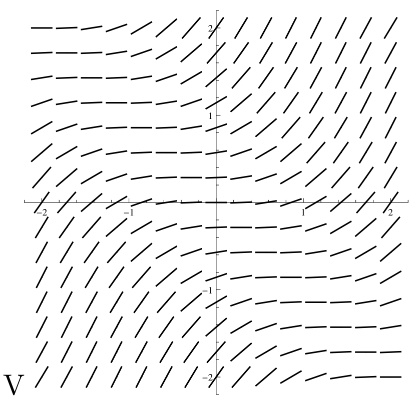
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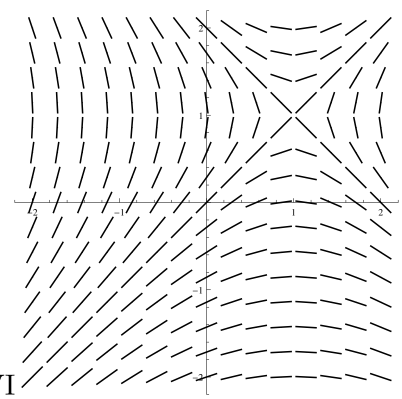
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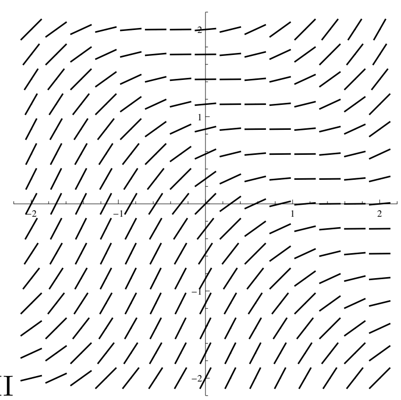
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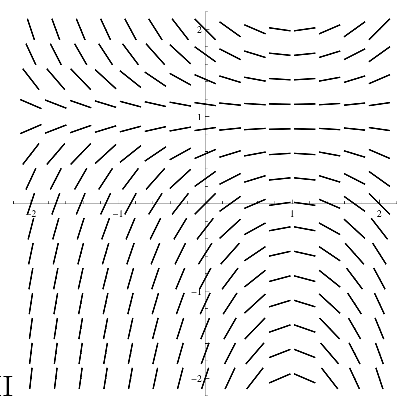

VIII

Equation	I, II, III, IV, V, VI, VII, or VIII	Equation	I, II, III, IV, V, VI, VII, or VIII
$y' = (y-1)^2(x-1)$		$y' = y^2 - x^2$	
$y' = \dfrac{y-1}{x-1}$		$y' = 1 - \sin(x+y)$	
$y' = \dfrac{x-1}{y-1}$		$y' = 1 - \cos(x+y)$	

5. (12 points)

(a) Show all steps in solving the initial value problem

$$\frac{dx}{dt} = x\cos t - \cos t, \quad x(0) = 2$$

(b) Show all steps in solving the initial value problem

$$\frac{dy}{dx} = -xe^{y+x^2}, \quad y(0) = 0$$

6. (8 points) In a certain country the population grows according to natural growth with relative growth rate $k = \frac{1}{20}$ per year, but internal strife also encourages *emigration* (that is, departures) at a constant rate of 2 million people per year.

   (a) Set up (but do not solve) a differential equation for $P(t)$, the population of the country in millions of people at time t, measured in years.

   (b) Find the equilibrium values of population based on your equation in part (a).

   (c) Suppose the population at time $t = 0$ is 75 million; solve the differential equation to find an expression for the population (in millions of people) after t years.

7. (14 points) A charged particle in an idealized physics experiment is confined by electromagnetic forces to move along a (theoretically) infinite line so that at any time t, in seconds, its position $x(t)$, in meters, to the right of a fixed origin is governed by the differential equation

$$\frac{dx}{dt} = x - \frac{x^3}{4}.$$

(a) Find the equilibrium solutions of the differential equation.

(b) At what positions is the particle moving to the right? To the left? Standing still? Justify your answers.

(c) Suppose $x(0) = 5$. What is the behavior of the particle as $t \to \infty$? Justify your answer. (*Hint:* you don't have to solve the differential equation.)

For easy reference, the differential equation is: $\dfrac{dx}{dt} = x - \dfrac{x^3}{4}$.

(d) Now suppose instead $x(0) = 1$. Use Euler's method with step size $\frac{1}{2}$ to estimate the value $x(1)$. Show your steps, but you don't need to simplify your answer.

(e) In reality, the experiment takes place only between $x = -2$ and $x = 2$ meters, not $x = \pm\infty$. Determine those positions located between $x = -2$ and $x = 2$ when the particle is moving the fastest. (*Hint:* first use the differential equation to find an expression for $\dfrac{d^2x}{dt^2}$.)

