Solutions to Math 42 Second Exam — February 20, 2014

1. (12 points)

   (a) Evaluate $\int_0^\infty xe^{-x^2}\,dx$ or explain why its value does not exist; show all reasoning.

   (5 points) The function $f(x) = xe^{-x^2}$ is continuous at 0, so we have an improper integral of type I and we should compute

   $$\lim_{t\to\infty}\int_0^t xe^{-x^2}\,dx.$$

   We use u-substitution: let $u = -x^2$ so that $du = -2x\,dx$.
   The bounds of integration become $u(0) = 0$ and $u(t) = -t^2$, hence we obtain

   $$\lim_{t\to\infty}\int_0^{-t^2} e^u\left(-\frac{du}{2}\right) = -\frac{1}{2}\lim_{t\to\infty} e^u\Big]_0^{-t^2} =$$

   $$= -\frac{1}{2}\lim_{t\to\infty}\left(e^{-t^2} - 1\right) = -\frac{1}{2}(0-1) = \boxed{\frac{1}{2}}.$$

   (b) Determine whether $\int_0^1 \frac{\ln(1+x)}{x^2}dx$ converges or diverges; give complete reasoning.

   (7 points) The function $f(x) = \frac{\ln(1+x)}{x^2}$ is continuous and positive for all $x > 0$, but is not continuous at 0 so we have an improper integral of type II. Here are three possible solutions:
   Solution 1 (Limit Comparison Theorem): We take $g(x) = \frac{1}{x}$ as a reference function, also continuous and positive for all $x > 0$: we have

   $$\lim_{x\to 0^+}\frac{\frac{\ln(1+x)}{x^2}}{\frac{1}{x}} = \lim_{x\to 0^+}\frac{\ln(1+x)}{x^2}\cdot\frac{x}{1} = \lim_{x\to 0^+}\frac{\ln(1+x)}{x}.$$

   The last fraction is an indeterminate form of type $\frac{0}{0}$ and both $\ln(1+x)$ and x are continuous and positive when we take the limit for $x \to 0^+$, hence using l'Hôpital's Rule we obtain

   $$\lim_{x\to 0^+}\frac{\ln(1+x)}{x} = \lim_{x\to 0^+}\frac{\frac{1}{1+x}}{1} = \lim_{x\to 0^+}\frac{1}{1+x} = \frac{1}{1+0} = 1 > 0.$$

   Therefore by the Limit Comparison test for improper integrals of type II, our original integral converges if and only if

   $$I = \int_0^1 \frac{1}{x}\,dx$$

   converges.
   It remains to compute the integral above. We have

   $$I = \lim_{t\to 0^+}\int_t^1 \frac{1}{x}\,dx = \lim_{t\to 0^+}\ln x\Big]_t^1 = \lim_{t\to 0^+}(\ln 1 - \ln t) = -(0-\infty) = +\infty,$$

   hence $\int_0^1 \frac{\ln(1+x)}{x^2}\,dx$ $\boxed{\text{diverges}}$ too.
   Solution 2 (direct computation): We can do the indefinite integral by parts, using $u = \ln(1+x)$, $dv = x^{-2}\,dx$, so that $du = \frac{1}{1+x}dx$, $v = -\frac{1}{x}$:

   $$\int\frac{\ln(1+x)}{x^2}dx = -\frac{\ln(1+x)}{x} + \int\frac{dx}{x(1+x)}$$

Using the partial fraction decomposition $\dfrac{1}{x(x+1)} = \dfrac{1}{x} - \dfrac{1}{x+1}$ the latter integral becomes

$$\int \frac{dx}{x(1+x)} = \int \frac{dx}{x} - \int \frac{dx}{x+1} = \ln x - \ln(x+1) + C$$

so that

$$\begin{aligned}\int \frac{\ln(1+x)}{x^2}dx &= -\frac{\ln(1+x)}{x} + \int \frac{dx}{x(1+x)} \\ &= -\frac{\ln(1+x)}{x} + \ln x - \ln(x+1) + C\end{aligned}$$

