Solutions to Math 42 Final Exam — March 19, 2012

1. (12 points) Consider the initial value problem

$$\begin{cases} \dfrac{dy}{dx} = -\ln y \\ y(0) = 2 \end{cases}$$

(a) Use Euler's method with step size 0.1 to find an approximation to $y(0.3)$. Show your steps, but you do not need to simplify your answer.

(4 points) We begin with the initial values $x_0 = 0,\ y_0 = 2$ and compute three steps:

n	x_n	y_n
0	0	2
1	0.1	$2 - 0.1(\ln 2)$
2	0.2	$[2 - 0.1(\ln 2)] - 0.1\ln[2 - 0.1(\ln 2)]$
3	0.3	$[2 - 0.1(\ln 2)] - 0.1\ln[2 - 0.1(\ln 2)]$ $- 0.1\ln\left(\ [2 - 0.1(\ln 2)] - 0.1\ln[2 - 0.1(\ln 2)]\ \right)$

Thus, Euler's method finds that $y(0.3)$ is approximately

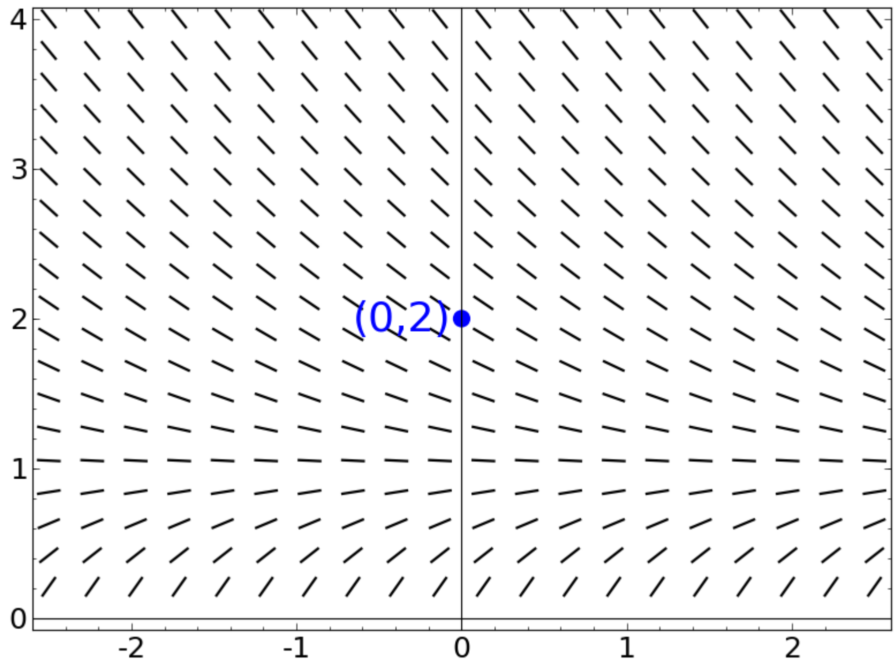
$$\begin{aligned} y_3 = {} & [2 - 0.1(\ln 2)] - 0.1\ln[2 - 0.1(\ln 2)] \\ & - 0.1\ln\left(\ [2 - 0.1(\ln 2)] - 0.1\ln[2 - 0.1(\ln 2)]\ \right) \end{aligned}$$

(b) Find the equilibrium solutions of the differential equation $\frac{dy}{dx} = -\ln y$.

(2 points) An equilibrium solution has $y(x) = C$ for some constant C, so that $dy/dx = 0$. Substituting this into the differential equation, we find $0 = -\ln C$, so $C = 1$. Thus, the only equilibrium solution is the function $y(x) = 1$.

(c) The following picture gives a direction field for the differential equation $\frac{dy}{dx} = -\ln y$. Draw on the picture:

- the equilibrium solutions for $\frac{dy}{dx} = -\ln y$, and
- an approximation to the graph of the solution curve having $y(0) = 2$.



(3 points) Picture to come

(d) What is the behavior of the solution to the above initial value problem as $x \to +\infty$ and as $x \to -\infty$? Briefly explain your reasoning.

(3 points) As $x \to \infty$, we have that $y(x)$ approaches 1; the function $y(x)$ is decreasing and concave up whenever $y > 1$, and y will approach the equilibrium solution.

As $x \to -\infty$, $y(x)$ approaches $+\infty$. Since $\dfrac{dy}{dx}$ is negative and $\dfrac{d^2y}{dx^2} = -\dfrac{1}{y}\dfrac{dy}{dx}$ is positive for $y > 1$, $y(x)$ will increase without bound as x moves left from the point $(0, 2)$.

