Solutions to Math 42 First Exam — January 28, 2014

1. (12 points) Compute each of the following integrals, showing all of your reasoning.

   (a) $\displaystyle\int_0^2 t^2 e^t\,dt$

   (6 points) Let I be the integral we have to compute. We start by integrating by parts: let $u = t^2$, $dv = e^t\,dt$ so that $du = 2t\,dt$, $v = e^t$. We get

$$I = t^2e^t\Big]_0^2 - \int_0^2 2te^t\,dt = 4e^2 - 2\int_0^2 te^t\,dt.$$

   Call this last integral J: we integrate by parts again, taking $f = t$, $dg = e^t dt$ so that $df = dt$ and $g = e^t$. We get

$$J = te^t\Big]_0^2 - \int_0^2 e^t\,dt = te^t - e^t\Big]_0^2 = 2e^2 - e^2 + 1 = e^2 + 1.$$

   Plugging this in the first equality that we found for I, we obtain

$$I = 4e^2 - 2\left(e^2+1\right) = 2e^2 - 2.$$

   (b) $\displaystyle\int \sec^4 2\theta \tan^4 2\theta\,d\theta$

   (6 points) Let I be the integral we have to compute. Recall the trigonometric formula

$$\sec^2 t = 1 + \tan^2 t$$

   which we use to write our integral in the following form:

$$I = \int \tan^4 2\theta\left(\tan^2 2\theta + 1\right)\sec^2 2\theta\,d\theta.$$

   Now we use the following substitution: let $x = \tan 2\theta$ so that $dx = 2\sec^2 2\theta\,d\theta$. We get

$$I = \int x^4(x^2+1)\frac{dx}{2} = \frac{1}{2}\int x^4 + x^6\,dx = \frac{1}{2}\left(\frac{x^5}{5} + \frac{x^7}{7} + C\right).$$

   It remains to substitute back for $x = \tan 2\theta$, and we obtain

$$I = \frac{1}{2}\left(\frac{\tan^5 2\theta}{5} + \frac{\tan^7 2\theta}{7} + C\right).$$

2. (13 points) Compute each of the following integrals, showing all of your reasoning.

(a) $\int \frac{x^2}{\sqrt{4-x^2}}\,dx$

(7 points) Let I be the integral we have to compute. As the square root term is of the type $\sqrt{a^2-x^2}$, the trigonometric substitution we need to use is $x = a\sin\theta$. In this particular problem we have $a = 2$ so we set $x = 2\sin\theta$ which gives $dx = 2\cos\theta\,d\theta$. We get

$$I = \int \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}}(2\cos\theta\,d\theta) = \int \frac{4\sin^2\theta}{2\cos\theta} 2\cos\theta\,d\theta = 4\int \sin^2\theta\,d\theta.$$

Now we recall (and apply) the following trigonometric formula:

$$\sin^2 t = \frac{1-\cos 2t}{2}.$$

We get

$$I = 4\int \frac{1-\cos 2\theta}{2}\,d\theta = 2\left(\theta - \frac{sin 2\theta}{2} + C\right).$$

It remains to substitute back for x. By reversing $x = 2\sin\theta$ we get $\theta = \arcsin\frac{x}{2}$, hence

$$I = 2\left(\arcsin\frac{x}{2} - \frac{sin 2\arcsin\frac{x}{2}}{2} + C\right).$$

(b) $\int_{-1}^{1} \frac{dx}{e^x+e^{-x}}$

(6 points) Let I be the integral we have to compute. We start by some algebraic manipulation:

$$\frac{1}{e^x+e^{-x}} = \frac{1}{e^x+\frac{1}{e^x}} = \frac{1}{\frac{e^{2x}+1}{e^x}} = \frac{e^x}{e^{2x}+1}.$$