Now we have

$$\begin{aligned}\int_0^1 \frac{\ln(1+x)}{x^2}dx &= \lim_{a\to 0^+}\left[\frac{\ln(1+x)}{x} + \ln x - \ln(x+1)\right]_a^1 \\ &= \lim_{a\to 0^+}\left[\ln 2 + 0 - \ln 2 - \frac{\ln(1+a)}{a} - \ln a + \ln(a+1)\right] \\ &= \lim_{a\to 0^+}\left[-\frac{\ln(1+a)}{a} + \ln\left(\frac{a+1}{a}\right)\right] = \infty, \text{ so the integral } \boxed{\text{diverges}}, \text{ since}\end{aligned}$$

$\lim\limits_{a\to 0^+}\left(-\dfrac{\ln(1+a)}{a}\right) = -1$ (using l'Hôpital as above) and $\lim\limits_{a\to 0^+}\ln\left(\dfrac{a+1}{a}\right) = \infty$.

Solution 3 (Comparison Theorem): We should strongly suspect that this integral diverges, because near the integrand's discontinuity of $x = 0$, the function $\ln(1+x)$ is approximately x (this is one of our basic linear approximations, or you can think of x as the start of the Taylor series for $\ln(1+x)$), which means that $\dfrac{\ln(1+x)}{x^2}$ is approximately $\dfrac{x}{x^2} = \dfrac{1}{x}$; and we can verify (as in solution 1) that the improper integral $\displaystyle\int_0^1 \frac{dx}{x}$ diverges. This is not a proof, but it helps guide us toward making a successful comparison.

The one complication here is that $\ln(1+x) \le x$ for small x (again think about how this expresses that the graph of the linear approximation lies above the graph of $\ln(1+x)$, since the latter is concave down), so we can't apply the Comparison Theorem to $\dfrac{\ln(1+x)}{x^2}$ and $\dfrac{1}{x}$, since the latter is larger and its integral diverges. But there is a simple fix: we claim that

$$(*) \qquad \frac{\ln(1+x)}{x^2} \ge \frac{1}{2x} \quad \text{for all} \quad 0 < x \le 1.$$

If true, the Comparison Test would apply; and $\displaystyle\int_0^1 \frac{\ln(1+x)}{x^2}dx$ $\boxed{\text{diverges}}$, since $\displaystyle\int_0^1 \frac{dx}{2x}$ diverges.

To see why $(*)$ is true, note that $(*)$ follows if we could show that

$$(**) \qquad \ln(1+x) \ge \frac{x}{2} \quad \text{for all} \quad 0 \le x \le 1.$$

But the function $f(x) = \ln(1+x) - \dfrac{x}{2}$ satisfies $f(0) = 0$, and

$$f'(x) = \frac{1}{1+x} - \frac{1}{2} > 0 \quad \text{for all} \quad 0 \le x < 1,$$

so f is increasing on $[0,1)$ and is therefore nonnegative on this interval (and it is also nonnegative at $x = 1$ by direct computation), so the inequality $(**)$ is true, and this implies $(*)$.

2. (10 points) Determine, with justification, whether each series converges. *If the series converges, find its sum.*

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{4^{n-1} - 5^{n+2}}{3^{2n}}$

(5 points) Note that

$$\sum_{n=1}^{\infty} \frac{4^{n-1}}{3^{2n}} = \frac{1}{9} \sum_{n=1}^{\infty} \frac{4^{n-1}}{9^{n-1}}$$

is a geometric series with common ratio $\frac{4}{9} < 1$, so it is convergent. It converges to

$$\frac{1}{9} \times \frac{1}{1 - \frac{4}{9}} = \frac{1}{9} \times \frac{9}{5} = \frac{1}{5}$$

Similarly,

$$\sum_{n=1}^{\infty} \frac{5^{n+1}}{3^{2n}} = \frac{5^3}{9} \sum_{n=1}^{\infty} \frac{5^{n-1}}{9^{n-1}}$$

is also a geometric series with common ratio $\frac{5}{9} < 1$, so it is convergent. It converges to

$$\frac{5^3}{9} \frac{1}{1 - \frac{5}{9}} = \frac{5^3}{9} \frac{9}{4} = \frac{5^3}{4} = \frac{125}{4}$$

