Solutions to Math 42 Final Exam — March 18, 2013

1. (12 points) Evaluate each of the following, or explain why its value does not exist; show all reasoning.

   (a) $\displaystyle\int_0^2 \frac{5}{x^3}\,dx$

   (6 points) The integral is improper, because the integrand has a discontinuity at 0.

$$\begin{aligned}\int_0^2 \frac{5}{x^3}\,dx &= 5\int_0^2 \frac{1}{x^3}\,dx \\ &= 5\lim_{t\to 0^+}\int_t^2 \frac{1}{x^3}\,dx \\ &= 5\lim_{t\to 0^+}\left(-\frac{1}{2}x^{-2}\right)\Big|_t^2 \\ &= 5\lim_{t\to 0^+}\left(-\frac{1}{8}+\frac{1}{2t^2}\right) \\ &= \infty\end{aligned}$$

   Thus, the integral diverges (i.e., its value does not exist).

   (b) $\displaystyle\int_0^\infty \frac{e^x}{e^{2x}+1}\,dx$

   (6 points) Let $u = e^x$. Then we have $du = e^x dx$, and hence

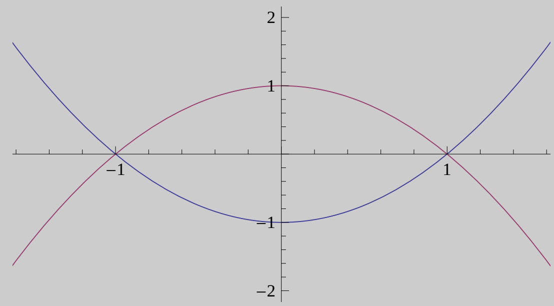
$$\begin{aligned}\int_0^\infty \frac{e^x}{e^{2x}+1}\,dx &= \int_1^\infty \frac{1}{u^2+1}\,du \\ &= \lim_{t\to\infty}\int_1^t \frac{1}{u^2+1}\,du \\ &= \lim_{t\to\infty} \arctan u\big|_1^t \\ &= \lim_{t\to\infty} (\arctan t - \arctan 1) \\ &= \frac{\pi}{2}-\frac{\pi}{4} \\ &= \boxed{\frac{\pi}{4}}\end{aligned}$$

2. (12 points) Consider the region R in the xy-plane bounded by the curves $y = x^2 - 1$ and $y = 1 - x^2$.

(a) Suppose S_1 is the solid obtained by rotating R about the line $x = 2$. Set up two different integrals representing the volume of S_1. In each case, cite the method used and justify with a sketch, but don't evaluate either integral.

(8 points) The region R is the area between the two parabolas, as in the figure below. The points of intersection occur when

$$\begin{aligned} x^2 - 1 = y &= 1 - x^2 \\ \iff 2x^2 &= 2 \\ \iff x &= \pm 1. \end{aligned}$$



The rotation axis is $x = 2$, so in the disk/washer method, our slices are horizontal, and in the cylindrical shell method, our slices are vertical.

Disk/Washer Method: Since our slices are horizontal, we need to find the x coordinates for points on the parabolas in terms of y. For the top half, we have that $y = 1 - x^2$, so solving for x yields $x_{\text{left}} = -\sqrt{1-y}$ and $x_{\text{right}} = \sqrt{1-y}$. Our cross sections are washers, so the cross sectional area is $\pi R_{outer}^2 - \pi R_{inner}^2$. So our integral for the volume of the top half of S_1 is

$$\int_0^1 \left(\pi R_{outer}^2 - \pi R_{inner}^2\right) dy = \int_0^1 \pi \left(2 - \left(-\sqrt{1-y}\right)\right)^2 - \pi \left(2 - \sqrt{1-y}\right)^2 dy.$$

Similarly, we can solve for x for the bottom half, which is bounded by the parabola $y = x^2 - 1$, and we have $x = \pm\sqrt{y+1}$. Then, the volume for the bottom half of S_1 is

$$\int_{-1}^0 \left(\pi R_{outer}^2 - \pi R_{inner}^2\right) dy = \int_{-1}^0 \pi[2 - (-\sqrt{y+1})]^2 - \pi(2 - \sqrt{y+1})^2 dy.$$

Thus, the volume of S_1 is given by

$$\int_0^1 \pi(2 + \sqrt{1-y})^2 - \pi(2 - \sqrt{1-y})^2 dy + \int_{-1}^0 \pi(2 + \sqrt{y+1})^2 - \pi(2 - \sqrt{y+1})^2 dy.$$

Cylindrical Shells Method: Recall that the volume is computed by $V = \int 2\pi r h\, dx$, where x ranges over the coordinates of vertical slices of R, namely from -1 to 1. In this case, the radius of a shell is the corresponding slice's distance from the line $x = 2$, so $r(x) = 2 - x$. The height of a shell is the height of the corresponding slice, so $h(x) = y_{\text{top}} - y_{\text{bottom}} = (1 - x^2) - (x^2 - 1) = 2 - 2x^2$.

