Solutions to Math 42 Second Exam — February 21, 2013

1. (10 points) Determine, with justification, whether each series converges. *If the series converges, find its sum.*

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$

   (5 points) We see that $n^2 - 1 = (n+1)(n-1)$, so we will use a partial fraction decomposition:

   $$\frac{1}{n^2-1} = \frac{1}{(n+1)(n-1)} = \frac{A}{n+1} + \frac{B}{n-1}.$$

   We multiply through by $(n+1)(n-1)$ and group like terms:

   $$1 = A(n-1) + B(n+1) = An - A + Bn + B = (A+B)n + (B-A)n.$$

   Thus, we have that $A + B = 0$ and $B - A = 1$. Adding these two equations yields $2B = 1$, so that $B = \frac{1}{2}$ and $A = -B = -\frac{1}{2}$. So we can rewrite our sum as

   $$\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \sum_{n=2}^{\infty} \left(-\frac{1}{2(n+1)} + \frac{1}{2(n-1)} \right) = \frac{1}{2} \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n+1} \right)$$

   Writing out the first few partial sums, we notice telescoping:

   $$\begin{aligned}
   s_2 &= \tfrac{1}{2}\left(1 - \tfrac{1}{3}\right) \\
   s_3 &= \tfrac{1}{2}\left[\left(1 - \tfrac{1}{3}\right) + \left(\tfrac{1}{2} - \tfrac{1}{4}\right)\right] = \tfrac{1}{2}\left(1 + \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{4}\right) \\
   s_4 &= \tfrac{1}{2}\left[\left(1 - \tfrac{1}{3}\right) + \left(\tfrac{1}{2} - \tfrac{1}{4}\right) + \left(\tfrac{1}{3} - \tfrac{1}{5}\right)\right] = \tfrac{1}{2}\left(1 + \tfrac{1}{2} - \tfrac{1}{4} - \tfrac{1}{5}\right) \\
   s_5 &= \tfrac{1}{2}\left[\left(1 - \tfrac{1}{3}\right) + \left(\tfrac{1}{2} - \tfrac{1}{4}\right) + \left(\tfrac{1}{3} - \tfrac{1}{5}\right) + \left(\tfrac{1}{4} - \tfrac{1}{6}\right)\right] = \tfrac{1}{2}\left(1 + \tfrac{1}{2} - \tfrac{1}{5} - \tfrac{1}{6}\right) \\
   &\vdots \\
   s_n &= \tfrac{1}{2}\left(1 + \tfrac{1}{2} - \tfrac{1}{n} - \tfrac{1}{n+1}\right).
   \end{aligned}$$

   Thus,

   $$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{2}\left(1 + \frac{1}{2}\right) = \boxed{\frac{3}{4}}.$$

   (b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4^n}{(-5)^{n-1}} - \frac{1}{2^n} \right)$

   (5 points) We will work with each part of the sum separately. First,

   $$\sum_{n=1}^{\infty} \frac{4^n}{(-5)^{n-1}} = \sum_{n=0}^{\infty} \frac{4^{n+1}}{(-5)^n} = 4\sum_{n=0}^{\infty} \frac{4^n}{(-5)^n} = 4\sum_{n=0}^{\infty} \left(-\frac{4}{5}\right)^n.$$

   This is a geometric series with $r = -\frac{4}{5}$, so it converges to $4\left(\frac{1}{1-(-\frac{4}{5})}\right) = 4\left(\frac{1}{(\frac{9}{5})}\right) = 4\left(\frac{5}{9}\right) = \frac{20}{9}$. The second part is also a geometric series

   $$\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} = \frac{1}{2}\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n.$$

   Since $r = \frac{1}{2}$, this also converges to $\frac{1}{2}\left(\frac{1}{1-\frac{1}{2}}\right) = \frac{1}{2}\left(\frac{1}{(\frac{1}{2})}\right) = 1$. Since each of the two series converges, then the original series (the difference of the two geometric series) converges to $\frac{20}{9} - 1 = \boxed{\frac{11}{9}}$.