Thus,

$$\sum_{n=1}^{\infty} \frac{4^{n-1}}{3^{2n}} - \frac{5^{n+1}}{3^{2n}}$$

is convergent since it is a difference of two convergent series and it converges to

$$\frac{1}{5} - \frac{125}{4} = -\frac{621}{20}$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + \sqrt{n+1}}$

(5 points) Note that for any integer n,

$$\sqrt{n} + \sqrt{n+1} < \sqrt{n} + \sqrt{n+3n} = \sqrt{n} + 2\sqrt{n} = 3\sqrt{n}$$

so

$$\frac{1}{\sqrt{n} + \sqrt{n+1}} > \frac{1}{3\sqrt{n}} > 0$$

We know that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges since it is a p-series with $p = 1/2 < 1$. Thus, by Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$ diverges as well.

Alternatively, one can use Limit Comparison Test by comparing it with $b_n = \frac{1}{\sqrt{n}}$. Clearly, all terms are positive. Next, we check the limit condition;

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}+\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} = \frac{1}{1 + \sqrt{\frac{n+1}{n}}} = \frac{1}{2} \neq 0$$

Thus, by LCT, either both series converges or both diverges. But we know that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges since it is a p-series with $p = \frac{1}{2} < 1$. Therefore, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$ diverges as well.

3. (10 points) Determine whether each of the series below converges or diverges. Indicate clearly which tests you use and how you apply them.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{n^{2/3}}{n^{3/2}-n+1}$

(5 points) **Comparison Test.** Let $a_n = \frac{n^{2/3}}{n^{3/2}-n+1}$ be the n-th term and let $b_n = \frac{n^{2/3}}{n^{3/2}}$ be the reference series. Notice that $b_n = \frac{n^{2/3}}{n^{3/2}} = n^{-5/6}$ forms the p-series for $p = 5/6$. Since $5/6 < 1$, the series $\sum_{n=2}^{\infty} b_n$ diverges by the p-series test.

We claim that $0 < b_n < a_n$ for all $n \geq 2$. To see this, notice for all $n \geq 2$, we have $-n+1 < 0$, which implies

$$n^{3/2} - n + 1 < n^{3/2}, \tag{1}$$

and for all $n > 1$ we have $n^{3/2} > n$, which implies $n^{3/2} - n + 1$ is positive. Thus we can flip (1) to obtain

$$0 < \frac{1}{n^{3/2}} < \frac{1}{n^{3/2}-n+1}.$$

Multiplying this inequality by $n^{2/3}$, we get $0 < b_n < a_n$, for all $n \geq 2$. Since both a_n, b_n are positive and $\sum_{n=2}^{\infty} b_n$ diverges, the Comparison Test applies, concluding that $\sum_{n=2}^{\infty} a_n$ diverges.

Note: we need to check that $n^{3/2} - n + 1$ is positive in order to flip (1). For example, $-2 < 3$ but $\frac{1}{3} \not< -\frac{1}{2}$.

Limit Comparison Test. Let a_n and b_n be the same as before. Calculate $\lim\limits_{n\to\infty} \frac{a_n}{b_n}$:

$$\begin{aligned}
&\lim_{n\to\infty} \frac{a_n}{b_n} \\
&= \lim_{n\to\infty} \frac{\frac{n^{2/3}}{n^{3/2}-n+1}}{\frac{n^{2/3}}{n^{3/2}}} \\
&= \lim_{n\to\infty} \frac{n^{2/3}}{n^{3/2}-n+1} \frac{n^{3/2}}{n^{2/3}} \\
&= \lim_{n\to\infty} \frac{n^{13/6}}{n^{13/6} - n^{5/3} + n^{2/3}} \\
&= \lim_{n\to\infty} \frac{1}{1 - n^{-1/2} + n^{-3/2}} \\
&= 1.
\end{aligned}$$

To apply the Limit Comparison Test, we further check that a_n and b_n are positive for all $n \geq 2$. b_n is clearly positive for all $n \geq 2$, and a_n is positive for all $n \geq 2$ because $n^{3/2} > n$. By the Limit Comparison Test the series $\sum_{n=2}^{\infty} a_n$ and $\sum_{n=2}^{\infty} b_n$ both converge or both diverge. The latter diverges by the p-series test for $p = 5/6 < 1$. Hence $\sum_{n=2}^{\infty} a_n$ also diverges.