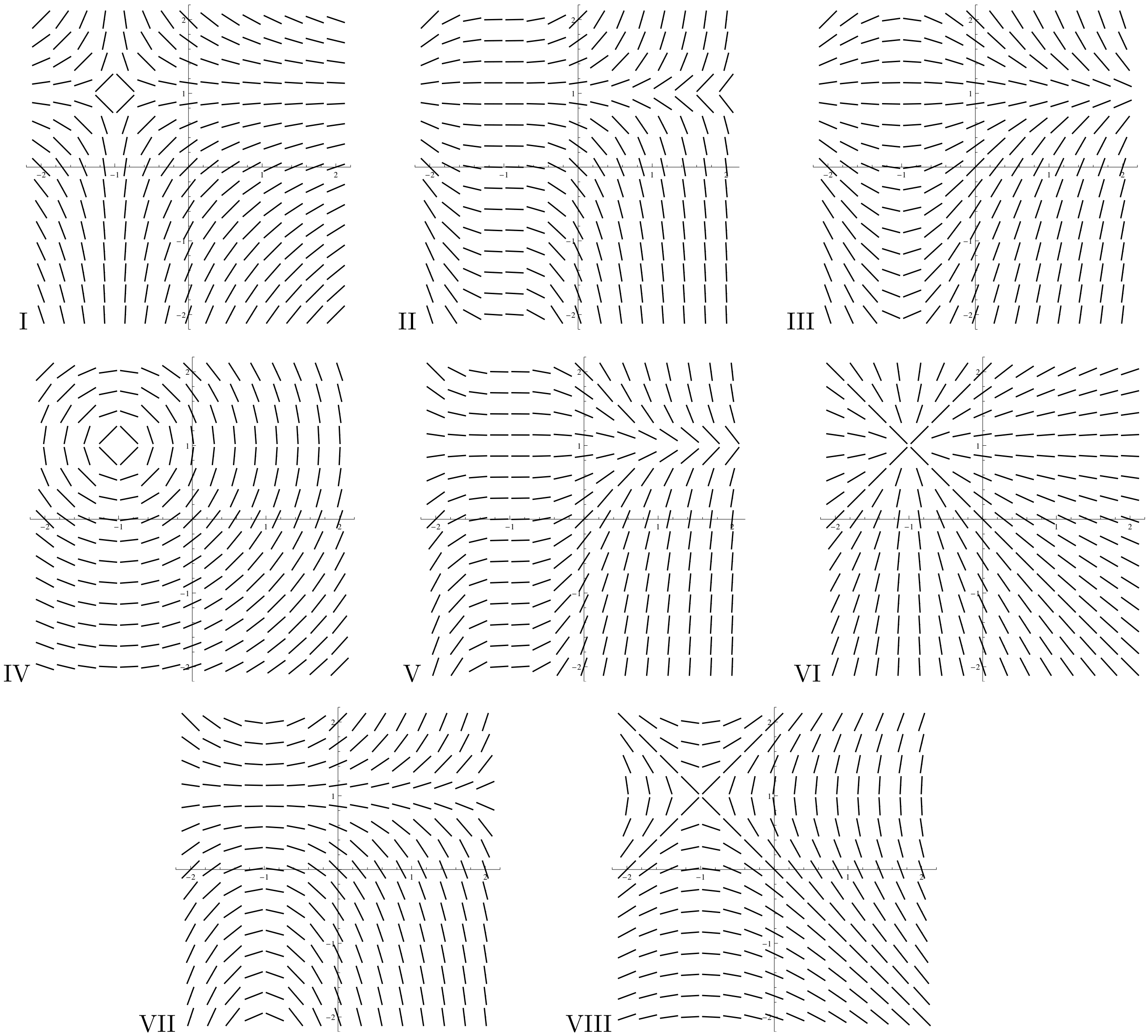
Thus, the volume of S_1 is given by $\boxed{\int_{-1}^1 2\pi(2-x)(2-2x^2)dx}$.

(b) Suppose S_2 is a solid whose base is R, and whose cross-sections perpendicular to the y-axis are equilateral triangles. Set up, but do not evaluate, an integral that gives the volume of S_2. (Note: the area of an equilateral triangle of side length s is $\frac{1}{4}s^2\sqrt{3}$.)

