Solutions to Math 42 First Exam — January 31, 2013

1. (12 points) Evaluate each of the following, or explain why its value does not exist; show all reasoning.

(a) $\displaystyle\int_1^e (\ln x)^2 \, dx$

(6 points) Let $u = \ln x$. We have $du = \frac{1}{x}dx$ and $x = e^u$, which implies $dx = e^u du$.Then

$$\begin{aligned}
\int_1^e (\ln x)^2 dx &= \int_0^1 u^2 e^u du \\
&= u^2 e^u \Big|_0^1 - \int_0^1 e^u (2u) du \qquad (v = u^2, w = e^u) \\
&= e - \left(2ue^u \Big|_0^1 - \int_0^1 (e^u) 2du \right) \qquad (v = 2u, w = e^u) \\
&= e - \left(2e - 2e^u \Big|_0^1 \right) \\
&= e - 2.
\end{aligned}$$

(b) $\displaystyle\int_{-1}^1 \frac{e^x}{e^x - 1} \, dx$

(6 points) Notice that the domain of the integrand is $e^x \neq 1$ and $-1 \leq x \leq 1$. Therefore, we have $-1 \leq x < 0$ or $0 < x \leq 1$. This implies that the integral is improper, and we have

$$\begin{aligned}
\int_{-1}^1 \frac{e^x}{e^x - 1} dx &= \int_{-1}^0 \frac{e^x}{e^x - 1} dx + \int_0^1 \frac{e^x}{e^x - 1} dx \\
&= \lim_{a \to 0^-} \int_{-1}^a \frac{e^x}{e^x - 1} dx + \lim_{b \to 0^+} \int_b^1 \frac{e^x}{e^x - 1} dx \\
&= \lim_{a \to 0^-} \int_{e^{-1}-1}^a \frac{du}{u} + \lim_{b \to 0^+} \int_b^{e-1} \frac{du}{u} \qquad (u = e^x - 1, \ du = e^x \, dx) \\
&= \lim_{a \to 0^-} (\ln |a| - \ln |e^{-1} - 1|) + \lim_{b \to 0^+} (\ln(e - 1) - \ln b)
\end{aligned}$$

Since $\displaystyle\lim_{a \to 0^-} \ln |a| = -\infty$ does not exist, we know the improper integral is divergent.

2. (13 points) Compute each of the following integrals, showing all of your reasoning.

(a) $\int \sec^3 x \tan x \, dx$

(6 points) Using the substitution $u = \sec x$, $du = \sec x \tan x \, dx$, we have

$$\begin{aligned}\int \sec^3 x \tan x \, dx &= \int u^2 \, du \\ &= \frac{1}{3}u^3 + C \\ &= \frac{1}{3}\sec^3 x + C\end{aligned}$$

(b) $\int \frac{dt}{t^2\sqrt{9-t^2}}$

(7 points) Using the substitution $t = 3\sin\theta$, $dt = 3\cos\theta \, d\theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, we have

$$\begin{aligned}\int \frac{dt}{t^2\sqrt{9-t^2}} &= \int \frac{3\cos\theta \, d\theta}{9\sin^2\theta\sqrt{9-9\sin^2\theta}} \\ &= \int \frac{\cos\theta}{3\sin^2\theta\sqrt{9\cos^2\theta}} \, d\theta \\ &= \int \frac{\cos\theta}{(3\sin^2\theta)(3\cos\theta)} \, d\theta \\ &= \frac{1}{9}\int \frac{d\theta}{\sin^2\theta} \\ &= \frac{1}{9}\int \csc^2\theta \, d\theta \\ &= -\frac{1}{9}\cot\theta + C \\ &= -\frac{1}{9}\frac{\cos\theta}{\sin\theta} + C \\ &= -\frac{1}{9}\frac{3\cos\theta}{3\sin\theta} + C \\ &= -\frac{1}{9}\frac{\sqrt{9-9\sin^2\theta}}{3\sin\theta} + C \\ &= -\frac{1}{9}\frac{\sqrt{9-t^2}}{t} + C \\ &= -\frac{\sqrt{9-t^2}}{9t} + C\end{aligned}$$

3. (8 points) Show all your steps in computing $\int \frac{x^6+16}{x^6-16x^2}\,dx$

The degree of the numerator is equal to the degree of the denominator, so we perform long division:

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad\quad\quad 1 \\ x^6 \quad -16x^2 & \; x^6 \quad\quad\quad\quad +16 \\ & -\;(x^6 \quad -16x^2) \\ \hline & \quad\quad\quad 16x^2 \quad +16 \end{array}$$