Problem instructions, repeated: Determine whether each of the series below converges or diverges. Indicate clearly which tests you use and how you apply them.

(b) $\sum_{n=1}^{\infty} \frac{2^n\, n!}{(2n-1)!}$

(5 points) Use the Ratio Test. Let $a_n = \frac{2^n n!}{(2n-1)!}$.

$$\begin{aligned}
&\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \\
&= \lim_{n\to\infty} \frac{\frac{2^{n+1}(n+1)!}{(2n+1)!}}{\frac{2^n n!}{(2n-1)!}} \\
&= \lim_{n\to\infty} \frac{2^{n+1}(n+1)!}{(2n+1)!} \frac{(2n-1)!}{2^n n!} \\
&= \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \frac{(n+1)!}{n!} \frac{(2n-1)!}{(2n+1)!} \\
&= \lim_{n\to\infty} \frac{2}{1} \frac{n+1}{1} \frac{1}{2n(2n+1)} \\
&= \lim_{n\to\infty} \frac{1+1/n}{n(2+1/n)} \\
&= 0 < 1.
\end{aligned}$$

By the Ratio Test, the series converges absolutely, and hence it converges.

4. (10 points) One of three possibilities holds for each of the series below: (i) it converges absolutely, (ii) it converges but does not converge absolutely, or (iii) it diverges. Determine which possibility holds for each of the series below; indicate clearly which tests you use and how you apply them.

   (a) $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n \ln n}{2n}$

(6 points) Let $b_n = \dfrac{\ln n}{2n}$ which is positive for $n \geq 3$.

Then the given series is an alternating series $\displaystyle\sum_{n=3}^{\infty}(-1)^n b_n$, and thus we use the alternating series test for convergence.

1. As $x \to \infty$, $\ln x$ and x both tends to infinity.

   By l'Hôpital's Rule $\displaystyle\lim_{x\to\infty} \frac{\ln x}{2x} = \lim_{x\to\infty} \frac{(\ln x)'}{(2x)'} = \lim_{x\to\infty} \frac{\frac{1}{x}}{2} = 0.$

   Therefore $\displaystyle\lim_{n\to\infty} b_n = 0.$

2. Let $f(x) = \dfrac{\ln x}{2x}$.

   Then $f'(x) = \dfrac{1}{2} \cdot \dfrac{x(\ln x)' - (\ln x)x'}{x^2} = \dfrac{1 - \ln x}{2x^2} < 0$ when $x \geq 3$ because $\ln x > \ln e = 1$.

   Therefore $f(x)$ is decreasing and $f(n+1) < f(n)$; and thus $b_{n+1} < b_n$ for all $n \geq 3$.

Thus by the alternating series test, the series converges.
We now test if the series converges absolutely, which is to test if the series

$$\sum_{n=3}^{\infty} \left| \frac{(-1)^n \ln n}{2n} \right| = \sum_{n=3}^{\infty} \frac{\ln n}{2n}$$

converges.
We have $\dfrac{\ln n}{2n} > \dfrac{1}{2n} > 0$ for $n \geq 3$ and we know that $\displaystyle\sum_{n=3}^{\infty} \frac{1}{2n}$ diverges from the p-series Test.