2. (10 points) A large cylindrical tank is positioned upright, with its circular base parallel to the ground. The tank is filled with water, which is being drained through a hole at the base of the tank. Torricelli's law states that, in this situation, the volume of water in the tank decreases at a rate proportional to the square root of the height of the water in the tank.

   Initially, at time $t = 0$ seconds, the water occupies a height of 10 meters in the tank. Moreover, it is experimentally determined that at this time, the height is decreasing at a rate of $-\frac{1}{100}$ m/s.

   (a) *Set up* an initial value problem satisfied by $h(t)$, the height of the water in the tank after t seconds. (A complete answer should include a differential equation that does not depend on unknown constants; but finding a solution is not necessary for part (a).)

   (5 points) Volume of water in a **cylindrical** tank is proportional to its height, hence the rate of change of volume is proportional to the rate of change of height. Accordingly, by Torricelli's Law given in the question, $h(t)$ satisfies the following differential equation

$$h'(t) = -k\sqrt{h(t)}$$

   for some constant $k > 0$. Next we find out the constant k: using the initial condition at $t = 0$, we have $h(0) = 10$ and $h'(0) = -\frac{1}{100}$. Substituting these into the differential equation, we have

$$h'(0) = -k\sqrt{h(0)} \quad \Rightarrow \quad -\frac{1}{100} = -k\sqrt{10} \quad \Rightarrow \quad k = \frac{1}{100\sqrt{10}}.$$

   Hence, $h(t)$ satisfies the following initial value problem:

$$\frac{dh}{dt} = -\frac{1}{100\sqrt{10}}\sqrt{h}, \quad h(0) = 10.$$

   (b) How long does it take for the tank to become empty? Show all reasoning.

   (5 points) We solve the initial value problem in (a) by separation of variables:

$$\begin{aligned} \frac{1}{\sqrt{h}}dh &= -\frac{1}{100\sqrt{10}}dt \\ \int \frac{1}{\sqrt{h}}dh &= -\int \frac{1}{100\sqrt{10}}dt \\ 2\sqrt{h} &= -\frac{1}{100\sqrt{10}}t + C. \end{aligned}$$

   We solve C using the initial condition: put $t = 0$, we have $h = 10$ and so

$$2\sqrt{10} = 0 + C \quad \Rightarrow \quad C = 2\sqrt{10}.$$

   One can then find out the solution $h(t)$:

$$\begin{aligned} 2\sqrt{h(t)} &= -\frac{1}{100\sqrt{10}}t + 2\sqrt{10} \\ \sqrt{h(t)} &= \sqrt{10} - \frac{t}{200\sqrt{10}} \\ h(t) &= \left(\sqrt{10} - \frac{t}{200\sqrt{10}}\right)^2. \end{aligned}$$

   At the time t that the tank becomes empty, we $h(t) = 0$, i.e.

$$\sqrt{10} - \frac{t}{200\sqrt{10}} = 0 \quad \Rightarrow \quad t = 200 \cdot \sqrt{10} \cdot \sqrt{10} = 2000.$$

   Therefore, the tank becomes empty after 2000 seconds.

3. (12 points)

(a) Show all steps in solving the initial value problem

$$\begin{cases} \dfrac{dx}{dt} = x + xt - \sqrt{x} - t\sqrt{x} \\ x(0) = \dfrac{1}{4} \end{cases}$$

(6 points) Using separation of variables:

$$\begin{aligned} \frac{dx}{dt} &= x + xt - \sqrt{x} - t\sqrt{x} \\ &= (x - \sqrt{x})(1+t) \\ \frac{dx}{x - \sqrt{x}} &= (1+t)\,dt \\ \int \frac{dx}{x-\sqrt{x}} &= \int 1 + t\,dt = t + \frac{t^2}{2} + C \end{aligned}$$

Use the substitution $u = \sqrt{x}$ on the left-hand-side. So $x = u^2$ and $dx = 2u\,du$:

$$\begin{aligned} \int \frac{dx}{x - \sqrt{x}} &= \int \frac{2u\,du}{u^2 - u} \\ &= \int \frac{2\,du}{u-1} = 2\ln|u-1| + D = 2\ln|\sqrt{x} - 1| + D \end{aligned}$$

So we have

$$\begin{aligned} 2\ln|\sqrt{x}-1| + D &= t + \frac{t^2}{2} + C \\ \ln|\sqrt{x} - 1| &= \frac{t}{2} + \frac{t^2}{4} + \frac{C-D}{2} \end{aligned}$$

Next we exponentiate; let's also combine our constants of integration into one constant E:

$$\begin{aligned} |\sqrt{x} - 1| &= e^{\frac{t}{2}+\frac{t^2}{4}+E} \\ \sqrt{x} - 1 &= \pm e^{\frac{t}{2}+\frac{t^2}{4}} e^E \end{aligned}$$