Hence our integral is

$$I = \int_{-1}^{1} \frac{e^x}{e^{2x}+1}\,dx.$$

Now we use the following substitution: let $u = e^x$ so that $du = e^x dx$. The bounds of integration become $u(1) = e$ and $u(-1) = e^{-1}$. We obtain

$$I = \int_{e^{-1}}^{e} \frac{du}{u^2+1} = \arctan u\Big]_{e^{-1}}^{e} = \arctan e - \arctan e^{-1}.$$

3. (8 points) Show all your steps in computing $\displaystyle\int \frac{x^6-9}{x^4-x^2}\,dx$

Since the degree of the numerator is higher than the denominator, we need to do long division.

$$\begin{array}{r|l} & x^2 \quad +1 \\ \hline x^4-x^2 & x^6 \qquad\qquad -9 \\ & -(x^6 - x^4) \\ & \quad x^4 \\ & \quad -(x^4 - x^2) \\ & \qquad x^2 - 9 \end{array}$$

That is,

$$\frac{x^6-9}{x^4-x^2} = x^2 + 1 + \frac{x^2-9}{x^4-x^2}$$

Next, we set the partial fraction for the remainder term. We factor

$$x^4 - x^2 = x^2(x-1)(x+1)$$

so the partial fraction is of the form

$$\begin{aligned} \frac{x^2-9}{x^4-x^2} &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{x-1} \\ &= \frac{Ax(x-1)(x+1) + B(x-1)(x+1) + Cx^2(x-1) + Dx^2(x+1)}{x^4-x^2} \\ &= \frac{(A+C+D)x^3 + (B-C+D)x^2 - Ax - B}{x^4-x^2} \end{aligned}$$

Comparing the coefficients on the numerator,

$$\begin{aligned} A + C + D &= 0 \\ B - C + D &= 1 \\ -A &= 0 \\ -B &= -9 \end{aligned}$$

Solving this system yields,

$$A = 0, B = 9, C = 4, D = -4$$

Alternatively, one can solve for the coefficients by plugging value for x. From

$$x^2 - 9 = Ax(x-1)(x+1) + B(x-1)(x+1) + Cx^2(x-1) + Dx^2(x+1)$$

Plugging in the following values for x,

$$\begin{array}{llll} x=-1; & -8 = 2D & \Rightarrow & D = -4 \\ x = 1; & -8 = -2C & \Rightarrow & C = 4 \\ x = 0; & -9 = -B & \Rightarrow & B = 9 \\ x = 2; & -5 = 6A + 3B + 4C + 12D = 6A + 27 + 16 - 48 = 6A - 5 & \Rightarrow & A = 0 \end{array}$$

Taking the integral, we get

$$\int \frac{x^6-9}{x^4-x^2}dx = \int \left(x^2 + 1 + \frac{9}{x^2} + \frac{4}{x+1} - \frac{4}{x-1}\right) dx = \frac{x^3}{3} + x - \frac{9}{x} + 4\ln|x+1| - 4\ln|x-1| + C$$

4. (8 points) Let $f(x) = e^{1/x}$. In this problem, we study approximations of the following integral:

$$\int_1^3 e^{1/x}\,dx \qquad\qquad \text{(Note: don't evaluate this integral.)}$$

(a) Show that if this integral is approximated using the Midpoint Rule with $n = 10$ subintervals, then the approximation is accurate to within 0.1 units. Explain all reasoning; you may make use of the fact that $f''(x) = \dfrac{e^{1/x}(1+2x)}{x^4}$.

(4 points)
Method 1: Note that the function

$$f''(x) = \frac{e^{1/x}(1+2x)}{x^4} = \frac{e^{1/x}}{x^4} + 2\frac{e^{1/x}}{x^3}$$

is the sum of two positive, decreasing functions of x. (In each case, the denominator is a positive, increasing function, and the numerator is a positive, decreasing function.) Hence $f''(x)$ is a decreasing function on $[1, 3]$ and it reaches its maximal value at $x = 1$ for $1 \le x \le 3$. So we have

$$|f''(x)| \le f''(1) = 3e \quad \text{for all} \quad 1 \le x \le 3.$$

Using the error bound of midpoint rule with $n = 10$ and taking $K_2 = 3e$, we get

$$|E_M| \le \frac{3e(3-1)^3}{24 \times 10^2} \le \frac{30(3-1)^2}{24 \times 10^2} = 0.1.$$