So

$$\int \frac{x^6+16}{x^6-16x^2}\,dx = \int 1 + \frac{16x^2+16}{x^6-16x^2}\,dx$$

We'll now decompose the remainder term. Using difference-of-squares, the denominator factors as:

$$\begin{aligned} x^6-16x^2 = x^2(x^4-16) &= x^2(x^2-4)(x^2+4) && \text{(factoring difference of squares)} \\ &= x^2(x+2)(x-2)(x^2+4) && \text{(difference of squares again)} \end{aligned}$$

Since x^2+4 has no real roots, it's irreducible. Now we can set up the partial fraction decomposition:

$$\frac{16x^2+16}{x^6-16x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2} + \frac{D}{x-2} + \frac{Ex+F}{x^2+4}$$

Clearing fractions by multiplying by the denominator of the left-hand side, we obtain:

$$16x^2+16 = \begin{cases} Ax(x+2)(x-2)(x^2+4) + B(x+2)(x-2)(x^2+4) + Cx^2(x-2)(x^2+4) \\ +Dx^2(x+2)(x^2+4) + Ex^3(x+2)(x-2) + Fx^2(x+2)(x-2) \end{cases} \tag{1}$$

$$= \begin{cases} Ax(x^2-4)(x^2+4) + B(x^2-4)(x^2+4) + Cx^2(x-2)(x^2+4) \\ +Dx^2(x+2)(x^2+4) + Ex^3(x^2-4) + Fx^2(x^2-4) \end{cases} \tag{2}$$

We want to solve for $A,\ldots,F$. The roots of the denominator are 0, 2, and -2. If we plug these numbers in for x, we'll get simpler relations. For example, plugging $x=0$ into both sides of equation (1), it becomes $16=-16B$, so $\boxed{B=-1}$. Similarly, plugging in $x=2$ yields $80=128D$, so $\boxed{D=\frac{5}{8}}$. Finally, plugging in $x=-2$ yields $80=-128C$, so $\boxed{C=-\frac{5}{8}}$.

It remains to find A, E, and F. We must build equations for these by equating coefficients on each side of equation (1). But we only need three relations, so rather than fully multiply out the right side of equation (1), we can just start to build relations one by one — starting with the coefficients of the highest powers x^5, x^4, and x^3, which on the left side happen to be all zero. Using either (1) or (2) and the values for B, C, D, we obtain:

$$\begin{aligned} 0 &= A+C+D+E = A+E && \text{(coefficients of } x^5) \\ 0 &= B-2C+2D+F = \tfrac{3}{2}+F && \text{(coefficients of } x^4) \\ 0 &= 4C+4D-4E = -4E && \text{(coefficients of } x^3) \end{aligned}$$

We can stop here, because it now follows that $\boxed{E=0,\ A=0,\ F=-\frac{3}{2}}$. Combining the results of the long division and partial fraction decomposition, we have

$$\begin{aligned} \int \frac{x^6+16}{x^6-16x^2}\,dx &= \int 1 - \frac{1}{x^2} - \frac{5}{8}\frac{1}{x+2} + \frac{5}{8}\frac{1}{x-2} - \frac{3}{2}\frac{1}{x^2+4}\,dx \\ &= x + \frac{1}{x} - \frac{5}{8}\ln|x+2| + \frac{5}{8}\ln|x-2| - \frac{3}{2}\cdot\frac{1}{2}\arctan\frac{x}{2} + C \\ &= x + \frac{1}{x} + \frac{5}{8}\ln\left|\frac{x-2}{x+2}\right| - \frac{3}{4}\arctan\frac{x}{2} + C \end{aligned}$$

(You can integrate $\frac{1}{x^2+4}$ using either the substitution $x=2\tan\theta$, or alternatively $u=x/2$.)

4. (5 points) *Set up* the complete partial fraction decomposition of the following rational expression, in terms of undetermined constants A, B, etc., as appropriate. (*Do not* attempt to clear fractions or determine the values of any of the constants!)

$$\frac{x^8+x^4+1}{(x+2)^2(x-3)(x^2+1)^2(x^2+4x+5)} =$$

First note that the numerator has degree eight while the denominator has degree nine, so no long division is needed. Since the denominator factors as a product of a repeated linear factor, a linear factor, a repeated irreducible polynomial, and the irreducible polynomial x^2+4x+5 (irreducibility can be determined by computing its discriminant, which is negative), the partial fraction decomposition has the form

$$\frac{x^8+x^4+1}{(x+2)^2(x-3)(x^2+1)^2(x^2+4x+5)} = \frac{A}{x+2}+\frac{B}{(x+2)^2}+\frac{C}{x-3}+\frac{Dx+E}{x^2+1}+\frac{Fx+G}{(x^2+1)^2}+\frac{Hx+I}{x^2+4x+5}.$$

(The *discriminant* of the quadratic polynomial ax^2+bx+c is defined to be b^2-4ac.)