For simplicity, let $F = \pm e^E$.

$$\sqrt{x} - 1 = Fe^{\frac{t}{2}+\frac{t^2}{4}} \iff \sqrt{x} = 1 + Fe^{\frac{t}{2}+\frac{t^2}{4}}$$

Since we've gotten rid of the absolute value signs, now is a good time to use the initial condition to determine the constant F. We set $t = 0$ and $x = \frac{1}{4}$:

$$\sqrt{\frac{1}{4}} = 1 + Fe^0 \iff \frac{1}{2} = 1 + F \iff F = -\frac{1}{2}$$

So

$$\begin{aligned} \sqrt{x} &= 1 - \frac{1}{2} e^{\frac{t}{2}+\frac{t^2}{4}} \\ x &= \left(1 - \frac{1}{2} e^{\frac{t}{2}+\frac{t^2}{4}}\right)^2 \end{aligned}$$

And that is our solution.

(If we had waited until this point to solve for F, we would have obtained two solutions, $F = \frac{1}{2}$ and $F = \frac{3}{2}$. But only the former satisfies the differential equation.)

(b) Show all steps in solving the initial value problem

$$\begin{cases} \dfrac{dx}{dt} = x \ln\left(x^{(3t^2)}\right) \\ x(0) = e \end{cases}$$

(6 points) We use a property of logarithms to separate variables:

$$\begin{aligned} \frac{dx}{dt} &= x \ln\left(x^{3t^2}\right) \\ &= x(3t^2)\ln x \\ \frac{dx}{x \ln x} &= 3t^2\,dt \\ \int \frac{dx}{x \ln x} &= \int 3t^2\,dt \\ &= t^3 + C \end{aligned}$$

To compute the integral on the left-hand-side, use the substitution $u = \ln x$. So $x = e^u$ and $dx = e^u\,du$.

$$\begin{aligned} \int \frac{dx}{x \ln x} &= \int \frac{e^u\,du}{e^u u} \\ &= \int \frac{du}{u} \\ &= \ln|u| + D \\ &= \ln|\ln x| + D \end{aligned}$$

So, combining our constants of integration into a single constant E, we have

$$\begin{aligned} \ln|\ln x| &= t^3 + E \\ |\ln x| &= e^{t^3} e^E \\ \ln x &= \pm e^E e^{t^3} \end{aligned}$$

Let $F = \pm e^E$:

$$\ln x = Fe^{t^3}$$

Now that the absolute value signs are gone, it's a good time to use the initial condition to solve for F. We set $t = 0$, $x = e$:

$$\begin{aligned} \ln e &= Fe^0 \\ 1 &= F \end{aligned}$$

So we have

$$\begin{aligned} \ln x &= 1e^{t^3} \\ &= e^{t^3} \\ x &= e^{e^{t^3}} \end{aligned}$$

4. (14 points) In River Logistyx, there exist many species of fish. Among them are well-known species W and X.

   (a) Species W, measured in thousands of individuals, approximately follows a logistic model with constant of proportionality $k = 2$, and carrying capacity 0.5. The initial population of W is one thousand individuals, i.e. $W(0) = 1$.

   Set up the corresponding initial value problem for W, and give its solution.

   (4 points) Since the carrying capacity is $M = 1/2$ and $k = 2$, the initial value problem is

$$\frac{dW}{dt} = 2W(1 - 2W)$$
$$W(0) = 1$$

   We know that $W(t) = 0$ is not a solution because it doesn't satisfy the initial condition. Therefore the solution of this logistic equation has the form

$$W(t) = \frac{1/2}{1 + Ae^{-2t}}$$

   for some constant A. Using the initial condition, we see that

$$1 = \frac{1}{2 + 2A}$$

   which implies that $A = -1/2$. Therefore the solution to the initial value problem is

$$W = \frac{1/2}{1 - (1/2)e^{-2t}}.$$

   (b) What happens to the population of W in the long term? Explain.

   (2 points) Since $\lim_{t\to\infty} e^{-2t} = 0$,

$$\lim_{t\to\infty} W(t) = \lim_{t\to\infty} \frac{1/2}{1 - (1/2)e^{-2t}} = \frac{1/2}{1 - 0} = \frac{1}{2}.$$

   This means that in the long term, as time goes to infinity, the population of W goes to 500 fish.