2. (10 points) Determine whether each of the series below converges or diverges. Indicate clearly which tests you use and how you apply them.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+4}{5n^3-1}$

(5 points) Notice that for $n \geq 1$, we have $5n^3 - 1 > 0$, so

$$\frac{2n^2+4}{5n^3-1} \geq \frac{2n^2}{5n^3} = \frac{2}{5n} = \frac{2}{5}\left(\frac{1}{n}\right) > 0.$$

We know that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges because it is a p-series with $p = 1$ (it is also the harmonic series). So letting $b_n = \frac{2}{5n}$, we have that

$$\sum_{n=1}^{\infty} b_n = \frac{2}{5}\sum_{n=1}^{\infty} \frac{1}{n},$$

which diverges. Then, if $a_n = \frac{2n^2+4}{5n^3-1}$, we have that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms, $a_n > b_n$, and $\sum_{n=1}^{\infty} b_n$ diverges. Hence,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{2n^2+4}{5n^3-1}$$

also diverges by the Comparison Test.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{(n+1)!}$

(5 points) We will use the Ratio Test on $a_n = \dfrac{(-2)^n}{(n+1)!}$. We have that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-2)^{n+1}}{(n+2)!} \cdot \frac{(n+1)!}{(-2)^n} \right| = \lim_{n\to\infty} \left| \frac{-2}{n+2} \right| = 0$$

because the denominator goes to ∞ while the numerator stays constant. Hence,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1,$$

so by the Ratio Test, the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-2)^n}{(n+1)!}$$

converges absolutely, and therefore converges.

3. (10 points) Determine whether each of the series below converges or diverges. Indicate clearly which tests you use and how you apply them.

(a) $\sum_{n=1}^{\infty} 2ne^{-n}$

(5 points)

Method 1: Ratio Test

We will apply the ratio test to $a_n = 2ne^{-n}$. We have that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2(n+1)e^{-(n+1)}}{2ne^{-n}}\right| = \lim_{n\to\infty}\left|\frac{2(n+1)e^{n}}{2ne^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{2n+2}{ne}\right|$$
$$= \lim_{n\to\infty}\left|\frac{2+\frac{2}{n}}{2e}\right| = \frac{1}{e}\lim_{n\to\infty}\left|\frac{2+\frac{2}{n}}{2}\right| = \frac{1}{e},$$

since $\lim_{n\to\infty}\frac{2}{n} = 0$. Since $\frac{1}{e} < 1$, the Ratio Test tells us that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 2ne^{-n}$ converges.

Method 2: Integral Test

We let $f(x) = 2xe^{-x}$, so that $f(n) = a_n = 2ne^{-n}$. Now $f(x)$ is continuous and positive for $x \geq 1$ because x and e^{-x} are continuous functions, and x is positive when $x \geq 1$, and e^{-x} is positive for all x. We can show that $f(x)$ is decreasing on $[1, \infty)$ by showing that $f'(x) < 0$ for $x > 1$:

$$f'(x) = 2x(e^{-x})' + 2(x)'e^{-x} = -2xe^{-x} + 2e^{-x} = 2(1-x)e^{-x},$$

and if $x > 1$, then $1 - x < 0$ and e^{-x} is always positive, so the product $2(1-x)e^{-x}$ is negative. Now we consider the integral

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} 2xe^{-x}\,dx.$$

We will integrate by parts letting $u = 2x$ and $dv = e^{-x}$ so that $du = 2dx$ and $v = -e^{-x}$. Then

$$\int_1^{\infty} 2xe^{-x}dx = \lim_{t\to\infty}\int_1^{t} 2xe^{-x}dx = \lim_{t\to\infty} 2x(-e^{-x})]_1^t - \int_1^t -2e^{-x}dx$$
$$= \lim_{t\to\infty} -2xe^{-x}]_1^t + \int_1^t 2e^{-x}dx = \lim_{t\to\infty} -2xe^{-x}]_1^t - 2e^{-x}]_1^t$$
$$= \lim_{t\to\infty} -2(x+1)e^{-x}]_1^t = \lim_{t\to\infty} -2(t+1)e^{-t} + 4e^{-1}.$$

But now,

$$\lim_{t\to\infty} -2(t+1)e^{-t} = \lim_{t\to\infty}\frac{-2(t+1)}{e^t}$$

is an indeterminate form, so we apply L'Hospital's rule:

$$\lim_{t\to\infty}\frac{-2(t+1)}{e^t} = \lim_{t\to\infty}\frac{-2}{e^t} = 0.$$

Hence, the improper integral converges, so by the Integral Test, the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 2ne^{-n}$$

converges.

