Math 42 Final Exam — March 18, 2013

Name: ______________________________ **SUID#:** ____________

Circle your section:				
Kenji Kozai 02 (10-10:50am) 07 (1:15-2:05pm)	Jeremy Leach 04 (11-11:50am) 09 (2:15-3:05pm)	Xiaodong Li 05 (11-11:50am) 10 (2:15-3:05pm)	Mike Lipnowski 03 (10-10:50am) 08 (1:15-2:05pm)	Nisan Stiennon ACE

- Complete the following problems. In order to receive full credit, please show all of your work and justify your answers. You do not need to simplify your answers unless specifically instructed to do so. You may use any result proved in class or the text, but be sure to clearly state the result before using it, and to verify that all hypotheses are satisfied.
- Please check that your copy of this exam contains 16 numbered pages and is correctly stapled.
- This is a closed-book, closed-notes exam. No electronic devices, including cellphones, headphones, or calculation aids, will be permitted for any reason.
- **You have 3 hours.** Your organizer will signal the times between which you are permitted to be writing, including anything on this cover sheet, and to have the exam booklet open. During these times, the exam and all papers must remain in the testing room. When you are finished, you must hand your exam paper to a member of teaching staff.
- Paper not provided by teaching staff is prohibited. If you need extra room for your answers, use the back side of a page or one of the extra sheets provided in this packet, and clearly indicate that your answer continues there. Do not unstaple or detach pages from this exam.
- Please sign the following:

> "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ______________________________

The following boxes are strictly for grading purposes. Please do not mark.

Question:	1	2	3	4	5	6	7	8	9	10	11	**Total**
Points:	12	12	15	13	12	14	14	16	10	12	11	141
Score:												

1. (12 points) Evaluate each of the following, or explain why its value does not exist; show all reasoning.

(a) $\displaystyle\int_0^2 \frac{5}{x^3}\,dx$

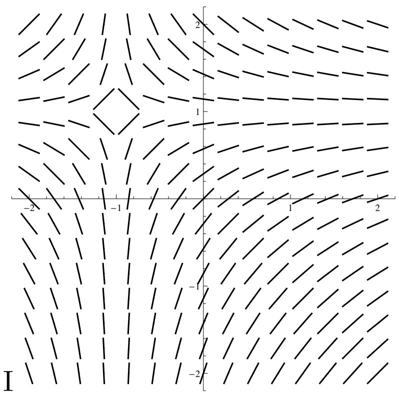
(b) $\displaystyle\int_0^\infty \frac{e^x}{e^{2x}+1}\,dx$

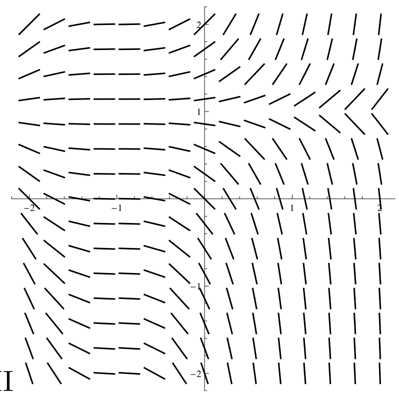
2. (12 points) Consider the region R in the xy-plane bounded by the curves $y = x^2 - 1$ and $y = 1 - x^2$.

   (a) Suppose S_1 is the solid obtained by rotating R about the line $x = 2$. Set up two different integrals representing the volume of S_1. In each case, cite the method used and justify with a sketch, but don't evaluate either integral.

   (b) Suppose S_2 is a solid whose base is R, and whose cross-sections perpendicular to the y-axis are equilateral triangles. Set up, but do not evaluate, an integral that gives the volume of S_2. (Note: the area of an equilateral triangle of side length s is $\frac{1}{4}s^2\sqrt{3}$.)

3. (15 points) Each of the eight direction fields below corresponds to exactly one of the equations given; the scale on each is $-2 \le x \le 2$, $-2 \le y \le 2$. Determine the equation that corresponds to each direction field. No justification is necessary.