(c) Fish of the species X are considered a very tasty treat by people in Lotka City, and are heavily fished at a rate of one thousand individuals each year. Under these conditions, the population of X, measured in thousands of individuals, approximately satisfies the initial value problem

$$\begin{cases} X' = 2X\left(1 - \dfrac{X}{2}\right) - 1 \\ X(0) = 2 \end{cases}$$

Solve this initial value problem for X, showing all steps.

(4 points)

$$\frac{dX}{dt} = 2X\left(1 - \frac{X}{2}\right) - 1 = 2X - X^2 - 1 = -(X-1)^2$$

Assuming that $X \neq 1$, we can use separation of variables to get

$$\int -(X-1)^{-2}\, dX = \int dt$$
$$(X-1)^{-1} = t + C$$

As long as $t + C \neq 0$, we can solve for X:

$$X - 1 = \frac{1}{t+C}$$
$$X = \frac{1}{t+C} + 1$$

Now we use the initial condition $X(0) = 2$ to find C:

$$2 = X(0) = \frac{1}{C} + 1 \implies C = 1$$

Therefore the solution to our initial value problem is

$$X(t) = \frac{1}{t+1} + 1.$$

(d) Use your answer to (c) to explain what happens to the population of X in the long term.

(2 points)

$$\lim_{t\to\infty} X(t) = \lim_{t\to\infty} \frac{1}{t+1} + 1 = 1$$

So in the long term, the population of X goes to one thousand individuals.

(e) Are there other initial conditions $X(0)$ for which the species eventually goes extinct? Explain completely.

(2 points) Yes. If, for example, $X(0) = 1/2$, we can again use separation of variables to see that the solution to the initial value problem is

$$X = \frac{1}{t+C} + 1$$

for some constant C, and so

$$\frac{1}{2} = \frac{1}{C} + 1 \implies c = -2,$$

hence the solution is

$$X = \frac{1}{t-2} + 1$$

We can see that for this solution, $X(1) = 0$; in fact, as $t \to 2$, $X(t)$ goes to $-\infty$. This tells us that the solution becomes zero at some point and stays negative. It doesn't make physical sense to have a negative population of fish, so what the model is really telling us is that the population of fish becomes extinct at $t = 1$. This makes intuitive sense – we start with fewer than 1000 fish, and the Lotkan people are eating 1000 fish a year, so it makes sense that the fish population will die out.

Note: it's not enough to say that for this initial condition X is always decreasing – a function can be always decreasing, but still not go to zero!

5. (15 points) An environmental act has been passed in Volterra County declaring species X from River Logistyx an endangered species. Consequently, all fishing of this species has been halted. In the meantime, an invasive species Y has been introduced into the river from a nearby industrial aquaculture farm. (*Note:* Despite our storyline, this problem does not depend in any way on Problem 4.)

The populations of species X and Y, measured in thousands, follow closely the following system of differential equations:

$$\begin{aligned} X' &= 2X\left(1 - \frac{X}{2}\right) - 2XY \\ Y' &= Y - XY \end{aligned}$$

(a) Describe the nature of the relationship between the species X and Y: is it one of competition, cooperation, or predation? If the relationship is one of predation, indicate which species is the predator and which is the prey. Explain every part of your answer.

(3 points) Species X and Y are in competition. Since the product XY is approximately proportional to the number of interactions between the two species in a fixed amount of time, the terms "$-2XY$" and "$-XY$" in the expressions for X' and Y' respectively indicate that the growth rate of each species decreases by amounts proportional to the number of interactions between X and Y. Since both species suffer diminished growth with more of the other present, this indicates a competitive relationship.

(b) Find all equilibrium solutions of the above system. (You may omit negative X, Y here.)

(4 points) With X and Y both constant, $X' = Y' = 0$. Factoring the expression for Y', we find this implies

$$Y' = Y(1 - X) = 0,$$

so we have that either $Y = 0$ or $X = 1$. In the case $X = 1$, the equation $X' = 0$ becomes

$$2\left(1 - \frac{1}{2}\right) - 2Y = 0,$$

which implies that $Y = 1/2$. Alternatively in the case $Y = 0$, the equation $X' = 0$ becomes

$$2X\left(1 - \frac{X}{2}\right) = 0,$$

so $X = 0$ or $X = 2$. It follows that the only equilibrium solutions are:

$$\boxed{(X,Y) = \left(1, \frac{1}{2}\right), \quad (X,Y) = (0,0), \quad (X,Y) = (2,0)}.$$

For quick reference, here again is the system:

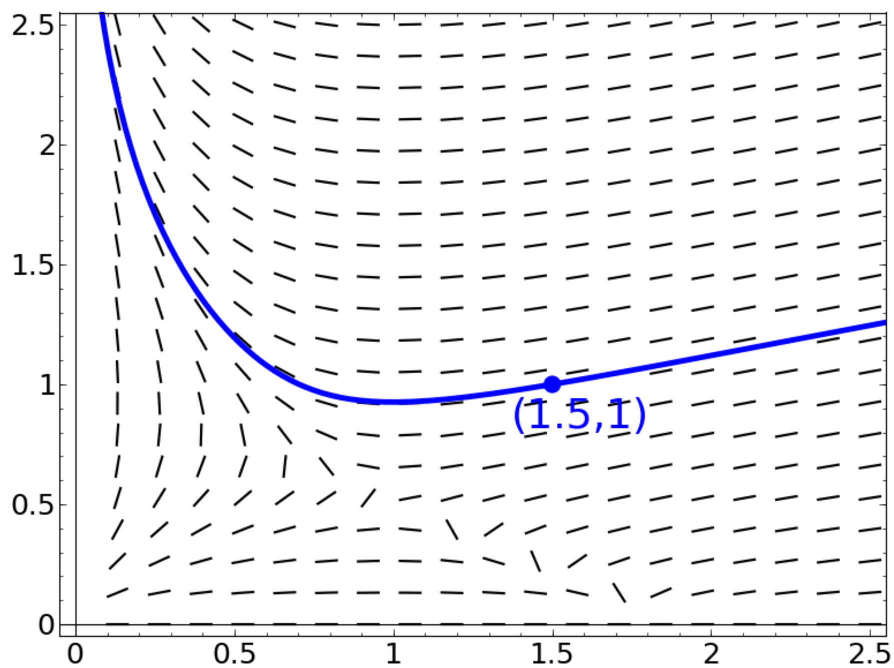
$$X' = 2X\left(1 - \frac{X}{2}\right) - 2XY$$
$$Y' = Y - XY$$

(c) The fish populations are measured to be $X(0) = 1.5$ and $Y(0) = 1$ at a certain point in time. For each of the two populations, determine if it is increasing, decreasing, or not changing size at that moment. Explain all your reasoning.

(2 points) Note that $X' = 2X(1 - X/2 - Y)$ and $Y' = Y(1 - X)$. Thus, at this moment, $X'(0) = 2(1.5)(1 - (1.5/2) - 1) < 0$ since the last factor is negative and the other factors are positive. This implies that X is decreasing at this point. Also, $Y'(0) = 1(1 - 1.5) < 0$, which means that Y is also decreasing at this moment.

(d) Below is a picture of a direction field for the system of differential equations above satisfied by X and Y. On it is drawn the phase trajectory corresponding to the initial condition $X(0) = 1.5$, $Y(0) = 1$ considered in part (c). Make the following marks on the diagram:

- draw the equilibrium solution(s) found in part (b); and
- use part (c) to draw an arrow indicating the direction for the phase trajectory.



(3 points) Diagram to come. Arrow points along curve, to the left.

(e) Use the above parts to describe the eventual fate of the species X and Y, starting from the initial condition $X(0) = 1.5$, $Y(0) = 1$.

(3 points) From the initial point, the trajectory is directed downward and to the left, but then turns upward while still moving to the left, ultimately approaching the Y axis while climbing higher and higher; this suggests that as time elapses, X will approach a size of 0 (i.e. the species will die out), and species Y will increase in size without bound.

6. (10 points) Let R be the bounded region in the xy-plane enclosed between the curves $y = x^4 - 1$ and $y = 2x^4 - 2$.

(a) Suppose S_1 is the solid generated by rotating R about the y-axis. Set up an integral in terms of a single variable representing the volume of S_1. Cite the method used. Do not evaluate the integral.

(5 points) The two curves $y = x^4 - 1$ and $y = 2x^4 - 2$ intersect at $(1, 0), (-1, 0)$. Note that the region R is symmetric about the y axis. The volume can be found using the method of cylindrical shells parallel to the y axis (integrating along the x axis). Let $h(x)$ denote the height of the cylindrical shell at radius x. Then $h(x)$ is given by $h(x) = (x^4 - 1) - (2x^4 - 2) = (1 - x^4)$. The surface area of this cylindrical shell is $A(x) = 2\pi x h(x)$.

The volume of the solid is thus given by:

$$\boxed{V = \int_0^1 A(x)dx = \int_0^1 2\pi x(1 - x^4)dx.}$$

Note: Points were taken off if the method was not correctly named or the limits of integration were incorrect.

(b) Suppose S_2 is the solid generated by rotating R about the line $y = -3$. Set up an integral in terms of a single variable representing the volume of S_2. Cite the method used. Do not evaluate the integral.