Therefore using the comparison test on two series $\displaystyle\sum_{n=3}^{\infty} \frac{\ln n}{2n}$ and $\displaystyle\sum_{n=3}^{\infty} \frac{1}{2n}$, the series $\displaystyle\sum_{n=3}^{\infty} \frac{\ln n}{2n}$ diverges as well.
Answer: (ii) the series converges, but does not converge absolutely

Problem instructions, repeated: One of three possibilities holds for each of the series below: (i) it converges absolutely, (ii) it converges but does not converge absolutely, or (iii) it diverges. Determine which possibility holds for each of the series below; indicate clearly which tests you use and how you apply them.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{2+(-1)^n}$

(4 points) Let $a_n = \dfrac{\arctan n}{2+(-1)^n}$.

For odd n, say $n = 2k-1$, $a_{2k-1} = \arctan(2k-1)$ whereas on even n, say $n = 2k$, $a_{2k} = \dfrac{\arctan 2k}{3}$.

Then taking limits for odd and even n separately, $\displaystyle\lim_{k\to\infty} a_{2k} = \frac{\pi}{6}$ whereas $\displaystyle\lim_{k\to\infty} a_{2k-1} = \frac{\pi}{2}$.

Thus $\displaystyle\lim_{n\to\infty} a_n$ does not exist, since it may not approach to two distinct number.

Using the test for divergence, the series $\sum a_n$diverges.

Alternate solution:

For any $n \geq 1$, we have $\arctan n \geq \arctan 1 \geq \dfrac{\pi}{4}$ and $1 \leq 2+(-1)^n \leq 3$.

Therefore $\dfrac{\arctan n}{2+(-1)^n} \geq \dfrac{\pi}{12}$ for all $n \geq 1$.

Since $\sum_{n=1}^{\infty} \frac{\pi}{12}$ diverges, so is $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{2+(-1)^n}$ from the comparison test.

Answer: (iii) the series diverges.

5. (12 points) Find, with complete justification, the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(3-2x)^n}{(n^3+2)\,5^n}$$

The first thing to do is compute the radius of convergence by using the ratio test (6 points). Let

$$a_n = \frac{(3-2x)^n}{(n^3+2)5^n},$$

we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(3-2x)^{n+1}}{((n+1)^3+2)\,5^{n+1}}\cdot\frac{(n^3+2)5^n}{(3-2x)^n}\right| = \lim_{n\to\infty}\left(\left|\frac{3-2x}{5}\right|\cdot\frac{n^3+2}{(n+1)^3+2}\right).$$

Now $\frac{|3-2x|}{5}$ is independent of n, so we can take it outside of the limit sign, while in the fraction $\frac{n^3+2}{(n+1)^3+2}$ both the numerator and the denominator are polynomials of the same degree (3), hence the limit as $n\to\infty$ is the quotient of the coefficients of the terms of highest degree (that is, degree 3). Thus we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{3-2x}{5}\right|\cdot\frac{1}{1} = \left|\frac{3-2x}{5}\right|.$$

The ratio test says that the power series converges absolutely for

$$\left|\frac{3-2x}{5}\right| < 1$$

and diverges for

$$\left|\frac{3-2x}{5}\right| > 1.$$

In particular,

$$\left|\frac{3-2x}{5}\right| < 1$$

is true if and only if

$$-5 < 3-2x < 5$$

which is equivalent to

$$4 > x > -1.$$

Now we have to check the endpoints 4 and -1 (3 points each).

For $x=-1$, one gets

$$\sum_{n=0}^{\infty}\frac{(3-2(-1))^n}{(n^3+2)5^n} = \sum_{n=0}^{\infty}\frac{(3+2)^n}{(n^3+2)5^n} = \sum_{n=0}^{\infty}\frac{1}{n^3+2}.$$

Letting now $b_n = \frac{1}{n^3+2}$, it is clear that

$$0 < \frac{1}{n^3+2} < \frac{1}{n^3} \text{ for all } n \geq 1;$$

moreover

$$\sum_{n=1}^{\infty}\frac{1}{n^3}$$

is a convergent p-series for $p = 3 > 1$, hence by comparison test also

$$\sum_{n=0}^{\infty} \frac{1}{n^3 + 2}$$

converges, which means that the endpoint -1 belongs to the interval of convergence.