I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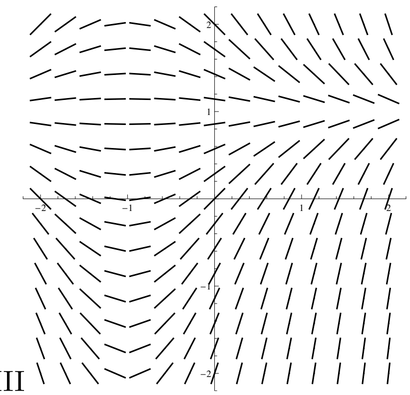
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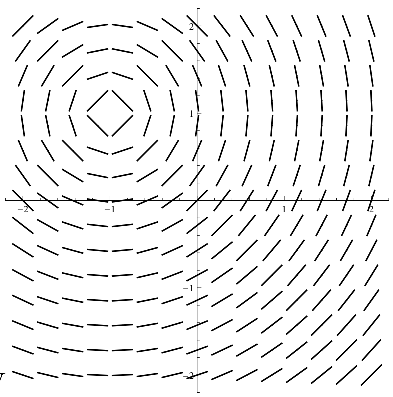
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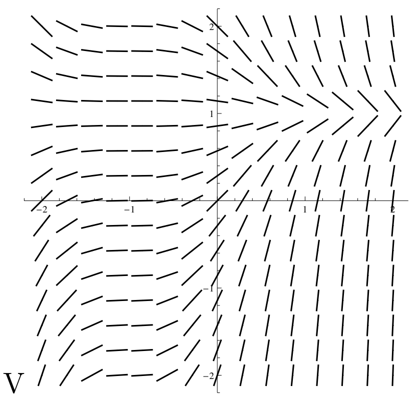
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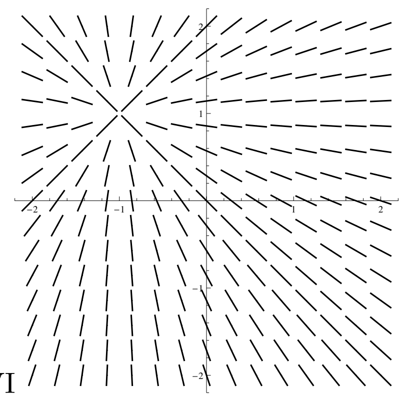
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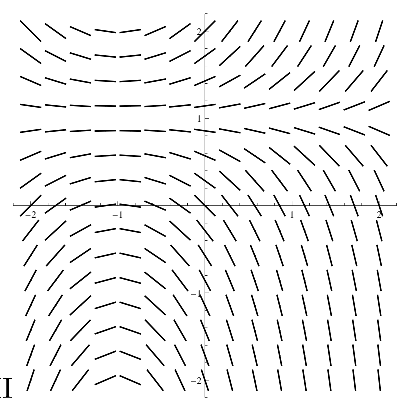
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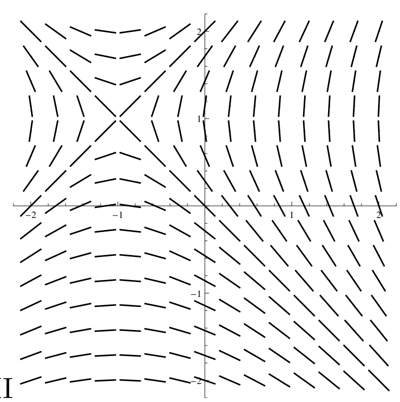
VIII

Equation	I, II, III, IV, V, VI, VII, or VIII	Equation	I, II, III, IV, V, VI, VII, or VIII
$y' = (y-1)(x+1)$		$y' = -(y-1)(x+1)$	
$y' = (y-1)(x+1)^2$		$y' = -(y-1)(x+1)^2$	
$y' = \dfrac{y-1}{x+1}$		$y' = -\dfrac{y-1}{x+1}$	
$y' = \dfrac{x+1}{y-1}$		$y' = -\dfrac{x+1}{y-1}$	

4. (13 points) At noon, a tank contains 10 kg of salt, dissolved in 100 L of water. At this time, pure water begins entering the tank at a rate of 8 L/min; the solution is kept thoroughly mixed and drains from the tank at the rate of 10 L/min.

   (a) Find an expression for the volume of the fluid in the tank t minutes after noon.

   (b) Set up an initial value problem for $y(t)$, the amount of salt in the tank t minutes after noon. Be sure to state your initial condition, including the units involved.

   (c) By solving the initial value problem, find the amount of salt in the tank 25 minutes after noon.

5. (12 points) A population is modeled by the following differential equation: $\frac{dP}{dt} = \frac{1}{2}P - \frac{1}{16}P^4$

(a) Find the equilibrium solutions of the differential equation.

(b) Determine the value of P for which the population is growing fastest; explain all reasoning. (Hint: first use the differential equation to find an expression for $\frac{d^2P}{dt^2}$.)

For quick reference, the differential equation for P is: $\dfrac{dP}{dt} = \dfrac{1}{2}P - \dfrac{1}{16}P^4$

(c) Suppose $P(0) = 1$. Use Euler's method with step size 1 to estimate the value $P(2)$. Show your steps, but you don't need to simplify your answer.

(d) For $P(t)$ satisfying the initial value problem $\begin{cases} \dfrac{dP}{dt} = \dfrac{1}{2}P - \dfrac{1}{16}P^4 \\ P(0) = 1 \end{cases}$, what is the behavior of $P(t)$ as $t \to \infty$? Justify your answer.