(4 points) Our triangular cross-sections are horizontal, being perpendicular to the y-axis, so their side lengths $s(y)$ are given by $s(y) = x_{\text{right}} - x_{\text{left}}$, where $-1 \le y \le 1$. We calculated previously that for the top half of R, these x values are given by $\pm\sqrt{1-y}$, and for the bottom half of R, they are given by $\pm\sqrt{y+1}$. Thus, using the triangle area formula, the volume of S_2 is

$$\int_0^1 A(y)dy + \int_{-1}^0 A(y)dy = \boxed{\int_0^1 \frac{\sqrt{3}}{4}\left(\sqrt{1-y} - (-\sqrt{1-y})\right)^2 dy + \int_{-1}^0 \frac{\sqrt{3}}{4}\left(\sqrt{1+y} - (-\sqrt{1+y})\right)^2 dy}$$

3. (15 points) Each of the eight direction fields below corresponds to exactly one of the equations given; the scale on each is $-2 \le x \le 2$, $-2 \le y \le 2$. Determine the equation that corresponds to each direction field. No justification is necessary.



First notice that VI and VIII are the only direction fields such that $y' = -1$ along the diagonal line $y = -x$. One easily checks that the differential equations

$$y' = \frac{y-1}{x+1} \quad \text{and} \quad y' = \frac{x+1}{y-1}$$

are the only ones on the list with this property. Next look at the horizontal line $y = 1$ in VI. Away from the point $(-1, 1)$, the slopes approach 0 as we get close to this line. This means VI must correspond to the first equation above, and VIII to the second.

Now since $y' = -1$ along the line $y = -x$ for the two equations above, the slopes along this line must be $+1$ for the direction fields of

$$y' = -\frac{y-1}{x+1} \quad \text{and} \quad y' = -\frac{x+1}{y-1}.$$

Only I and IV have this property. Again observing that the slopes in I approach 0 near the line $y = 1$, we conclude that I corresponds to the first of these and hence IV corresponds to the second.

Now in the remaining direction fields, only II and VII have positive slopes in the region where $y > 1$ and $x > -1$ (near the upper right corner of the picture). This property narrows down our remaining choices to

$$y' = (y-1)(x+1) \quad \text{or} \quad y' = (y-1)(x+1)^2.$$

The first of these has the additional property that all slopes are also positive near the lower left corner, in the region where $y < 1$ and $x < -1$. Hence this equation corresponds to VII and the second to II. Nearly identical reasoning leads us to conclude that the first differential equation in the right hand column corresponds to III and the second equation in the right hand column corresponds to V. Our table should thus be:

Equation	I, II, III, IV, V, VI, VII, or VIII	Equation	I, II, III, IV, V, VI, VII, or VIII
$y' = (y-1)(x+1)$	**VII**	$y' = -(y-1)(x+1)$	**III**
$y' = (y-1)(x+1)^2$	**II**	$y' = -(y-1)(x+1)^2$	**V**
$y' = \frac{y-1}{x+1}$	**VI**	$y' = -\frac{y-1}{x+1}$	**I**
$y' = \frac{x+1}{y-1}$	**VIII**	$y' = -\frac{x+1}{y-1}$	**IV**

4. (13 points) At noon, a tank contains 10 kg of salt, dissolved in 100 L of water. At this time, pure water begins entering the tank at a rate of 8 L/min; the solution is kept thoroughly mixed and drains from the tank at the rate of 10 L/min.

   (a) Find an expression for the volume of the fluid in the tank t minutes after noon.

   (2 points) The initial volume of fluid in the tank is 100 L. Since water enters the tank at a constant rate of 8 L/min and drains at a constant rate of 10 L/min, the volume of fluid in the tank is decreasing at a constant rate of 2 L/min. Therefore, the amount of fluid in the tank at time t equals

   $$(100 - 2t) \text{ Liters.}$$

   (b) Set up an initial value problem for $y(t)$, the amount of salt in the tank t minutes after noon. Be sure to state your initial condition, including the units involved.

   (5 points) Let $y(t)$ denote the amount of salt in the tank t minutes after noon, measured in kg. $y(t)$ satisfies a differential equation

   $$dy/dt = \text{rate in} - \text{rate out}.$$

   We are told that the fluid entering the tank is *pure water*. Therefore,

   $$\text{rate in} = \left(8\ \frac{\text{L}}{\text{min}}\right) \times \left(0\ \frac{\text{kg}}{\text{L}}\right) = 0\ \frac{\text{kg}}{\text{min}}.$$

   The rate out is the concentration of salt in the mixture at time t multiplied by the rate that it drains from the tank. Therefore,

   $$\text{rate out} = \left(10\ \frac{\text{L}}{\text{min}}\right) \times \left(\frac{y(t)}{100-2t}\ \frac{\text{kg}}{\text{L}}\right) = \frac{10y(t)}{100-2t}\ \frac{\text{kg}}{\text{min}}.$$

   Therefore,

   $$\begin{cases} dy/dt = -10y/(100-2t) \\ y(0) = 10. \end{cases}$$

   (c) By solving the initial value problem, find the amount of salt in the tank 25 minutes after noon.