Method 2: For $1 \le x \le 3$, we have

$$\max_{1\le x\le 3}\left|\frac{e^{1/x}(1+2x)}{x^4}\right| \le \max_{1\le x\le 3}\left(e^{1/x}\right) \times \max_{1\le x\le 3}|1+2x| \times \max_{1\le x\le 3}\left(\frac{1}{x^4}\right)$$
$$\le e \times 7 \times 1 = 7e$$

By taking $K_2 = 7e$, one can still get $|E_M| \le 0.1$ by a similar calculation as above.

(b) How many subintervals n would ensure that the Midpoint Rule approximation is accurate to within 10^{-5} units? Give a valid n in simplified form. (As long as you justify your answer, you do not have to find the best possible value.)

(4 points) To ensure $|E_M| \le 10^{-5}$, we only need to find n such that

$$\frac{K_2(3-1)^3}{24 \times n^2} \le 10^{-5},$$

where K_2 is the bound of $|f''(x)|$ computed in part (a).
By taking $K_2 = 3e$, we have

$$n^2 \ge 10^5 e$$

and hence $n \ge \sqrt{10^5 e} = 100\sqrt{10e}$. Since $6 > \sqrt{30} > \sqrt{10e}$, we see that $\boxed{n = 600}$ is an example of a valid n.

Grading notes: Most people had confusing/incorrect inequalities throughout their solutions, and did not give an example valid n in simplified form. This was graded with lenience this time, but on subsequent exams it won't be.

5. (9 points) An *ice core* is a long, thin cylinder of solid ice removed from a glacier by a vertical drilling process. One such core taken from an Antarctic ice sheet is 1200 cm in length; it is studied to determine the varying properties of glacier ice whose depth, x, ranges from 0 cm (i.e., the surface) to 1200 cm. For each such x, a thin cross section of the core is circular of constant radius 10 cm, and has mass density $d(x)$ g/cm^3, according to the values given in the following table:

x	0	200	400	600	800	1000	1200
$d(x)$	0.71	0.87	0.89	0.905	0.912	0.916	0.917

(a) Set up an integral involving the function d which expresses the total mass of the ice core in grams.

(3 points) The cross section at x has the area $\pi r^2 = 100\pi$ for each x. So the mass is

$$\int_0^{1200} 100\pi d(x)\, dx$$

(b) Write an expression, involving only numbers, that approximates your integral from part (a) by the Trapezoidal Rule, using all of the data in the table. Do not simplify your answer.

(3 points) Noting that $n = 6, \Delta x = \frac{1200-0}{6} = 200$, so the approximation by Trapezoid rule is given by

$$\begin{aligned} T_6 &= (100\pi)\frac{200}{2}(0.71 + 2 \times 0.87 + 2 \times 0.89 + 2 \times 0.905 + 2 \times 0.912 + 2 \times 0.916 + 0.917) \\ &= 10000\pi\,(0.71 + 2 \times 0.87 + 2 \times 0.89 + 2 \times 0.905 + 2 \times 0.912 + 2 \times 0.916 + 0.917) \end{aligned}$$

(c) Write an expression, involving only numbers, that approximates your integral from part (a) by Simpson's Rule, using all of the data in the table. Do not simplify your answer.