(5 points) We will use the method of washers parallel to the y axis to find the volume of this solid. Since R is rotated around $y = -3$, the washer at $x = a$ has inner radius $= 2x^4 - 2 - (-3) = 2x^4 + 1$ and outer radius $= x^4 - 1 - (-3) = x^4 + 2$. Thus the volume of the solid is given by,

$$\boxed{V = \int_{-1}^1 \pi((x^4 + 2)^2 - (2x^4 + 1)^2)dx.}$$

7. (9 points) A sample of genetic material to be analyzed is placed in a test tube and then centrifuged; assume that the test tube is a cylinder of radius 0.5 cm. At the end of the process, the sample occupies a 1 cm-high portion at the bottom of the tube. The density $\rho(z)$, in grams per cubic centimeter, of the sample at each point is assumed to depend only on the height z measured from the base of the tube.

(a) Write an integral involving the function ρ which expresses the total mass of the sample in grams.

(3 points) The surface area of each horizontal cross section of the cylinder is $\pi r^2 = 0.25\pi$. Thus the infinitesimal volume of a cross section at height z is given by $0.25\pi\rho(z)\Delta z$ and hence the total mass is:

$$\boxed{M = 0.25\pi \int_0^1 \rho(z)dz \quad \text{grams.}}$$

(b) If $\rho(z) = e^{-z}\cos z$, evaluate the integral for mass that you found in part (a). Give complete reasoning, but you do not have to simplify your answer.

(6 points) If $\rho(z) = e^{-z}\cos(z)$ then,

$$M = 0.25\pi \int_0^1 e^{-z}\cos(z)dz.$$

We find the integral using integration by parts twice. We start with $u = \cos(z), dv = e^{-z}$. Thus, $du = -\sin(z), v = -e^{-z}$. Thus,

$$\begin{aligned}
M &= 0.25\pi \int_0^1 e^{-z}\cos(z)dz \\
&= 0.25\pi\left([-\cos(z)e^{-z}]_0^1 - \int_0^1 e^{-z}\sin(z)\right) \\
&= 0.25\pi\left((1-\cos(1)e^{-1}) - \int_0^1 e^{-z}\sin(z)\right)
\end{aligned}$$

Evaluate this integral again using integration by parts with the choice: $dv = e^{-z}, u = \sin(z)$. Hence $v = -e^{-z}, du = \cos(z)$. Thus,

$$\begin{aligned}
M &= 0.25\pi\left((1-\cos(1)e^{-1}) - \int_0^1 e^{-z}\sin(z)\right) \\
&= 0.25\pi\left((1-\cos(1)e^{-1}) - ([-e^{-z}\sin(z)]_0^1 + \int_0^1 e^{-z}\cos(z))\right) \\
&= 0.25\pi\left((1-\cos(1)e^{-1}) + (e^{-1}\sin(1)\right) + M.
\end{aligned}$$

Thus,

$$2M = 0.25\pi\left(1 - \frac{\cos(1)}{e} + \frac{\sin(1)}{e}\right).$$

And hence,

$$\boxed{M = 0.125\pi\left(1 - \frac{\cos(1)}{e} + \frac{\sin(1)}{e}\right)}$$

8. (13 points) The annual rainfall (in meters) in Lotka City approximately follows a probability distribution given by:

$$f(x) = \begin{cases} 0 & x < 0 \\ \dfrac{4}{\pi(x^2+1)^2} & x \geq 0 \end{cases}$$

(You do not have to prove that this is a valid probability density function.)

(a) What is the mean annual rainfall in Lotka City? Show all reasoning.

(5 points) The mean rainfall in Lotka city is:

$$\begin{aligned} \mu &= \int_{-\infty}^{\infty} f(x) \\ &= 0 + \int_0^{\infty} \frac{4x}{\pi(x^2+1)^2} dx \\ &= \lim_{t\to\infty} \int_0^t \frac{4x}{\pi(x^2+1)^2} dx \end{aligned}$$

Using u-substitution with $u = x^2 + 1, du = 2xdx$, we get

$$\begin{aligned} \mu &= \lim_{t\to\infty} \int_1^t \frac{2}{\pi u^2} du \\ &= \frac{2}{\pi} \lim_{t\to\infty} [-\frac{1}{u}]_1^t \\ &= \frac{2}{\pi} \lim_{t\to\infty} (1 - \frac{1}{t}) \\ &= \boxed{\frac{2}{\pi}} \end{aligned}$$

(b) Find the probability that a given year's rainfall in Lotka City will be greater than 1 meter.