   (6 points) We separate variables:

   $$dy/y = -10/(100-2t).$$

   Integrating, we see that

   $$\log|y| = 5\log|100-2t| + C,$$

   for some constant C. Plugging in $t = 0$, we find that

   $$C = \log|10| - 5\log|100| = \log|10^{-9}|.$$

   Exponentiating, we find that

   $$|y(t)| = |100-2t|^5 \cdot 10^{-9}.$$

   Since the amount of salt $y(t)$ is always positive,

   $$y(t) = |100-2t|^5 \cdot 10^{-9}.$$

   So, when $t = 25min$, we compute that

   $$y(25) = |100 - 2 \cdot 25|^5 \cdot 10^{-9}.$$

   Therefore, there are $50^5 \cdot 10^{-9}$ kg of salt in the tank 25 minutes after noon.

5. (12 points) A population is modeled by the following differential equation: $\frac{dP}{dt} = \frac{1}{2}P - \frac{1}{16}P^4$

(a) Find the equilibrium solutions of the differential equation.

(2 points) The equilibrium solutions are constant solutions, so they occur when $\frac{dP}{dt} = \frac{1}{2}P - \frac{1}{16}P^4 = 0$. Factoring, we see that

$$\frac{1}{2}P(1 - \frac{1}{8}P^3) = 0.$$

Thus, they are when $P = 0$ and when $1 - \frac{1}{8}P^3 = 0$. Solving for the latter, we see that $P^3 = 8$, so $P = 2$. Our two equilibrium solutions are $P = 0$ and $P = 2$.

(b) Determine the value of P for which the population is growing fastest; explain all reasoning. (Hint: first use the differential equation to find an expression for $\frac{d^2P}{dt^2}$.)

(4 points) We find that

$$\begin{aligned}\frac{d^2P}{dt^2} &= \frac{1}{2}\frac{dP}{dt} - \frac{4}{16}P^3\frac{dP}{dt} \\ &= \frac{1}{2}(1 - \frac{1}{2}P^3)\frac{dP}{dt} \\ &= \frac{1}{2}(1 - \frac{1}{2}P^3)(\frac{1}{2}P - \frac{1}{16}P^4).\end{aligned}$$

If we set $\frac{d^2P}{dt^2} = 0$, this will tell us where the inflection points are. If we set $\frac{1}{2}P - \frac{1}{16}P^4 = 0$, we get the roots $P = 0$ and $P = 2$ (which we found in the previous part). We see that the only other possibility is if $1 - \frac{1}{2}P^3 = 0$. Solving for P, we get that $P = \sqrt[3]{2}$. We can see that this has to be a maximum for $\frac{dP}{dt}$ because $\frac{dP}{dt}$ is positive for $0 < P < 2$, and $\frac{dP}{dt} = 0$ if $P = 0$ or $P = 2$.

Note: Many people wrote that $\frac{d^2P}{dt^2} = \frac{1}{2} - \frac{4}{16}P^3$. This is not correct as $\frac{1}{2} - \frac{4}{16}P^3$ is the derivative of $\frac{d^2P}{dt^2}$ with respect to P, not with respect to t.

For quick reference, the differential equation for P is: $\dfrac{dP}{dt} = \dfrac{1}{2}P - \dfrac{1}{16}P^4$

(c) Suppose $P(0) = 1$. Use Euler's method with step size 1 to estimate the value $P(2)$. Show your steps, but you don't need to simplify your answer.

(3 points) Our step size is $h = 1$, $F(t, P) = \frac{1}{2}P - \frac{1}{16}P^4$, $P_0 = P(0) = 1$, and $t_0 = 0$. Then, Euler's method gives us

$$\begin{aligned} P_1 = P_0 + hF(t_0, P_0) &= 1 + 1[\frac{1}{2}(1) - \frac{1}{16}(1)^4] \\ &= 1 + \frac{1}{2} - \frac{1}{16} = \frac{23}{16}, \end{aligned}$$

which is our approximation for $P(t_1) = P(1)$. Then, our next approximation gives

$$P_2 = P_1 + hF(t_1, P_1) = \frac{23}{16} + 1[\frac{1}{2}(\frac{23}{16}) - \frac{1}{16}(\frac{23}{16})^4].$$

This is our approximation for $P(t_2) = P(2)$.

(d) For $P(t)$ satisfying the initial value problem $\begin{cases} \dfrac{dP}{dt} = \dfrac{1}{2}P - \dfrac{1}{16}P^4 \\ P(0) = 1 \end{cases}$, what is the behavior of $P(t)$ as $t \to \infty$? Justify your answer.