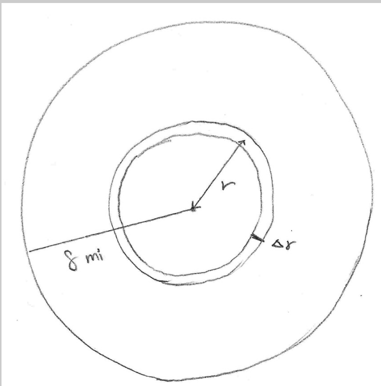
(3 points) Again, $n = 6, \Delta x = 200$, so the approximation by Simpson's rule is given by

$$\begin{aligned} S_6 &= (100\pi)\frac{200}{3}(0.71 + 4 \times 0.87 + 2 \times 0.89 + 4 \times 0.905 + 2 \times 0.912 + 4 \times 0.916 + 0.917) \\ &= \frac{20000\pi}{3}(0.71 + 4 \times 0.87 + 2 \times 0.89 + 4 \times 0.905 + 2 \times 0.912 + 4 \times 0.916 + 0.917) \end{aligned}$$

6. (8 points) The city of Centropolis has the shape of a circle 8 miles in radius; the population density at a point r miles from the city center is given by $d(r) = \frac{10^5}{\pi} e^{-r^2/2}$ people per square mile.

(a) Set up, but do not evaluate, an integral that gives the total population of the city.

(3 points) Divide the circle into many concentric rings with small thickness Δr.



The area of each ring is

$$\text{circumference} \cdot \text{thickness} = 2\pi r \Delta r.$$

Since the density is a function of r, on each ring the density is approximately uniform. Thus, we can apply the formula

$$\text{mass} \;=\; \text{area} \cdot \text{density}$$

to find the population on each ring as $(2\pi r \Delta r) \cdot d(r)$. The total population is then approximately the sum of population of all rings:

$$\text{total population} \;\approx \sum_{\text{rings}} 2\pi r d(r) \Delta r.$$

Finally, when Δr tends to zero, this approximation becomes exact, and the summation becomes an integral:

$$\text{total population} \;= \int_0^8 2\pi r d(r) dr = \int_0^8 2\pi r \frac{10^5}{\pi} e^{-\frac{r^2}{2}}\, dr.$$

The lower and upper bounds are 0 and 8 because we go from the the center ($r = 0$) to the edge ($r = 8$).

(b) Find the distance R with the property that 20,000 people live within R miles of the city center. (Your final expression for R should not involve integral symbols.)

(5 points) From part (a), the total population withing R miles of the city center is

$$\int_0^R 2\pi r \frac{10^5}{\pi} e^{-\frac{r^2}{2}}\, dr.$$

We wish to find an R so that this quantity is 20,000. Hence we set up the equation

$$20{,}000 = \int_0^R 2\pi r \frac{10^5}{\pi} e^{-\frac{r^2}{2}}\, dr. \tag{1}$$

First solve the integral on the right hand side:

$$\int_0^R 2\pi r \frac{10^5}{\pi} e^{-\frac{r^2}{2}}\, dr = 2 \cdot 10^5 \int_0^R r e^{-\frac{r^2}{2}}\, dr.$$

Let $u = -\frac{r^2}{2}$. $du = -rdr$ and the lower and upper bounds for u becomes $-\frac{0^2}{2} = 0$ and $-\frac{R^2}{2}$.

Continuing, we have

$$
\begin{aligned}
&= 2 \cdot 10^5 \int_0^{-\frac{R^2}{2}} -e^u du \\
&= -2 \cdot 10^5 e^u \Big|_0^{-\frac{R^2}{2}} \\
&= -2 \cdot 10^5 (e^{-\frac{R^2}{2}} - e^0) \\
&= -2 \cdot 10^5 (e^{-\frac{R^2}{2}} - 1).
\end{aligned}
$$

Next, plug this back to (1) to solve for R:

$$
\begin{aligned}
&20{,}000 = -2 \cdot 10^5 (e^{-\frac{R^2}{2}} - 1) \\
&\Rightarrow -0.1 = e^{-\frac{R^2}{2}} - 1 \\
&\Rightarrow 0.9 = e^{-\frac{R^2}{2}} \\
&\Rightarrow \ln(0.9) = -\frac{R^2}{2} \\
&\Rightarrow \sqrt{-2\ln(0.9)} = R.
\end{aligned}
$$