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{2\cos n}{n^4-1}$

(5 points) The terms $a_n = \dfrac{2\cos n}{n^4-1}$ are not always positive, so a regular Comparison or Limit Comparison Test will not work. Instead, we will first check the series for absolute convergence. We can see that

$$|a_n| = \left|\frac{2\cos n}{n^4-1}\right| \le \frac{2}{n^4-1}$$

for $n \ge 2$. Letting $b_n = \frac{2}{n^4-1}$, we can now see that $|a_n| \le b_n$, so we will try to show that b_n converges. Now, $b_n = \frac{2}{n^4-1}$ looks like $\frac{1}{n^4}$, so taking $c_n = \frac{1}{n^4}$, we will apply the Limit Comparison Test. Note that $b_n > 0$ and $c_n > 0$ for $n \ge 2$, so we can apply the test. Then

$$\lim_{n\to\infty} \frac{b_n}{c_n} = \lim_{n\to\infty} \frac{2}{n^4-1} \cdot \frac{n^4}{1} = \lim_{n\to\infty} \frac{2n^4}{n^4-1} = \lim_{n\to\infty} \frac{2}{1-\frac{1}{n^4}}.$$

Then $\frac{1}{n^4}$ goes to 0 as n goes to infinity, so

$$\lim_{n\to\infty} \frac{2}{1-\frac{1}{n^4}} = 2.$$

In particular, $\lim_{n\to\infty} \frac{b_n}{c_n}$ is not 0, so either both $\sum_{n=2}^{\infty} b_n$ and $\sum_{n=2}^{\infty} c_n$ converge, or both series diverge. Notice that

$$\sum_{n=2}^{\infty} c_n = \sum_{2}^{\infty} \frac{1}{n^4}$$

is a p-series with $p = 4 > 1$, so it converges. Hence, $\sum_{n=2}^{\infty} b_n$ converges. But

$$\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{2}{n^4-1} \ge \sum_{n=2}^{\infty} \left|\frac{2\cos n}{n^4-1}\right| = \sum_{n=2}^{\infty} |a_n|$$

and both b_n and $|a_n|$ are positive, so by the comparison test, $\sum_{n=2}^{\infty} |a_n|$ converges. Therefore,

$$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{2\cos n}{n^4-1}$$

converges absolutely, and so it converges.

4. (15 points) (Two pages)

(a) Show that the series $s = \displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3}$ converges.

(4 points) If $n \geq 1$, start with the observation that

$$0 \leq |\sin n| \leq 1$$

Squaring all sides and dividing by $n^3 > 0$, we obtain

$$0 \leq \sin^2 n \leq 1 \quad \Longrightarrow \quad 0 \leq \frac{\sin^2 n}{n^3} \leq \frac{1}{n^3}$$

Therefore we can use the Comparison Test: We have $a_n = \frac{\sin^2 n}{n^3}$ and $b_n = \frac{1}{n^3}$. We've shown that $a_n \leq b_n$ for all $n \geq 1$ and that a_n and b_n are nonnegative for all $n \geq 1$. Now $\sum b_n = \sum_{n=1}^{\infty} \frac{1}{n^3}$ converges because it is a p-series with $p = 3 > 1$. So by the Comparison Test, $\sum a_n = \sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3}$ converges as well.

(The Comparison Test as stated in our text requires that the terms a_n and b_n be *positive*, not merely nonnegative. We can show that they are positive: indeed, $\sin x = 0$ holds only when x is an integer multiple of π. But π is irrational, so no integer is equal to an integer multiple of π. So $\sin n \neq 0$ for all $n \geq 1$, and so $a_n = \frac{\sin^2 n}{n^3} \neq 0$ for all $n \geq 1$. Combining this with the fact that $a_n \geq 0$, we've shown that $a_n > 0$. And since $b_n \geq a_n$, we've shown that $b_n > 0$ as well.)