(3 points) It is possible to solve this differential equation using separation of variable and partial fraction decomposition, but it is extremely cumbersome and definitely unnecessary to answer the question at hand. Instead, we will analyze the derivative $\frac{dP}{dt}$. We noted in the first part of the problem that the equilibrium solutions are $P = 0$ and $P = 2$. We also see that at $P(0) = 1$, our derivative $\frac{dP}{dt}$ is positive since

$$\frac{dP}{dt} = \frac{1}{2}(1) - \frac{1}{16}(1) = \frac{7}{16}.$$

So the population $P(t)$ will increase from 1. We also know that $\frac{dP}{dt}$ is positive for all $0 < P < 2$, so the population $P(t)$ will continue to increase. As P approaches 2, $\frac{dP}{dt}$ will become smaller and smaller, so the population will slowly level off, approaching but never reaching $P = 2$.

6. (14 points)

(a) Show all steps in solving the initial value problem

$$\frac{dy}{dx} = xe^{y-x}, \quad y(0) = -\ln 2$$

(6 points) Separating variables, we may write the differential equation as

$$e^{-y}dy = xe^{-x}dx.$$

Integrating (by parts on the right), we find that

$$-e^{-y} = -xe^{-x} - e^{-x} + C.$$

Plugging in our initial value, we have

$$-e^{-(-\ln 2)} = -0 \cdot e^{-0} - e^{-0} + C \Rightarrow -2 = -1 + C \Rightarrow C = -1.$$

Solving for y thus gives us

$$\boxed{y(x) = -\ln(xe^{-x} + e^{-x} + 1)}.$$

(b) Show all steps in solving the problem

$$\frac{dz}{dt} + 2tz = z, \quad z(0) = 4.$$

(6 points) Separating variables, we write the equation as

$$\frac{dz}{z} = (1 - 2t)dt.$$

Integrating gives

$$\ln|z| = t - t^2 + C \Rightarrow z = Ce^{t-t^2}.$$

Plugging in our initial condition easily gives $C = 4$, so our solution is

$$\boxed{z(t) = 4e^{t-t^2}}$$

(c) Is there a function $z(t)$ satisfying the differential equation of part (b), but instead with initial value $z(0) = 0$? Explain.

(2 points) One easily checks that the equilibrium solution $\boxed{z(t) = 0}$ is a solution to the new initial value problem.

7. (14 points) The basin of a concrete "pond" on campus has the shape of a hemisphere, 10 meters in radius. Water is filled to a depth of h meters, which means that the water occupies the shape of an upside-down "cap" of height h, cut from a sphere of radius 10. (Take $h \leq 10$; the case $h = 10$ means the hemisphere is completely filled with water.)

   (a) Show that top surface of the water is a circle of area $A = \pi(20h - h^2)$ square meters.

   (3 points) The perpendicular distance from the center of the spherical basin to the water's surface is $10 - h$. Since the radius of the sphere is 10, the Pythagorean theorem says that the radius, r, of the circular cross section corresponding to the water's surface satisfies the equation

$$r^2 = 10^2 - (10 - h)^2 = 20h - h^2.$$

   The top surface of the water is therefore a circle of area

$$A = \pi r^2 = \pi \cdot (20h - h^2).$$

   (b) Show that the volume of water in the basin is $V = \pi\left(10h^2 - \frac{h^3}{3}\right)$ cubic meters.

   (3 points) By the method of cross sectional area, the volume of water in the basin is

$$\begin{aligned} V &= \int_0^h \text{cross sectional area at height } x dx \\ &= \int_0^h \pi \cdot (20x - x^2) dx \\ &= \pi \cdot (10h^2 - h^3/3). \end{aligned}$$

For quick reference, the top surface has area $A = \pi(20h - h^2)$; the water has volume $V = \pi\left(10h^2 - \frac{h^3}{3}\right)$.

(c) Water starts to evaporate from the pond; at any moment the volume of water is decreasing at a rate proportional to the area of the top surface. Initially, at time $t = 0$ days, the pond is filled to a depth of exactly 9 meters. Moreover, it is determined that at this exact moment, the height is decreasing at a rate of $-3/100$ meters per day.
Set up an initial value problem satisfied by $h(t)$, the depth of the water in the pond after t days. A complete answer should not depend on unknown constants, but finding a solution is not necessary. (Hint: first express $\frac{dV}{dt}$ in terms of $\frac{dh}{dt}$.)

(4 points) We are given that $dV/dt = k \cdot A(h)$, where k is some constant of proportionality. On the other hand, by the chain rule,

$$\begin{aligned} dV/dt &= dV/dh \cdot dh/dt \\ &= (20h - h^2) \cdot dh/dt \\ &= A(h) \cdot dh/dt. \end{aligned}$$

Equating these two expressions for dV/dt, we find that

$$A(h) \cdot dh/dt = k \cdot A(h).$$

After dividing both sides by $A(h)$, this implies that

$$dh/dt = k$$

for some constant k. On the other hand, we are told that

$$dh/dt|_{t=0} = -3/100.$$

Therefore, the constant k must equal $-3/100$. Since the initial height of water is 9, we are led to the initial value problem

$$\begin{cases} dh/dt &= -3/100 \\ h(0) &= 9. \end{cases}$$

(d) A water source is installed to replenish some of the evaporated water, at the constant rate of b cubic meters per day. Set up (but *don't solve*) a new differential equation for $h(t)$ in this situation, and then compute b so that the depth of the water may be maintained constantly at the value $h = 9$. Show all reasoning.