8. (15 points) In a certain closed ecosystem, let functions $x(t)$ and $y(t)$ represent the population sizes (in thousands of beings) of two species, X and Y, respectively; here the time t is measured in months. Suppose further that the population sizes are modeled by the equations

$$\frac{dx}{dt} = \frac{x}{8} - \frac{xy}{400}$$
$$\frac{dy}{dt} = \frac{y}{8} - \frac{y^2}{800} - \frac{xy}{400}$$

(a) Describe the nature of the relationship between the species X and Y: is it one of competition, cooperation, or predation? If the relationship is one of predation, indicate which species is the predator and which is the prey. Explain every part of your answer.

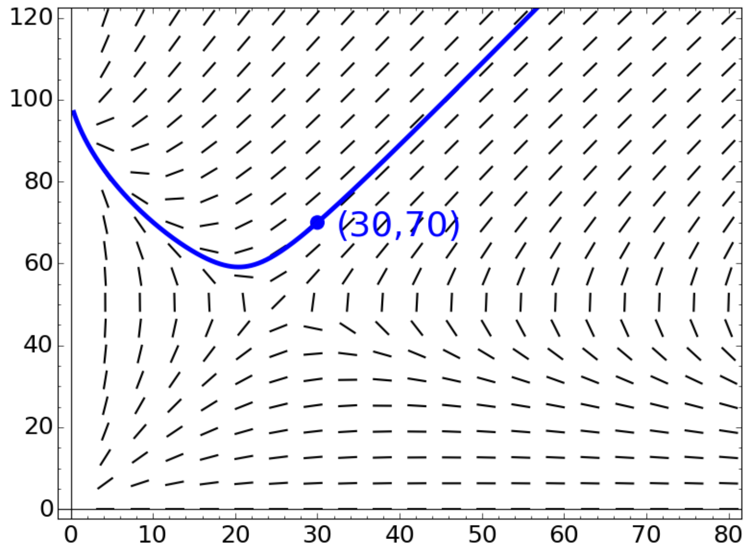
(b) Find all equilibrium solutions of the above system. (You may omit negative x, y here.)

For easy reference, here again is the system: $\begin{cases} \dfrac{dx}{dt} = \dfrac{x}{8} - \dfrac{xy}{400} \\ \dfrac{dy}{dt} = \dfrac{y}{8} - \dfrac{y^2}{800} - \dfrac{xy}{400} \end{cases}$

(c) Suppose that at time $t = 0$ months, we have $x(0) = 0$ and $y(0) = 20$. Solve for an explicit formula that gives the population size $y(t)$ in terms of t. What happens to x and y as t approaches infinity?

For easy reference, here again is the system: $\begin{cases} \dfrac{dx}{dt} = \dfrac{x}{8} - \dfrac{xy}{400} \\ \dfrac{dy}{dt} = \dfrac{y}{8} - \dfrac{y^2}{800} - \dfrac{xy}{400} \end{cases}$

(d) Now suppose instead that the species populations are measured to be $x(0) = 30$ and $y(0) = 70$ at time $t = 0$ months. Below is a picture of a direction field for the system of differential equations satisfied by species X and Y, and on it is drawn the phase trajectory corresponding to the initial condition $x(0) = 30$, $y(0) = 70$:



How should an arrow be drawn on the above trajectory so that it represents how the species' populations change as the time t increases? Explain your reasoning precisely. (*Hint*: for each of the two populations, determine conditions that predict if it is increasing, decreasing, or not changing size at a given moment.)

(e) Use the information provided above to describe the eventual fate of the species X and Y, starting from the initial condition $x(0) = 30$, $y(0) = 70$.

9. (12 points) Suppose the power series $\sum_{n=0}^{\infty} c_n x^n$ converges for $x = -4$, but diverges for $x = 6$; no other information about the values of c_n is given. Decide whether each of the following statements is either *always true* ("T"), or *always false* ("F"), or *sometimes true and sometimes false, depending on the situation* ("MAYBE"). Circle your answer. You do not need to provide justification.

(a) $\sum_{n=0}^{\infty} c_n 3^n$ converges. T F MAYBE

(b) $\sum_{n=0}^{\infty} c_n (-5)^n$ converges. T F MAYBE

(c) $\sum_{n=1}^{\infty} \frac{c_n}{n}$ converges absolutely. T F MAYBE

(d) $\lim_{n\to\infty} c_n = 0$. T F MAYBE

(e) $\lim_{n\to\infty} c_n 6^n = \infty$. T F MAYBE

(f) $\lim_{n\to\infty} c_{2n} 9^n = \infty$. T F MAYBE

10. (10 points) Show all steps in completing the parts below; if you wish, you may freely use without proof the fact that

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \text{for all } x.$$

(a) Compute the limit, or show that it does not exist: $\lim_{x \to 0} \frac{x^2 \cos x + 2\cos x - 2}{x^2 + 2\cos x - 2}$

For easy reference, $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ for all x, a fact which you do not have to prove.

(b) Determine with justification whether the series $\sum_{n=1}^{\infty} \left(1 - \cos\left(\frac{1}{n}\right)\right)$ converges or diverges.