5. (12 points)

(a) Evaluate $\int_e^{\infty} \frac{1}{x\,(\ln x)^2}\,dx$ or explain why its value does not exist; show all reasoning.

(6 points) Let $u = \ln x$, we have $du = \frac{1}{x}dx$. Then we have

$$\begin{aligned}\int_e^{\infty} \frac{1}{x(\ln x)^2}dx &= \int_1^{\infty} \frac{1}{u^2}du \\ &= \lim_{t\to\infty} \int_1^t \frac{du}{u^2} \\ &= \lim_{t\to\infty} (-\frac{1}{u})|_1^t \\ &= \lim_{t\to\infty} (-\frac{1}{t} + 1) \\ &= 1.\end{aligned}$$

(b) Determine whether $\int_0^{\pi} \frac{\cos^2 x + 1}{x^{2/3}}\,dx$ converges or diverges; give complete reasoning.

(6 points) Note that

$$0 \le \frac{\cos^2 x + 1}{x^{\frac{2}{3}}} \le \frac{2}{x^{\frac{2}{3}}}$$

and

$$\begin{aligned}\int_0^{\pi} \frac{2}{x^{\frac{2}{3}}}dx &= \lim_{t\to 0^+} \int_t^{\pi} \frac{2}{x^{\frac{2}{3}}}dx \\ &= \lim_{t\to 0^+} 6x^{\frac{1}{3}}|_t^{\pi} \\ &= \lim_{t\to 0^+} (6\pi^{\frac{1}{3}} - 6t^{\frac{1}{3}}) \\ &= 6\pi^{\frac{1}{3}},\end{aligned}$$

which is convergent. Therefore by the comparison theorem we have $\int_0^{\pi} \frac{\cos^2 x + 1}{x^{\frac{2}{3}}}dx$ converges.

6. (9 points) Put the following quantities in increasing order (from smallest number to largest). You do not need to justify your answer.

(A) $\int_2^6 e^{-t}\,dt$

(B) $e^{-2} + e^{-3} + e^{-4} + e^{-5}$

(C) $e^{-3} + e^{-4} + e^{-5} + e^{-6}$

(D) $e^{-2.5} + e^{-3.5} + e^{-4.5} + e^{-5.5}$

(E) The number 0

(F) $\frac{1}{2}\left(e^{-2} + 2e^{-3} + 2e^{-4} + 2e^{-5} + e^{-6}\right)$

(G) $\frac{1}{2}\left(e^{-2.25} + e^{-2.75} + e^{-3.25} + e^{-3.75} + e^{-4.25} + e^{-4.75} + e^{-5.25} + e^{-5.75}\right)$

(H) $\frac{1}{4}\left(e^{-2} + 2e^{-2.5} + 2e^{-3} + 2e^{-3.5} + 2e^{-4} + 2e^{-4.5} + 2e^{-5} + 2e^{-5.5} + e^{-6}\right)$

(I) $e^{-6} - e^{-2}$

- First note that $e^{-6} - e^{-2} < 0$, while all other nonzero quantities are positive. We thus know that our list should start with "$I < E$".
- Next, notice that all of the given sums have the form of a left endpoint (part B), right endpoint (part C), midpoint (parts D, G), or trapezoidal (parts F, H) approximation of the integral A. The number subdivisions is $n = 4$ in parts B, C, D, and F and $n = 8$ in parts G and H.
- Since the integrand e^{-x} is decreasing, it's easy to see that the right endpoint approximation C underestimates the integral while the left endpoint approximation B overestimates.
- Since the integrand e^{-x} is concave up, the trapezoidal approximation F overestimates; but note that it will fall between B and C (and therefore not be as great an overestimate as B): this is because for a fixed number of subintervals n, the trapezoidal approximation T_n is the *average* of the left and right endpoint approximations L_n and R_n.
- H is also a trapezoidal approximation, but with twice as many subintervals, so we expect it to be a better approximation of the integral A.
- This all suggests $I < E < C < A < H < F < B$.
- The only choices left we need to account for are D and G. These are both midpoint approximations, and we know that midpoint approximations underestimate integrals of functions that are concave up. G is a finer estimate than D, and C is clearly less than D, so putting this all together gives us

$$\boxed{I < E < C < D < G < A < H < F < B}$$

7. (12 points) Let $f(x) = \frac{1}{x} - \sin x$. In this problem, we study approximations of the following integral:

$$\int_1^5 \left(\frac{1}{x} - \sin x\right)\, dx \qquad \text{(Note: there is no need to evaluate integral)}$$