(8 points) The probability that Lotka receives a rainfall of more than 1 meter is:

$$P = \int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{4}{\pi(x^2+1)^2} dx.$$

Using u substitution with $x = \tan(u), dx = \sec^2(u)du$, we get,

$$\begin{aligned} P &= \lim_{t\to\pi/2} \int_{\pi/4}^t \frac{4\sec^2(u)}{\pi(\sec^2(u))^2} du \\ &= \frac{4}{\pi} \int_{\pi/4}^{\pi/2} \cos^(u)du \\ &= \frac{4}{\pi} \int_{\pi/4}^{\pi/2} \frac{1}{2}(1 + \cos(2u))du \\ &= \frac{2}{\pi}[u + \sin(2u)/2]_{\pi/4}^{\pi/2} \\ &= \boxed{\frac{1}{2} - \frac{1}{\pi}} \end{aligned}$$

9. (15 points) Determine whether each of the series below converges or diverges. Indicate clearly which tests you use and how you apply them.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{n! + 1}{(n+1)!}$

   (5 points) Using the fact that $(n+1)! = (n+1)n!$, we have

$$\sum_{n=0}^{\infty} \frac{n! + 1}{(n+1)!} = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} + \frac{1}{(n+1)!} \right)$$

   Now for each $n \geq 0$,

$$\frac{1}{n+1} + \frac{1}{(n+1)!} \geq \frac{1}{n+1} \geq 0$$

   Note that the series

$$\sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n}$$

   is the harmonic series and diverges by the p-series rule with $p = 1 \leq 1$. Therefore, by the Comparison Test, our original series diverges.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{3^{(n^2)}}$

(5 points) Let $a_n = \frac{(2n)!}{3^{(n^2)}}$. Applying the Ratio Test, we consider:

$$\begin{aligned} L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \frac{(2(n+1))!}{3^{(n+1)^2}} \frac{3^{n^2}}{(2n)!} \\ &= \lim_{n\to\infty} \frac{(2n+2)!}{(2n)!} \frac{3^{n^2}}{3^{n^2+2n+1}} \\ &= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{3^{2n+1}} \\ &= \lim_{n\to\infty} \frac{4n^2+6n+2}{3^{2n+1}} \end{aligned}$$

Then using L'Hopital's Rule twice on this $\frac{\infty}{\infty}$ form (and the fact that the derivative of a^x is $(\ln a)a^x$), we find:

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{8n+6}{2(\ln 3)3^{2n+1}} \\ &= \lim_{n\to\infty} \frac{8}{4(\ln 3)^2 3^{2n+1}} \\ &= 0 \end{aligned}$$

Since $L = 0 < 1$, the Ratio Test allows us to conclude the series converges.

(c) $\displaystyle\sum_{n=0}^{\infty}\left(1-\frac{\sqrt{n}}{\sqrt{n+1}}\right)$

(5 points) Write $a_n = 1 - \frac{\sqrt{n}}{\sqrt{n+1}}$. The Test for Divergence is inconclusive, since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1 - \sqrt{\frac{n}{n+1}} = \lim_{n\to\infty} 1 - \sqrt{\frac{1}{1+1/n}} = 1 - \sqrt{1} = 0.$$

In order to learn something precise about the size of a_n (with a goal in mind of using the Comparison Test or Limit Comparison Test), we will need to simplify it. To obtain a rational numerator, we use a conjugate expression:

$$\begin{aligned} a_n = 1 - \frac{\sqrt{n}}{\sqrt{n+1}} &= \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}} \\ &= \frac{(\sqrt{n+1}-\sqrt{n})}{\sqrt{n+1}} \cdot \frac{(\sqrt{n+1}+\sqrt{n})}{(\sqrt{n+1}+\sqrt{n})} \\ &= \frac{n+1-n}{n+1+\sqrt{n(n+1)}} \\ &= \frac{1}{n+1+\sqrt{n(n+1)}} \end{aligned}$$

Then since $\sqrt{n(n+1)} < \sqrt{(n+1)^2} = n+1$ for $n \geq 0$, we find that

$$a_n = \frac{1}{(n+1)+\sqrt{n^2+n}} > \frac{1}{(n+1)+(n+1)} = \frac{1}{2(n+1)} \geq 0$$

Now

$$\sum_{n=0}^{\infty} \frac{1}{2(n+1)} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n+1} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$$

is one half the harmonic series, which diverges by the p-series rule with $p = 1 \leq 1$. So by the Comparison Test, our original series diverges. (Note we checked the condition of the comparison test that required the terms to be positive.)

10. (10 points)

(a) Find, showing all your steps, the Taylor series for $\cos x$ with center 0.