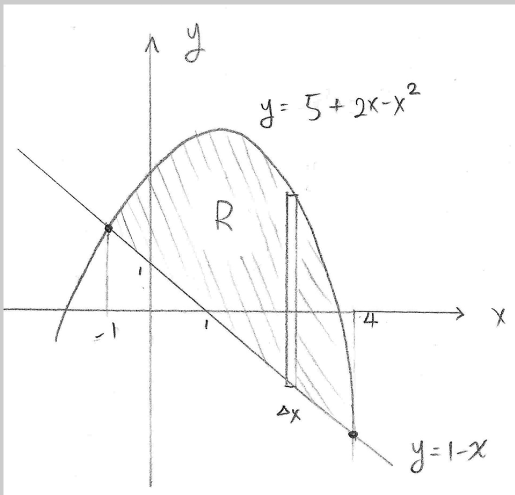
Notice that $\ln(0.9)$ is a negative number, so we are taking the square root of a positive number.

7. (8 points) Let R be the bounded region in the xy-plane enclosed by the curves $y = 5 + 2x - x^2$ and $y = 1 - x$.

   (a) Set up, but do not evaluate, an integral representing the area of R. As justification, draw a picture with a sample slice labeled.

(5 points) To find the x-values of the two intersections, we set the y's of the two equations $y = 5 + 2x - x^2$ and $y = 1 - x$ to be equal:

$$5 + 2x - x^2 = 1 - x \quad \Rightarrow \quad 4 + 3x - x^2 = 0 \quad \Rightarrow \quad x^2 - 3x - 4 = 0$$
$$\Rightarrow \quad (x+1)(x-4) = 0 \quad \Rightarrow \quad x = -1 \text{ or } 4.$$



To find the area of R, slice it into vertical rectangles with small width Δx. The area of each rectangle is

$$\begin{aligned} \text{height} \cdot \text{width} &= (y_{\text{upper}} - y_{\text{lower}}) \cdot \Delta x \\ &= \Big((5 + 2x - x^2) - (1 - x)\Big)\Delta x. \end{aligned}$$

The area of R is then approximately the sum of the areas of all rectangles.

$$\text{area of } R \approx \sum_{\text{rectangles}} \Big((5 + 2x - x^2) - (1 - x)\Big)\Delta x.$$

As Δx tends to zero, this approximation becomes exact and and the sum becomes the integral

$$\text{area of } R = \int_{-1}^{4} \Big((5 + 2x - x^2) - (1 - x)\Big)\,dx.$$

   (b) Suppose a V is a three-dimensional solid whose base is R, and whose cross-sections perpendicular to the x-axis are regular hexagons. Set up, but do not evaluate, an integral that gives the volume of V. (Note: the area of a regular hexagon of side length s is $\frac{3}{2}s^2\sqrt{3}$.)

(3 points) Consider a hexagonal cross-section with a small thickness Δx and side length s (side length of the hexagon). Its volume is

$$\text{area} \cdot \text{thickness} \;=\; \frac{3}{2}s^2\sqrt{3}\,\Delta x.$$

Since R is the base of V, s coincides with the height of rectangle from part (a). Namely,

$$s = (5 + 2x - x^2) - (1 - x).$$

Hence the volume of V is approximately

$$\text{volume of } V \approx \sum_{\text{cross-sections}} \frac{3}{2}\Big((5 + 2x - x^2) - (1 - x)\Big)^2 \sqrt{3}\,\Delta x.$$

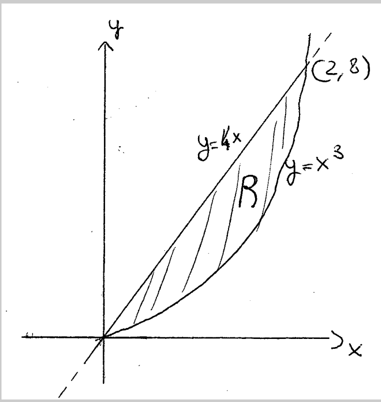
As Δx tends to zero, this approximation becomes exact and the sum becomes the integral

$$\text{volume of } V = \int_{-1}^{4} \frac{3}{2}\Big((5 + 2x - x^2) - (1 - x)\Big)^2 \sqrt{3}\,dx.$$

8. (12 points) Let R be the bounded region in the *first quadrant* of the xy-plane enclosed by the curves $y = x^3$ and $y = 4x$.