(b) We can approximate s by computing the tenth partial sum of the series, which is found to be

$$s_{10} = 0.8325298\ldots$$

Express the remainder (error) $R_{10} = s - s_{10}$ as a series of its own, and explain why R_{10} is positive.

(3 points)

$$R_{10} = s - s_{10} = \sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3} - \sum_{n=1}^{10} \frac{\sin^2 n}{n^3} = \sum_{n=11}^{\infty} \frac{\sin^2 n}{n^3}$$

R_{10} is positive because we've expressed it as a series all of whose terms are positive. (See the last paragraph of the solution to (a) for an explanation of why $\frac{\sin^2 n}{n^3}$ is positive for all $n \geq 1$.)

(c) Show that $R_{10} \leq \frac{1}{200}$. (Hint: Use the Comparison Test to relate R_{10} to a series that can be analyzed with the Integral Test.)

(4 points) We have

$$R_{10} = \sum_{n=11}^{\infty} \frac{\sin^2 n}{n^3} \leq \sum_{n=11}^{\infty} \frac{1}{n^3} \tag{1}$$

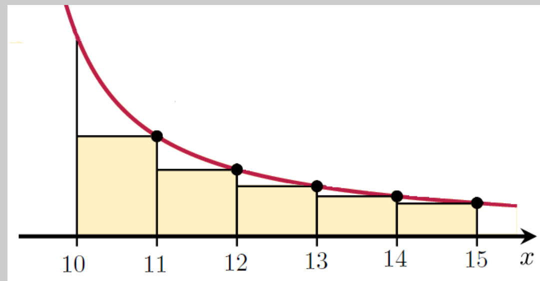
since $\frac{\sin^2 n}{n^3} \leq \frac{1}{n^3}$ for all $n \geq 1$ (see (a)). Now $f(x) = \frac{1}{x^3}$ is continuous, positive, and decreasing on $[10, \infty)$, so we can do things related to the Integral Test. Crucially, we can say that

$$\sum_{n=11}^{\infty} \frac{1}{n^3} \leq \int_{10}^{\infty} \frac{1}{x^3}\, dx \tag{2}$$

This is a consequence of the Integral Test Remainder Theorem, which says that the 10^{th} partial sum of $\sum_{n=1}^{\infty} \frac{1}{n^3}$ approximates the sum with an error bounded above by $\int_{10}^{\infty} \frac{1}{x^3}\, dx$. That is,

$$\sum_{n=11}^{\infty} \frac{1}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^3} - \sum_{n=1}^{10} \frac{1}{n^3} \leq \int_{10}^{\infty} \frac{1}{x^3}\, dx$$

Alternatively, (2) can be seen directly from the following graph of $y = \frac{1}{x^3}$; note that the integral is an overestimate of the sum (not to scale):



Putting together (1) and (2), we have

$$R_{10} \le \int_{10}^{\infty} \frac{1}{x^3}\,dx = \lim_{b\to\infty} \int_{10}^{b} \frac{1}{x^3}\,dx = \lim_{b\to\infty} \left[-\frac{1}{2x^2}\right]_{10}^{b} = \lim_{b\to\infty} \left(-\frac{1}{2b^2} + \frac{1}{200}\right) = \frac{1}{200}$$

Remark: Note that it would be incorrect to say that

$$\int_{11}^{\infty} \frac{\sin^2 x}{x^3}\,dx \le \sum_{n=11}^{\infty} \frac{\sin^2 n}{n^3} \le \int_{10}^{\infty} \frac{\sin^2 x}{x^3}\,dx$$

because $(\sin^2 x)/x^3$ is not a decreasing function.

(d) Put the information in parts (b) and (c) together to write a statement of the form

$$A \le \sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3} \le B$$

for appropriate values A and B. (A and B should be expressed as numbers, not as infinite summations or unevaluated integrals, but do not have to be in simplified form.)

(2 points) Recall $R_{10} = s - s_{10} = \left(\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3}\right) - s_{10}$ by definition. Using (b) and (c), we have

$$0 \;\le\; R_{10} \;\le\; \frac{1}{200} \quad\Longleftrightarrow\quad 0 \;\le\; \left(\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3}\right) - s_{10} \;\le\; \frac{1}{200}$$

$$\Longleftrightarrow\quad s_{10} \;\le\; \sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3} \;\le\; s_{10} + \frac{1}{200}$$

This last statement has the desired form, with $A = s_{10}$ and $B = s_{10} + \frac{1}{200}$.