6. (14 points)

   (a) Show all steps in solving the initial value problem

$$\frac{dy}{dx} = xe^{y-x}, \quad y(0) = -\ln 2$$

(b) Show all steps in solving the problem

$$\frac{dz}{dt} + 2tz = z, \quad z(0) = 4.$$

(c) Is there a function $z(t)$ satisfying the differential equation of part (b), but instead with initial value $z(0) = 0$? Explain.

7. (14 points) The basin of a concrete "pond" on campus has the shape of a hemisphere, 10 meters in radius. Water is filled to a depth of h meters, which means that the water occupies the shape of an upside-down "cap" of height h, cut from a sphere of radius 10. (Take $h \leq 10$; the case $h = 10$ means the hemisphere is completely filled with water.)

   (a) Show that top surface of the water is a circle of area $A = \pi(20h - h^2)$ square meters.

   (b) Show that the volume of water in the basin is $V = \pi\left(10h^2 - \frac{h^3}{3}\right)$ cubic meters.

For quick reference, the top surface has area $A = \pi(20h - h^2)$; the water has volume $V = \pi\left(10h^2 - \frac{h^3}{3}\right)$.

(c) Water starts to evaporate from the pond; at any moment the volume of water is decreasing at a rate proportional to the area of the top surface. Initially, at time $t = 0$ days, the pond is filled to a depth of exactly 9 meters. Moreover, it is determined that at this exact moment, the height is decreasing at a rate of $-3/100$ meters per day.

Set up an initial value problem satisfied by $h(t)$, the depth of the water in the pond after t days. A complete answer should not depend on unknown constants, but finding a solution is not necessary. (Hint: first express $\frac{dV}{dt}$ in terms of $\frac{dh}{dt}$.)

(d) A water source is installed to replenish some of the evaporated water, at the constant rate of b cubic meters per day. Set up (but *don't solve*) a new differential equation for $h(t)$ in this situation, and then compute b so that the depth of the water may be maintained constantly at the value $h = 9$. Show all reasoning.

8. (16 points) In a closed environment, let functions $x(t)$ and $y(t)$ represent the population sizes (measured in some unspecified counting units) of two species, X and Y, respectively. Here the time t is measured in months. Suppose that the population sizes are modeled by the system

$$\frac{dx}{dt} = -x + \frac{xy}{10}$$
$$\frac{dy}{dt} = y - \frac{y^2}{100} - \frac{xy}{100}$$

(a) Describe the nature of the relationship between species X and Y: is it one of competition, cooperation, or predator and prey, and how can you tell? (If the relationship is one of predator and prey, make sure to explain how to tell which species is which.)

(b) Find all equilibrium solutions to this system.

(c) Suppose that at time $t = 0$ months, we have $x(0) = 0$ and $y(0) = 50$. Solve for an explicit formula that gives the population size $y(t)$ in terms of t. What happens to x and y as t approaches infinity?

For quick reference, here is the system: $$\begin{cases} \dfrac{dx}{dt} &= -x + \dfrac{xy}{10} \\ \dfrac{dy}{dt} &= y - \dfrac{y^2}{100} - \dfrac{xy}{100} \end{cases}$$

(d) Suppose instead that at time $t = 0$ months, we have $x(0) = 70$ and $y(0) = 20$. Use the differential equations to predict approximate values for the sizes of the two populations in one month's time. (Hint: adapt Euler's method.)

(e) Is there any time $t > 0$ at which dx/dt changes sign? Is there any time at which dy/dt changes sign? Justify your answers. There's no need to discuss what happens as t approaches infinity.

9. (10 points) Determine whether each of the series below converges or diverges. Indicate clearly which tests you use and how you apply them.

   (a) $\displaystyle\sum_{n=1}^{\infty} \left((-1)^{n-1} \frac{\sqrt{n}}{n+1} \right)$

(b) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{5^n (n!)^2}{(2n)!}\right)$

10. (12 points) In each of the following parts, give an example of a series (by specifying the items requested in brackets) that has the given property or properties, or state that such a series cannot exist. *You do not need to justify your answers.* (Please treat each question as independent from the others; properties do not carry over from part (a) to part (b), etc.)

(a) The series $\sum_{n=1}^{\infty} a_n$ converges but does not converge absolutely. [Formula for a_n]

(b) The series $\sum_{n=1}^{\infty} a_n$ has positive terms and converges, but $\sum_{n=1}^{\infty} \frac{a_n}{n}$ diverges. [Formula for a_n]

(c) The power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has interval of convergence equal to $(-2, 2)$. [Value of a *and* formula for c_n]

(d) The power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence equal to 0. [Value of a *and* formula for c_n]

11. (11 points) Show all steps in completing the problem below.

(a) Use the fact that $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ for all x, a fact which you do *not* have to prove, to find a series that converges to the number

$$\int_0^1 x \sin(x^3)\, dx$$

(b) Write a partial sum of the series of part (a) that estimates the above integral to within 10^{-4}, and completely justify the accuracy of your partial sum.