(4 points) The modified differential equation for volume at time t is

$$\begin{aligned} \frac{dV}{dt} &= (\text{rate in}) - (\text{rate out}) = b - (\text{rate out from part (c)}) \\ &= b - \tfrac{3}{100} \cdot A(h). \end{aligned}$$

Supposing that b is chosen such that the height is constant and equal to 9, we must have that

- $dV/dt = 0$, because if the water height is constant, then so is the volume, according to (b).
- The area remains constant at $A(9)$, which equals $\pi(20 \cdot 9 - 9^2) = 99\pi$ according to (a).

Thus,

$$0 = \frac{dV}{dt} = b - \frac{3}{100} \cdot A(9) = b - \frac{3}{100} \cdot 99\pi.$$

Therefore, the unique value of b such that the depth of the water may be constantly maintained at $h = 9$ is $\boxed{b = 297\pi/100}$.

8. (16 points) In a closed environment, let functions $x(t)$ and $y(t)$ represent the population sizes (measured in some unspecified counting units) of two species, X and Y, respectively. Here the time t is measured in months. Suppose that the population sizes are modeled by the system

$$\frac{dx}{dt} = -x + \frac{xy}{10}$$
$$\frac{dy}{dt} = y - \frac{y^2}{100} - \frac{xy}{100}$$

(a) Describe the nature of the relationship between species X and Y: is it one of competition, cooperation, or predator and prey, and how can you tell? (If the relationship is one of predator and prey, make sure to explain how to tell which species is which.)

(2 points) We can tell by looking at the xy terms that this is a predator-prey system: The coefficient of the xy term in the equation for dx/dt is positive, so species x benefits from the presence of species y. And the coefficient of the xy term in the equation for dy/dt is negative, so species y suffers from the presence of species x. Thus x is the predator and y is the prey.

If the instead the xy term had a negative coefficient in the equation for dx/dt and a positive coefficient in the equation for dy/dt, then x would be the prey and y the predator. If they both had positive coefficients, this would be a cooperative system, and if they both had negative coefficients, the species would be in competition.

(b) Find all equilibrium solutions to this system.

(4 points) We suppose that $x(t)$ and $y(t)$ are constant functions (independent of t), so their derivatives are zero:

$$\begin{cases} 0 = -x + \frac{xy}{10} \\ 0 = y - \frac{y^2}{100} - \frac{xy}{100} \end{cases}$$

We can factor the first equation:

$$0 = x\left(-1 + \frac{y}{10}\right)$$

There are two cases: Either $x = 0$ or $-1 + y/10 = 0$.

Case 1:

$$x = 0$$

We can substitute this into the second equation in our system and get:

$$\begin{aligned} 0 &= y - \frac{y^2}{100} \\ &= y\left(1 - \frac{y}{100}\right) \end{aligned}$$

There are two possibilities: Either $y = 0$ or $1 - y/100 = 0$.

Sub-case 1a:

In this case, $x = 0$ and $y = 0$. So we have a solution $(x, y) = (0, 0)$.

Sub-case 1b:

We have $1 - \frac{y}{100} = 0$. Solving, we get $y = 100$. And we're in Case 1 where $x = 0$. So we have the solution $(x, y) = (0, 100)$.

Case 2:

We have $-1+y/10=0$. Solving, we get $y=10$. We can substitute this into the second equation in the system:

$$\begin{aligned}0 &= 10 - \frac{10^2}{100} - \frac{10x}{100}\\ 0 &= 9 - \frac{x}{10}\\ x &= 90\end{aligned}$$

So we have the solution $(x,y)=(90,10)$.

Thus there are three equilibrium solutions, and they are $(x(t),y(t))=(0,0)$, $(0,100)$, and $(90,10)$.

(c) Suppose that at time $t=0$ months, we have $x(0)=0$ and $y(0)=50$. Solve for an explicit formula that gives the population size $y(t)$ in terms of t. What happens to x and y as t approaches infinity?