(e) Use (d) to give an improved approximation for s. In a mathematically precise statement, express the error in this new approximation.

(2 points) From part (d) we know that s must lie in the interval $[A, B] = [s_{10}, s_{10} + \frac{1}{200}]$. Our best approximation for s will be the midpoint of this interval:

$$s \approx \frac{A+B}{2} = \frac{s_{10} + \left(s_{10} + \frac{1}{200}\right)}{2} = s_{10} + \frac{1}{400} = 0.8325298\ldots + 0.0025 = \boxed{0.8350298\ldots}$$

The error in this approximation is at most half the length of the interval:

$$|\text{error}| = |s - 0.8350298\ldots| \le \frac{B-A}{2} = \frac{\left(s_{10} + \frac{1}{200}\right) - s_{10}}{2} = \frac{1}{400} = 0.0025$$

5. (13 points) Find, with complete justification, the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{2^n(x+3)^n}{\sqrt{n+1}}$$

To find the interval of convergence for the given series, we apply the Ratio Test. This tells us that a series converges whenever $\lim_{n\to\infty} |a_{n+1}/a_n| < 1$, so we find that the series converges whenever

$$\begin{aligned} 1 > \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{2^{n+1}(x+3)^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{2^n(x+3)^n} \right| \\ &= 2|x+3| \lim_{n\to\infty} \sqrt{\frac{n+1}{n+2}} \\ &= 2|x+3|. \end{aligned}$$

In other words, we know the series converges whenever $|x+3| < \frac{1}{2}$; that is, whenever

$$-\frac{7}{2} < x < -\frac{5}{2}$$

The only other values of x at which this series might converge are those values where the Ratio Test is inconclusive, namely the two endpoints. We check them for convergence separately. First, we evaluate the series at $x = -5/2$:

$$\sum_{n=0}^{\infty} \frac{2^n(x+3)^n}{\sqrt{n+1}} \bigg|_{x=-5/2} = \sum_{n=0}^{\infty} \frac{2^n(1/2)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}.$$

There are a few ways we might show that this series diverges, but perhaps the fastest way is to note that it is just the p-series with $p = \frac{1}{2} \leq 1$ in disguise since

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$$

Now consider the series evaluated at the endpoint $x = -7/2$:

$$\sum_{n=0}^{\infty} \frac{2^n(x+3)^n}{\sqrt{n+1}} \bigg|_{x=-7/2} = \sum_{n=0}^{\infty} \frac{2^n(-1/2)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}.$$

This is a series with alternating signs, and letting $b_n = 1/\sqrt{n+1}$, it is clear that it satisfies the hypotheses of the Alternating Series Test (namely that $0 \leq b_{n+1} \leq b_n$ for all n, and that $\lim_{n\to\infty} b_n = 0$), so this series converges. We thus conclude that the interval of convergence is

$$\left[-\frac{7}{2}, -\frac{5}{2}\right) = \left\{x : -\frac{7}{2} \leq x < -\frac{5}{2}\right\}$$

6. (12 points) Suppose the power series $\sum_{n=0}^{\infty} c_n x^n$ converges for $x = 3$, but diverges for $x = -5$; no other information about the values of c_n is given. Decide which of the following series must converge, must diverge, or may either converge or diverge (inconclusive). Circle your answer. You do not need to justify your answers.

(2 points each) Let $f(x) = \sum_{n=0}^{\infty} c_n x^n$. The information above tells us partial information about the radius of convergence R of f about its center 0: specifically, we can conclude that $3 \leq R \leq 5$.

(a) $\sum_{n=0}^{\infty} 4^n c_n$ Converges Diverges **[Inconclusive]**

This is $f(x)$ evaluated at $x = 4$, which we do not know to lie inside or outside f's radius of convergence about the center 0. Thus, we don't know whether the series converges or diverges.

(b) $\sum_{n=0}^{\infty} (-2)^n c_n$ **[Converges]** Diverges Inconclusive

This is $f(x)$ evaluated at $x = -2$, which lies inside the smallest possible radius value R.