For $x = 4$, one gets

$$\sum_{n=0}^{\infty} \frac{(3 - 2(4))^n}{(n^3 + 2)5^n} = \sum_{n=0}^{\infty} \frac{(-5)^n}{(n^3 + 2)5^n} = \sum_{n=0}^{\infty} \frac{(-1)^n\, 5^n}{(n^3 + 2)5^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n^3 + 2)}.$$

We obtain then an alternating series and we apply the absolute convergence test: the series

$$\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{(n^3 + 2)} \right| = \sum_{n=0}^{\infty} \frac{1}{(n^3 + 2)}$$

converges as shown above, hence

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n^3 + 2)}$$

converges absolutely and thus converges (by absolute convergence test). In particular, the endpoint 4 also belongs to the interval of convergence.

The interval of convergence is then $[-1, 4]$.

6. (12 points) Match each function below with its power series, listed among the choices below. You do not need to justify your answers. (Not all of the series have a match, but every function has a match.)

(A) $-x^2 - \frac{x^4}{2} - \frac{x^6}{3} - \frac{x^8}{4} - \cdots$

(B) $1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots$

(C) $1 - \frac{2^2}{2!}x^2 + \frac{2^4}{4!}x^4 - \frac{2^6}{6!}x^6 + \cdots$

(D) $-x^3 - \frac{x^5}{2} - \frac{x^7}{3} - \frac{x^9}{4} - \cdots$

(E) $\frac{1}{10} + \frac{x}{100} + \frac{x^2}{1000} + \frac{x^3}{10000} + \cdots$

(F) $1 - x^2 + x^4 - x^6 + \cdots$

(G) $x - x^3 + x^5 - x^7 + \cdots$

(H) $x^3 - \frac{x^5}{2} + \frac{x^7}{3} - \frac{x^9}{4} + \cdots$

(I) $2x - \frac{2^3}{3!}x^3 + \frac{2^5}{5!}x^5 - \frac{2^7}{7!}x^7 + \cdots$

(J) $\frac{1}{10} - \frac{x}{100} + \frac{x^2}{1000} - \frac{x^3}{10000} + \cdots$

(K) $1 + 2x + \frac{2^2}{2!}x^2 + \frac{2^3}{3!}x^3 + \cdots$

(L) $1 - 2x + 3x^2 - 4x^3 + \cdots$

(2 points each)

Function	Series (choose one of (A) through (L))
$\cos(2x)$	**C**
$x\ln(1+x^2)$	**H**
e^{2x}	**K**
$\frac{1}{10+x}$	**J**
$\frac{1}{1+x^2}$	**F**
$\ln(1-x^2)$	**A**

7. (12 points) Let $f(x) = x^{5/4}$.

(a) Find $T_2(x)$, the degree-2 Taylor polynomial for f centered at 16. (Note: $16^{1/4} = 2$, so $16^{5/4} = 32$.)

(3 points) We have $f(x) = x^{5/4}$. $f'(x) = \frac{5}{4}x^{1/4}$, $f''(x) = \frac{5}{4} \cdot \frac{1}{4}x^{-3/4}$. Therefore

$$\begin{aligned} T_2(x) &= f(16) + \frac{f'(16)}{1!}(x-16) + \frac{f''(16)}{2!}(x-16)^2 \\ &= 32 + \frac{5}{2}(x-16) + \frac{5}{256}(x-16)^2. \end{aligned}$$

(b) Use T_2 to obtain an approximation for $17^{5/4}$.

(1 point)

$$17^{5/4} \approx T_2(17) = 32 + \frac{5}{2} + \frac{5}{256}.$$

(c) Determine the accuracy of your approximation from part (b), explaining the steps of your reasoning, and giving your answer in sentence form.