(4 points) Since $x(0)=0$, we can use the equation for dx/dt and find that $dx/dt(0)=0$. The population of species x is zero and it is staying that way. So $x(t)=0$ for all t. So we can focus on the second differential equation in the system and substitute $x=0$:

$$\frac{dy}{dt} = y - \frac{y^2}{100}$$

We can put this into the form of a logistic equation:

$$\frac{dy}{dt} = y\left(1 - \frac{y}{100}\right)$$

where $k=1$ and $M=100$. The solution to the logistic equation is:

$$y(t) = \frac{M}{1+Ae^{-kt}} = \frac{100}{1+Ae^{-t}}$$

where A is a constant. We find A by using the initial condition $y(0)=50$:

$$\begin{aligned}50 &= \frac{100}{1+Ae^{-0}}\\ 50 &= \frac{100}{1+A}\\ 1+A &= \frac{100}{50}\\ 1+A &= 2\\ A &= 1\end{aligned}$$

So

$$y(t) = \frac{100}{1+e^{-t}}$$

Taking the limit as $t\to\infty$,

$$\lim_{t\to\infty} y(t) = 100$$

So y approaches 100, which is its carrying capacity in the absence of species x. And since $x(t)=0$ for all t in this situation,

$$\lim_{t\to\infty} x(t) = 0$$

(Strictly speaking, we can't conclude that $x(t)=0$ for all t just because $x(0)=0$ and $dx/dt(0)=0$. But it turns out that in this case the intuitive outcome (if there are no members of species x then there will never be any members of species x) is correct.)

For quick reference, here is the system: $\begin{cases} \dfrac{dx}{dt} &= -x + \dfrac{xy}{10} \\ \dfrac{dy}{dt} &= y - \dfrac{y^2}{100} - \dfrac{xy}{100} \end{cases}$

(d) Suppose instead that at time $t = 0$ months, we have $x(0) = 70$ and $y(0) = 20$. Use the differential equations to predict approximate values for the sizes of the two populations in one month's time. (Hint: adapt Euler's method.)

(3 points) Let's use a step size of 1 to get a crude estimate. We use Euler's method except now we have two dependent variables x, y and one independent variable t:

$$t_0 = 0 \qquad x_0 = 70 \qquad y_0 = 20$$

$$t_1 = 1 \qquad \begin{aligned} x_1 &= x_0 + h\frac{dx}{dt}(x_0, y_0, t_0) \\ &= 70 + 1\left(-70 + \frac{(70)(20)}{10}\right) \\ &= 70 + (-70 + 140) \\ &= 140 \end{aligned} \qquad \begin{aligned} y_1 &= y_0 + h\frac{dy}{dt}(x_0, y_0, t_0) \\ &= 20 + 1\left(20 - \frac{20^2}{100} - \frac{(70)(20)}{100}\right) \\ &= 20 + (20 - 4 - 14) \\ &= 22 \end{aligned}$$

So $x(1) \approx 140$ and $y(1) \approx 22$.

(e) Is there any time $t > 0$ at which dx/dt changes sign? Is there any time at which dy/dt changes sign? Justify your answers. There's no need to discuss what happens as t approaches infinity.

(3 points) In the course of using Euler's method in part (d), we found that at time $t = 0$, $dx/dt = 70 > 0$ and $dy/dt = 2 > 0$. So we want to know whether these derivatives become negative at any point.

Using the system of differential equations and our answer to part (d), we can approximate the derivatives at time $t = 1$:

$$\frac{dx}{dt} \approx -140 + \frac{(140)(22)}{10} > 0$$
$$\frac{dy}{dt} \approx 22 - \frac{22^2}{100} - \frac{(140)(22)}{100} < 0$$

So according to our approximation, dy/dt has already changed sign at time $t = 1$.

Alternatively, we can just look at the equation for dy/dt:

$$\frac{dy}{dt} = \left(y - \frac{y^2}{100}\right) - \frac{xy}{100}$$

The first two terms is a quadratic polynomial in y, and the coefficient of the quadratic term is negative. (So the graph is a downward-opening parabola.) So the first two terms together have a maximum value. And the right term will become very negative for sufficiently large x and y. So we can predict that x and y can only grow so far before dy/dt becomes negative.

After dy/dt becomes negative, y decreases over time and x continues to increase. If we factor the equation for dx/dt,

$$\frac{dx}{dt} = x\left(-1 + \frac{y}{10}\right)$$

we see that dx/dt will remain positive unless $-1 + y/10 = 0$ at some point. That is, when y decreases to 10, dx/dt will change sign.

(This kind of reasoning can be extended to show that dx/dt and dy/dt will continue to change sign periodically.)

9. (10 points) Determine whether each of the series below converges or diverges. Indicate clearly which tests you use and how you apply them.