(a) Write an expression involving only numbers (and the sine function) that approximates the above integral using Simpson's Rule with 4 subintervals. You do *not* have to simplify this expression.

(3 points)

$$S_4 = \frac{1}{3}\left((1 - \sin 1) + 4\left(\frac{1}{2} - \sin 2\right) + 2\left(\frac{1}{3} - \sin 3\right) + 4\left(\frac{1}{4} - \sin 4\right) + \left(\frac{1}{5} - \sin 5\right)\right)$$

(b) How accurately does your expression of part (a) approximate the above integral? *State your final answer, involving rational numbers, in a complete sentence.* Show all reasoning.

(5 points) Taking four derivatives in a row, we find $f^{(4)}(x) = \dfrac{24}{x^5} - \sin x$, so

$$|f^{(4)}(x)| = \left|\frac{24}{x^5} - \sin x\right| \leq \frac{24}{x^5} + |\sin x| \leq 24 + 1 = 25$$

for all $1 \leq x \leq 5$, since $\frac{24}{x^5}$ is decreasing and $|\sin x|$ is bounded by 1. Thus, we may choose $K_4 = 25$. We get

$$|E_S| \leq \frac{25 \cdot 4^5}{180 \cdot 4^4} = \frac{100}{180} = \frac{5}{9}$$

We conclude that the approximation S_4 is correct to within $\frac{5}{9}$ of the actual value.

(c) Find a value of n which guarantees that a Simpson's Rule approximation of the above integral using n subintervals is accurate to within 10^{-8}. Your final answer should give a valid n in simplified form, and be fully justified, but it need not be optimal in any sense.

(4 points) We need to find a positive even integer n such that

$$|E_S| \leq \frac{25 \cdot 4^5}{180 n^4} \leq \frac{1}{10^8}$$

This is equivalent to

$$n^4 \geq \frac{25 \cdot 4^5 \cdot 10^8}{180} = 4^4 \cdot 10^8 \cdot \frac{100}{180} \qquad \Longrightarrow \qquad n \geq 4 \cdot 10^2 \cdot \left(\sqrt[4]{\frac{100}{180}}\right)$$

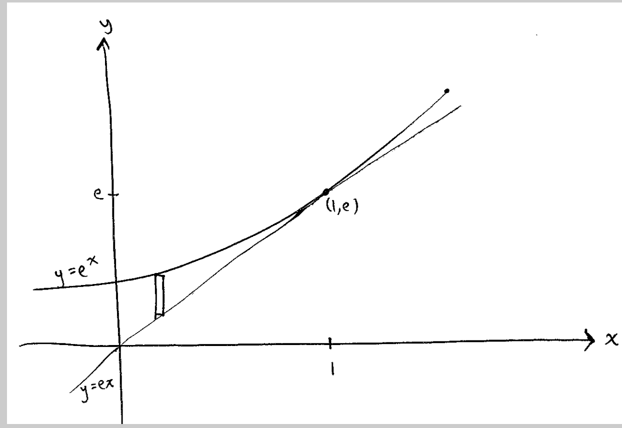
Thus, we may choose $n = 400$.

8. (13 points) Consider the region A in the xy-plane bounded by the curves $x = 0$, $y = e^x$, and $y = ex$.

(a) Set up, but do not evaluate, an integral in terms of a single variable that represents the area of A. Justify your answer by sketching a picture and labeling a sample slice.