(5 points) To find the Taylor series of $\cos x$ we need to find a general form for its derivatives.

$$\begin{aligned} f(x) &= \cos x \implies f(0) = \cos 0 = 1 \\ f'(x) &= -\sin x \implies f'(0) = \sin 0 = 0 \\ f''(x) &= -\cos x \implies f''(0) = -\cos 0 = -1 \\ f'''(x) &= \sin x \implies f'''(0) = \sin 0 = 0 \end{aligned}$$

Then, $f^{(4)}(x) = \cos x = f(x)$ so the pattern repeats. Therefore,

$$f^{(k)}(0) = \begin{cases} (-1)^{k/2} & \text{if } k \text{ is even, and} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

So the Taylor series of $\cos x$, namely $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$, will have nonzero terms only when k is even, i.e., when $k = 2n$ for some n. Thus, the Taylor series is

$$\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}}$$

(b) Use series to compute $\displaystyle\lim_{x\to 0} \frac{1-\cos(x^2)}{x\sqrt{1-\cos(3x^3)}}$.

(You may take for granted the fact that the Taylor series for $\cos x$ converges to $\cos x$.)

(5 points) First we compute:

$$\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (x^2)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n}$$

$$\cos(3x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (3x^3)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} 3^{2n} x^{6n}$$

Then

$$\begin{aligned} \lim_{x\to 0} \frac{1-\cos(x^2)}{x\sqrt{1-\cos 3x^3}} &= \lim_{x\to 0} \frac{1-(1-x^4/2!+x^8/4!-\cdots)}{x\sqrt{1-(1-3^2x^6/2!+3^4x^{12}/4!-\cdots)}} \\ &= \lim_{x\to 0} \frac{x^4/2!-x^8/4!+\cdots}{x\sqrt{3^2x^6/2!-3^4x^{12}/4!+\cdots}} \\ &= \lim_{x\to 0} \frac{x^4(1/2!-x^4/4!+\cdots)}{x\sqrt{x^6(3^2/2!-3^4x^6/4!+\cdots)}} \\ &= \lim_{x\to 0} \frac{x^4(1/2!-x^4/4!+\cdots)}{x^4\sqrt{3^2/2!-3^4x^6/4!+\cdots}} \\ &= \lim_{x\to 0} \frac{1/2!-x^4/4!+\cdots}{\sqrt{3^2/2!-3^4x^6/4!+\cdots}} = \frac{1/2}{\sqrt{3^2/2}} = \boxed{\frac{\sqrt{2}}{6}} \end{aligned}$$

11. (12 points) Show all steps in completing the problem below.

(a) Use the fact that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x, a fact which you do *not* have to prove, to find a series that converges to the number

$$\int_0^{1/2} \frac{e^{(-x^2)} - 1}{x}\, dx$$

(7 points) Write $f(x)$ for the above integrand. For $x \neq 0$, we have

$$f(x) = \frac{e^{(-x^2)} - 1}{x} = \frac{-1 + \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}}{x}$$

$$= \frac{-1 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}}{x} = \frac{\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{n!}}{x} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{n!}$$

Thus,

$$\int_0^{1/2} \frac{e^{(-x^2)} - 1}{x} = \lim_{t \to 0^+} \int_t^{1/2} \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{n!}$$

$$= \lim_{t \to 0^+} \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)n!} \Bigg|_t^{1/2} = \boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)n!(2^{2n})}}$$

(Note that as $x \to 0$, $f(x)$ approaches 0, so although f is undefined at 0 it has a removable discontinuity. As a result, we did not deduct credit for not evaluating this integral as an improper integral.)

(b) Write a partial sum of the series of part (a) that estimates the above integral to within 10^{-3}, and completely justify the accuracy of your partial sum.

(5 points) If we let $b_n = \frac{1}{(2n)n!(2^{2n})}$, which is positive for all $n \geq 1$, it follows that our series from (a) alternates signs and begins

$$s = \sum_{n=1}^{\infty} (-1)^n b_n = -\frac{1}{2 \cdot 1 \cdot 2^2} + \frac{1}{4 \cdot 2 \cdot 2^4} - \frac{1}{6 \cdot 6 \cdot 2^6} + \cdots$$

In fact, b_n is decreasing and approaches zero as $n \to \infty$, because its denominator is a product of positive, increasing expressions that are unbounded as $n \to \infty$. Thus, the conditions of the Alternating Series Test are satisfied, and so we may apply the Remainder Estimate. Note that

$$b_3 = \frac{1}{6 \cdot 6 \cdot 2^6} = \frac{1}{36 \cdot 64} < \frac{1}{1000}$$

so that the error R_2 in using the second partial sum as an approximation for s is sufficiently small in absolute value: $|R_2| = |s - s_2| \leq b_3 < 10^{-3}$; the desired partial sum is $\boxed{s_2 = -\frac{1}{8} + \frac{1}{128}}$.