(c) $\sum_{n=1}^{\infty} n 3^n c_n$ Converges Diverges **[Inconclusive]**

This is $3f'(x)$ evaluated at $x = 3$. We know f' has the same *radius* of convergence, R, as f; however, since $R \geq 3$, we might have $R = 3$. In the latter case, we would not know whether the endpoints — i.e., the values $x = \pm 3$ — lie in the *interval* of convergence of f'.

(d) $\sum_{n=0}^{\infty} \frac{6^n c_n}{n+1}$ Converges **[Diverges]** Inconclusive

This is $\frac{1}{6}\int_0^x f(t)\,dt$ evaluated at $x = 6$, which lies outside the largest possible radius of convergence about 0 for f, and thus for any antiderivative of f. Therefore, the series diverges.

(e) $\sum_{n=0}^{\infty} (c_n)^2$ **[Converges]** Diverges Inconclusive

Since $\sum_{n=0}^{\infty} 3^n c_n$ converges, we know $\lim_{n\to\infty} 3^n c_n = 0$, which means that for all sufficiently large n we have $|3^n c_n| < 1$, so that $0 \leq |c_n| < \frac{1}{3^n}$, and thus $0 \leq (c_n)^2 < \frac{1}{9^n}$. Thus, by the Comparison Test, $\sum_{n=0}^{\infty} (c_n)^2$ converges because $\sum_{n=0}^{\infty} \frac{1}{9^n}$ does (the latter is geometric with common ratio $\frac{1}{9} < 1$).

(f) $\sum_{n=0}^{\infty} n! c_n$ Converges **[Diverges]** Inconclusive

The series diverges, because $\lim_{n\to\infty} n! c_n \neq 0$. For, consider what would be true if $\lim_{n\to\infty} n! c_n = 0$: then for all sufficiently large n we'd have $|n! c_n| < 1$, or equivalently $|c_n| < \frac{1}{n!}$. But then the series $\sum_{n=0}^{\infty} (-5)^n c_n$ would converge absolutely by comparison with $\sum_{n=0}^{\infty} \frac{5^n}{n!}$ (convergent by Ratio Test), since $0 \leq |(-5)^n c_n| < \frac{5^n}{n!}$ for all sufficiently large n; and this directly contradicts the given fact that $\sum_{n=0}^{\infty} (-5)^n c_n$ diverges! Thus, $\lim_{n\to\infty} n! c_n \neq 0$; so $\sum_{n=0}^{\infty} n! c_n$ diverges by the Test for Divergence.

7. (12 points) Determine, showing all reasoning, a power series centered at 0 for each of the functions given below, and give the *radius of convergence*.

(a) $g(x) = \dfrac{x}{4+x^2}$

(6 points) We have that

$$g(x) = \frac{x}{4+x^2} = \frac{x}{4}\cdot\frac{1}{1-(-x^2/4)} = \frac{x}{4}\sum_{n=0}^{\infty}\left(\frac{-x^2}{4}\right)^n$$
$$= \frac{x}{4}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{4^n}$$
$$= \boxed{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{4^{n+1}}}.$$

To find the radius of convergence, we could apply the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+3}}{4^{n+2}}\cdot\frac{4^{n+1}}{(-1)^n x^{2n+1}}\right| = \left|\frac{x^2}{4}\right| < 1 \Rightarrow |x^2| = |x|^2 < 4 \Rightarrow |x| < 2.$$

Alternatively, we could observe that by the Geometric Series Rule, the steps we take above to convert $\frac{1}{1-(-x^2/4)}$ into a geometric series are only valid for $|-x^2/4| < 1$, or equivalently for $|x| < 2$.

Either way, the radius of convergence is $\boxed{2}$.

(b) $h(x) = \dfrac{1}{(1-x)^2}$

(6 points) We have (using the chain rule!) that

$$h(x) = \frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\sum_{n=0}^{\infty}x^n = \boxed{\sum_{n=1}^{\infty}nx^{n-1}}.$$

To find the radius of convergence, we could similarly apply the Ratio Test. However, we could also observe that since neither differentiation nor integration changes the radius of convergence of a series, and since the radius of convergence for the geometric series $\sum_{n=0}^{\infty} x^n$ is 1, then the radius of convergence for the given series is also $\boxed{1}$.