(4 points) The line $y = ex$ lies below the graph $y = e^x$ and is tangent at the point $(1, e)$. The area of the region, A, bounded by these two graphs and the line $x = 0$ is therefore

$$\int_0^1 (e^x - ex)dx.$$

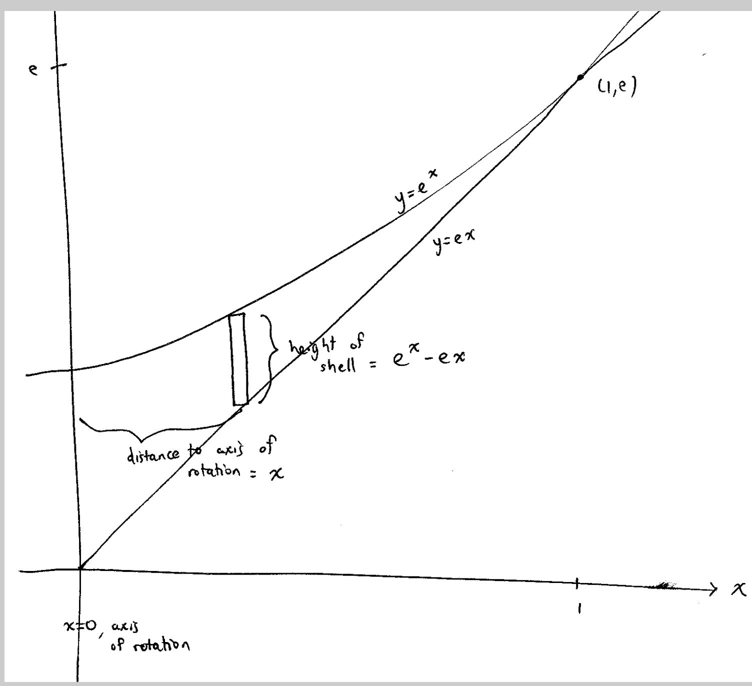


(b) Set up, but do not evaluate, an integral that represents the volume of the solid obtained by rotating A about the y-axis. Justify your answer by citing the method used, sketching a picture and labeling a sample slice.

(4 points) By the method of cylindrical shells, the volume of the solid obtained by rotation the given region about the y-axis is

$$2\pi \int_0^1 (\text{distance to axis of rotation}) \times (\text{height of shell}) \times (\text{width of shell})$$

$$= 2\pi \int_0^1 x(e^x - ex)dx.$$

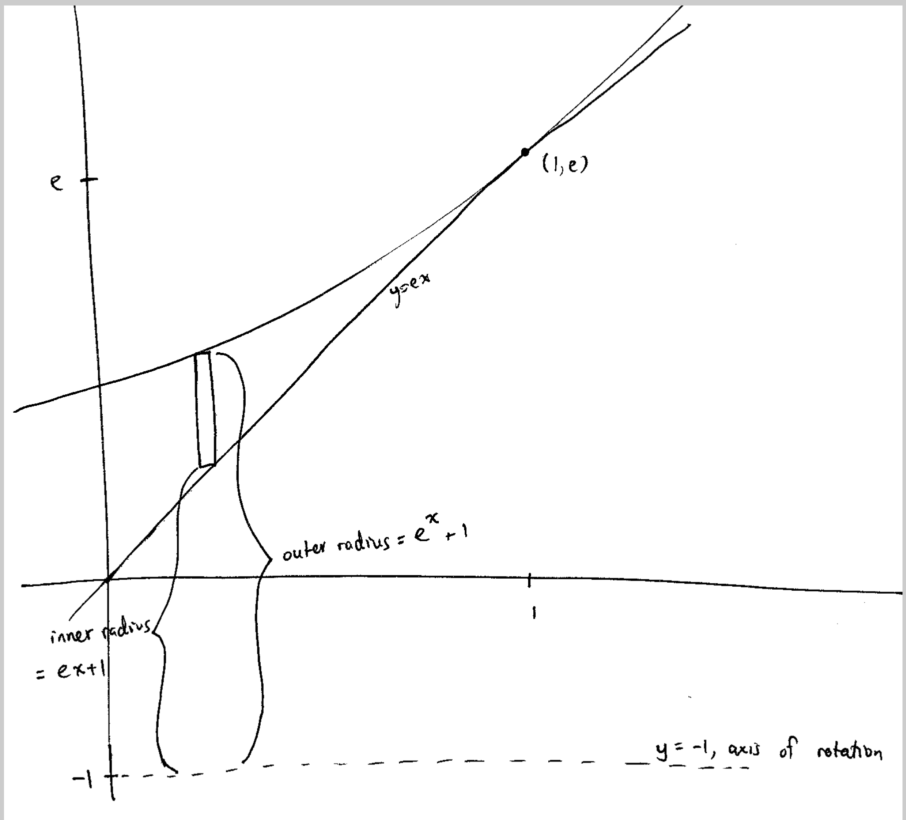


Quick reference: as before, A is the region bounded by the curves $x = 0$, $y = e^x$, and $y = ex$.