   (a) Set up (but do not evaluate) two distinct integrals, each in terms of a single variable, which represent the volume of the solid obtained by rotating R about the y-axis. Justify your answer by drawing pictures, labeling sample slices, and citing the methods used.

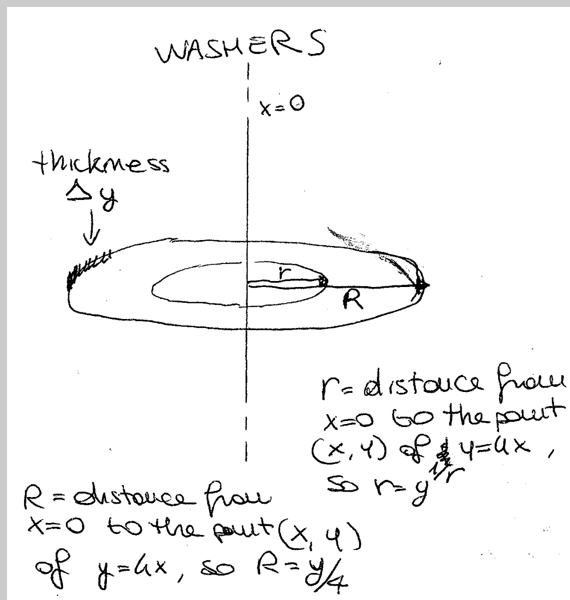
(7 points) Notice first that the problem is not asking to compute any integral, so no calculation will be shown in the solution.



We start by drawing a picture of the region R and finding the intersection points of the two curves (2 points). Setting $x^3 = 4x$ and recalling that we are interested in the region bounded by the two curves in the first quadrant, we found that the two intersections are $(0, 0)$ and $(2, 8)$.

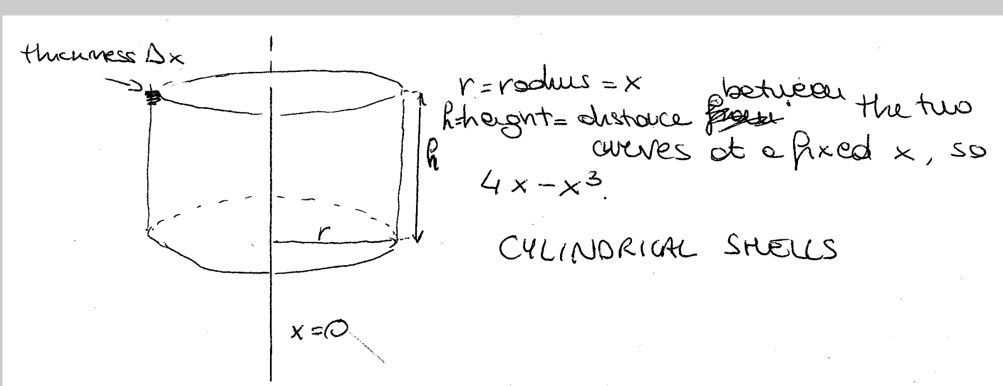
Let's use first the washer method (2.5 points). We slice perpendicularly to the y-axis, so the slice at height y will have outer radius $R = y^{1/3}$ and inner radius $r = \frac{y}{4}$. Hence we get

$$I = \int_0^8 \pi \left(R^2 - r^2\right)\, dy$$
$$= \int_0^8 \pi \left(\left(y^{1/3}\right)^2 - \left(\frac{y}{4}\right)^2\right)\, dy.$$