8. (12 points) Let $f(x) = \cos x$.

(a) Find $T_4(x)$, the degree-4 Taylor polynomial for f centered at $\frac{\pi}{4}$. $\left(\text{Hint: } \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}.\right)$ Show all steps of your reasoning.

(5 points) The first few derivatives of $f(x)$ and corresponding Taylor coefficients $c_n = \dfrac{f^{(n)}\left(\frac{\pi}{4}\right)}{n!}$:

$$\begin{aligned} f(x) &= \cos x &\implies\quad c_0 &= \frac{\cos\frac{\pi}{4}}{0!} = \frac{1}{\sqrt{2}} \\ f'(x) &= -\sin x &\implies\quad c_1 &= \frac{-\sin\frac{\pi}{4}}{1!} = -\frac{1}{\sqrt{2}} \\ f''(x) &= -\cos x &\implies\quad c_2 &= \frac{-\cos\frac{\pi}{4}}{2!} = -\frac{1}{2!\sqrt{2}} \\ f'''(x) &= \sin x &\implies\quad c_3 &= \frac{\sin\frac{\pi}{4}}{3!} = \frac{1}{3!\sqrt{2}} \\ f^{(4)}(x) &= \cos x &\implies\quad c_4 &= \frac{\cos\frac{\pi}{4}}{4!} = \frac{1}{4!\sqrt{2}} \end{aligned}$$

Therefore, the degree-4 Taylor polynomial of f, centered at $\pi/4$, is

$$T_4(x) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right)^1 - \frac{1}{2!\sqrt{2}}\left(x - \frac{\pi}{4}\right)^2 + \frac{1}{3!\sqrt{2}}\left(x - \frac{\pi}{4}\right)^3 + \frac{1}{4!\sqrt{2}}\left(x - \frac{\pi}{4}\right)^4.$$

(b) Use T_4 to obtain an approximation for $\cos\frac{\pi}{6}$. Give your approximation as an expression involving only numbers, not unevaluated trig functions; but you do not need to put it in simplified form.

(2 points) We take $T_4\left(\frac{\pi}{6}\right)$ as our approximation to $\cos\frac{\pi}{6}$. Namely,

$$\cos\frac{\pi}{6} \approx T_4\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(\frac{\pi}{6} - \frac{\pi}{4}\right)^1 - \frac{1}{2!\sqrt{2}}\left(\frac{\pi}{6} - \frac{\pi}{4}\right)^2 + \frac{1}{3!\sqrt{2}}\left(\frac{\pi}{6} - \frac{\pi}{4}\right)^3 + \frac{1}{4!\sqrt{2}}\left(\frac{\pi}{6} - \frac{\pi}{4}\right)^4$$

(c) Determine the accuracy of your approximation from part (b), explaining all your reasoning, and giving your final conclusion in sentence form. (Again, use only numbers, not unevaluated trig functions, but you don't need a simplified expression.)

(5 points) Taylor's Inequality, on the interval $\left[\frac{\pi}{6}, \frac{\pi}{3}\right] = \left[\frac{\pi}{4} - \frac{\pi}{12}, \frac{\pi}{4} + \frac{\pi}{12}\right]$ centered at $\frac{\pi}{4}$, tells us that

$$|f(x) - T_4(x)| \leq \frac{M}{5!}\left|x - \frac{\pi}{4}\right|^5 \quad \text{for any } x \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$$

provided that the number M is chosen to be *greater than or equal to the maximum value of* $|f^{(5)}(x)| = |\sin x|$ *on the interval* $[\pi/6, \pi/3]$. Because $|\sin x|$ is never greater than 1, we can certainly choose $M = 1$. Therefore,

$$\left|\cos\frac{\pi}{6} - T_4\left(\frac{\pi}{6}\right)\right| \leq \frac{1}{5!}\left|\frac{\pi}{6} - \frac{\pi}{4}\right|^5.$$

In words, this means that our approximation of $\cos\frac{\pi}{6}$ using $T_4\left(\frac{\pi}{6}\right)$ as in part (*b*) is accurate to within $\frac{1}{5!}\left|\frac{\pi}{6} - \frac{\pi}{4}\right|^5$.