(c) Set up, but do not evaluate, an integral that represents the volume of the solid obtained by rotating A about the line $y = -1$. Justify your answer by citing the method used, sketching a picture and labeling a sample slice.

(5 points) By the washer method, the volume of the solid obtained by rotating A about the line $y = -1$ is

$$\begin{aligned} &\pi \int_0^1 [(\text{inner radius})^2 - (\text{outer radius})^2] \times (\text{width of washer}) \\ = \ &\pi \int_0^1 [(e^x + 1)^2 - (ex + 1)^2] dx. \end{aligned}$$

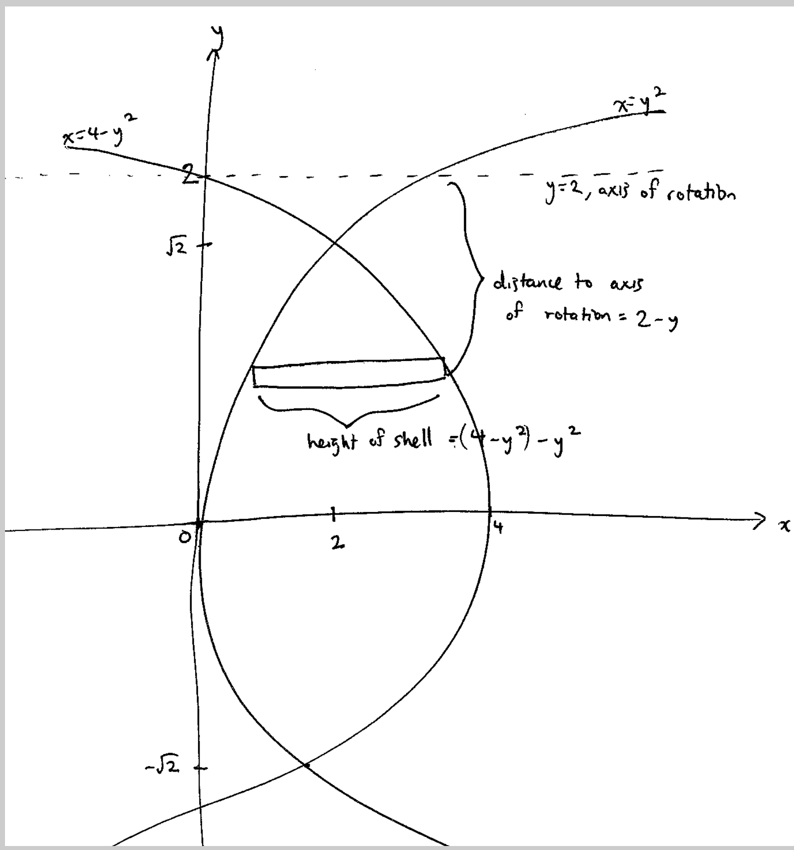


9. (10 points) Consider the region R in the xy-plane bounded by the curves $x = y^2$ and $x = 4 - y^2$.

(a) Set up, but do not evaluate, an integral in terms of a single variable that represents the volume of the solid obtained by rotating R about the line $y = 2$. Justify your answer by drawing a picture of R and labeling a sample slice.

(5 points) The two parabolas $x = y^2$ and $x = 4 - y^2$ intersect at $(2, \sqrt{2})$ and $(2, -\sqrt{2})$. Therefore, by the method of cyclindrical shells, the volume obtained by rotating the region R enclosed by the two parabolas about the line $y = 2$ is

$$\begin{aligned} & 2\pi \int_{-\sqrt{2}}^{\sqrt{2}} (\text{distance to axis of rotation}) \times (\text{height of shell}) \times (\text{width of shell}) \\ = \; & 2\pi \int_{0}^{1} (2 - y)[(4 - y^2) - y^2] dy. \end{aligned}$$



(b) Suppose a three-dimensional solid V has the following properties: it has R as its base; and cross-sections of V perpendicular to the y-axis are squares. Set up, but do not evaluate, an integral that gives the volume of V.

(5 points) The cross section at $y = y_0$ is a square of sidelength $(4 - y_0^2) - y_0^2$. Therefore, by the cross-sectional area method, the volume of the region with R as base and cross sections squares perpendicular to the y-axis is

$$\int_{-\sqrt{2}}^{\sqrt{2}} (\text{cross sectional area})\, dy = \int_{-\sqrt{2}}^{\sqrt{2}} ((4 - y^2) - y^2)^2 \, dy$$