Now we use cylindrical shells (2.5 points). The radius of each cylinder is the x coordinate of the slice, and recall that we are interested in the area of the lateral surface of this cylinder. At the slice correspondent to x, the circumference is $c = 2\pi x$ and the height is $h = 4x - x^3$, hence we obtain

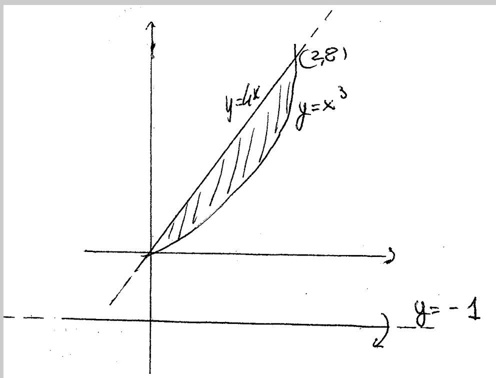
$$I = \int_0^2 c \cdot h\, dx = \int_0^2 (2\pi x)(4x - x^3)\, dx.$$



For easy reference, R is the bounded region in the *first quadrant* of the xy-plane enclosed by the curves $y = x^3$ and $y = 4x$.

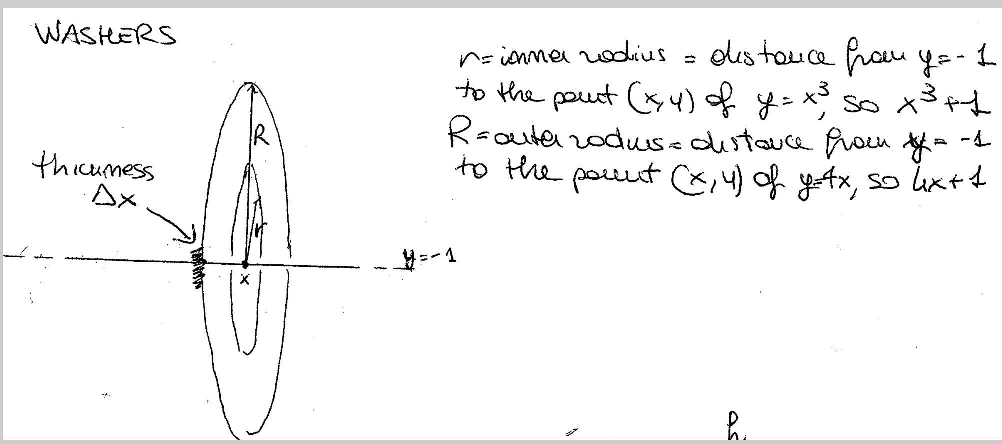
(b) Set up (but do not evaluate) two distinct integrals, each in terms of a single variable, which represent the volume of the solid obtained by rotating R about the line $y = -1$. Justify your answer by drawing pictures, labeling sample slices, and citing the methods used.

(5 points) This time we are rotating about the horizontal line $y = -1$.



Let's start by using the washer method (2.5 points). We slice perpendicularly to the line $y = -1$, so the slice at coordinate x will have outer radius $R = (4x+1)$ and inner radius $r = (x^3 + 1)$. Notice that each radius is the distance from the point having coordinate x on each curve to the line $y = -1$, so we have to add 1 to each radius. Hence we get

$$I = \int_0^2 \pi \left(R^2 - r^2\right) dx$$
$$= \int_0^2 \pi \left((4x+1)^2 - \left(x^3+1\right)^2\right) dx.$$



Now we use cylindrical shells (2.5 points). This time each cylinder has axis the line $y = -1$, so to get the radius we must add 1 to the y coordinate of the slice. Hence at the slice correspondent to y, the circumference is $c = 2\pi(y + 1)$ and the height is $h = y^{1/3} - \frac{y}{4}$. We obtain

$$I = \int_0^8 c \cdot h \, dy = \int_0^8 (2\pi(y+1)) \left(y^{1/3} - \frac{y}{4}\right) dy.$$