(a) $\sum_{n=1}^{\infty}\left((-1)^{n-1}\frac{\sqrt{n}}{n+1}\right)$

(5 points) Denote $b_n = \frac{\sqrt{n}}{n+1}$. We have $b_n \geq 0$. Also, we have

$$\begin{aligned} b_{n+1} - b_n &= \frac{\sqrt{n+1}}{n+2} - \frac{\sqrt{n}}{n+1} \\ &= \frac{(n+1)\sqrt{n+1} - (n+2)\sqrt{n}}{(n+1)(n+2)} \\ &= \frac{\sqrt{n^3+3n^2+3n+1} - \sqrt{n^3+4n^2+4n}}{(n+1)(n+2)} \\ &= \frac{1-n-n^2}{(n+1)(n+2)(\sqrt{n^3+3n^2+3n+1}+\sqrt{n^3+4n^2+4n})} \\ &< 0 \end{aligned}$$

And again,

$$\lim_{n\to+\infty}\frac{\sqrt{n}}{n+1} = \lim_{n\to+\infty}\frac{1}{\sqrt{n}+\frac{1}{\sqrt{n}}} = 0.$$

Therefore, by alternating series test, we have

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sqrt{n}}{n+1}$$

is convergent.

(b) $\sum_{n=1}^{\infty}\left(\frac{5^n(n!)^2}{(2n)!}\right)$

(5 points) Denote

$$a_n = \frac{5^n(n!)^2}{(2n)!}.$$

Then

$$\frac{a_{n+1}}{a_n} = a_{n+1}\cdot\frac{1}{a_n} = \frac{5^{(n+1)}((n+1)!)^2}{(2(n+1))!}\cdot\frac{(2n)!}{5^n(n!)^2} = \frac{5(n+1)^2}{(2n+2)(2n+1)}.$$

Therefore,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{5(n+1)^2}{(2n+2)(2n+1)} = \frac{5}{4} > 1.$$

Then by Ratio test, the series diverges.

10. (12 points) In each of the following parts, give an example of a series (by specifying the items requested in brackets) that has the given property or properties, or state that such a series cannot exist. *You do not need to justify your answers.* (Please treat each question as independent from the others; properties do not carry over from part (a) to part (b), etc.)

(a) The series $\sum_{n=1}^{\infty} a_n$ converges but does not converge absolutely. [Formula for a_n]

(3 points) $a_n = \frac{(-1)^n}{n}$ is an example. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by the Alternating Series Test, but it doesn't converge absolutely because the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series with $p = 1 \not> 1$.

(b) The series $\sum_{n=1}^{\infty} a_n$ has positive terms and converges, but $\sum_{n=1}^{\infty} \frac{a_n}{n}$ diverges. [Formula for a_n]

(3 points) Such a series cannot exist. We would have

$$0 < \frac{a_n}{n} < a_n$$

and so by the Comparison Test, if $\sum a_n$ converges, then $\sum \frac{a_n}{n}$ must converge as well.

(c) The power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ has interval of convergence equal to $(-2, 2)$. [Value of a *and* formula for c_n]

(3 points) $a = 0$ and $c_n = \frac{1}{2^n}$ is an example.

(d) The power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of convergence equal to 0. [Value of a *and* formula for c_n]

(3 points) For this to work, the sequence c_n has to grow faster than the exponential $(x-a)^n$, for any fixed value of x not equal to a. Both $c_n = n!$ and $c_n = n^n$ will work. a could be any number.

11. (11 points) Show all steps in completing the problem below.

(a) Use the fact that $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ for all x, a fact which you do *not* have to prove, to find a series that converges to the number

$$\int_0^1 x\sin(x^3)\,dx$$

(6 points) Since $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, we have

$$x\sin(x^3) = x\sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+4}}{(2n+1)!}.$$

Therefore

$$\begin{aligned}\int_0^1 x\sin(x^3)dx &= \int_0^1 \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+4}}{(2n+1)!}\right)dx \\ &= \sum_{n=0}^{\infty} \int_0^1 \frac{(-1)^n x^{6n+4}}{(2n+1)!}dx \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\frac{x^{6n+5}}{6n+5}\Big|_0^1 \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!(6n+5)}\end{aligned}$$

(b) Write a partial sum of the series of part (a) that estimates the above integral to within 10^{-4}, and completely justify the accuracy of your partial sum.

(5 points) We only consider the first 3 terms, i.e., $\int_0^1 x\sin(x^3)dx \approx \frac{1}{5} - \frac{1}{66} + \frac{1}{2040}$.
Next we want to justify the accuracy is within 0.0001. Indeed, The series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!(6n+5)}$$

is an alternating series since

$$\frac{1}{(2n+1)!(6n+5)}$$

is positive, decreasing and approaches to zero as n goes to infinity. This can be seen from the denominator, which is positive, increasing and approaches to infinity as n goes to infinity.
Notice that the fourth term

$$\frac{1}{7! \times 23} = \frac{1}{23\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} = \frac{1}{23\cdot 42\cdot 20\cdot 6} < \frac{1}{10\cdot 10\cdot 100} = \frac{1}{10^4} = 0.0001.$$

By the Alternating Series Test, we have $|R_2| \le b_3 < 0.0001$ (n begins from 0). Our claim is justified.