(4 points) From Taylor's inequality

$$|f(x) - T_2(x)| \le \frac{M|x-16|^3}{3!} \qquad |x-16| \le d$$

for any $d > 0$, where M is the upper bound of $|f'''(x)|$ on $[16-d, 16+d]$.
Because we need to estimate $|f(17) - T_2(17)|$ it is enough to take $d = 1$.
We have $f'''(x) = \frac{5}{4} \cdot \frac{1}{4} \cdot \frac{-3}{4}x^{-7/4}$. Therefore we need to find a constant M that satisfies

$$|f'''(x)| = \left|\frac{-15}{64x^{7/4}}\right| = \frac{15}{64x^{7/4}} \le M \qquad \text{(for all } x \text{ with } 15 \le x \le 17).$$

However, $x^{7/4}$ is increasing, so the whole expression is decreasing and attains its maximum at $x = 15$. Thus we may take M to be any value greater than $\frac{15}{64} \cdot 15^{-7/4} = \frac{15^{1/4}}{64 \cdot 15}$. It is clear that $15^{1/4} \le 16^{1/4} = 2$. Therefore we may take $M = \frac{1}{32 \cdot 15}$. Therefore,

$$|f(17) - T_2(17)| \le \frac{M|17-16|^3}{3!} = \frac{1}{32 \cdot 15 \cdot 6} = \frac{1}{2880} < \frac{1}{2000} = 0.0005.$$

Answer: The estimate in part (b) is accurate up to three decimal places, i.e., the absolute value of error is less than 0.0005.
Remark: You don't need to be this precise though.

(d) Find a set of values of x for which $x^{5/4} \approx T_2(x)$ with an error of no more than ± 0.01. (*Hint:* the set should be an interval of the form $[16-d, 16+d]$.)

(4 points) In the proof of (c), we showed that for $d = 1$,

$$|f'''(x)| = \left|\frac{-15}{64x^{7/4}}\right| \leq \frac{1}{32 \cdot 15} = M \qquad (15 \leq x \leq 17).$$

Therefore, for any $15 \leq x \leq 17$,

$$|f(x) - T_2(x)| \leq \frac{M|x-16|^3}{3!} \leq \frac{1}{6} \cdot \frac{1}{32} \cdot \frac{1}{15} \leq 0.01.$$

Therefore $x^{5/4} \approx T_2(x)$ correct within the error 0.01 for $15 \leq x \leq 17$.
Answer: $15 \leq x \leq 17$ is a range for which the estimate is good within 0.01.
Alternate solution (especially if the estimate in M in part (c) was not precise enough):
From the expression

$$|f(x) - T_2(x)| \leq \frac{M}{3!}|x-16|^3 \leq \frac{M}{3!}d^3 \qquad |x-16| \leq d$$

we have the error is less than $\dfrac{M}{6} \cdot (0.001)$ if we take $d = 0.1$. Therefore this estimate is good enough for 0.01 if we prove $M < 1$ in the range $15.9 \leq x \leq 16.1$. In fact, this is true, because

$$\left|\frac{-15}{64x^{7/4}}\right| \leq \frac{15}{64} < 1$$

if $x \geq 1$. Therefore, if $15.9 \leq x \leq 16.1$, then $|f(x) - T_2(x)| < 0.01$ as desired.
Answer: $15.9 \leq x \leq 16.1$ is a range for which the estimate is good to within ± 0.01.
Alternate solution 2:
We want to find the range $16 - d \leq x \leq 16 + d$ for which

$$|f(x) - T_2(x)| \leq \frac{M|x-16|^3}{3!} \qquad |x-16| \leq d$$

where M is the upper bound of $|f'''(x)|$ in the range $16-d \leq x \leq 16+d$. Since $|f'''| = 15x^{-7/4}/64$ is decreasing, it attains its maximum at $16 - d$. Therefore it is enough to find d that satisfies

$$\frac{\frac{15}{64}(16-d)^{-7/4}d^3}{3!} \leq 0.01$$

and any d that satisfies this inequality works. Therefore, if we put some small number, say $d = 1/2$ and use $(16-0.5)^{-7/4} < 1$,

$$\frac{15}{64(16-d)^{-7/4} \cdot 3!} \cdot \left(\frac{1}{2}\right)^3 = \frac{15}{64 \cdot 6 \cdot 8} < 0.01$$

and we have the error less than 0.01 for $15.5 \leq x \leq 16.5$.