Math 42 Final Exam — March 19, 2012

Name: ______________________________________ **SUID#:** _____________

Circle your section:				
Megan Bernstein 04 (11-11:50am) 07 (1:15-2:05pm)	Sukhada Fadnavis 02 (10-10:50am) 05 (11-11:50am)	Frederick Tsz-Ho Fong 08 (1:15-2:05pm) 09 (2:15-3:05pm)	Nisan Stiennon 03 (10-10:50am) 10 (2:15-3:05pm)	Tracy Nance ACE

- Complete the following problems. In order to receive full credit, please show all of your work and justify your answers. You do not need to simplify your answers unless specifically instructed to do so. You may use any result from class that you like, but if you cite a theorem be sure to verify the hypotheses are satisfied.
- Please check that your copy of this exam contains 17 numbered pages and is correctly stapled.
- This is a closed-book, closed-notes exam. No electronic devices, including cellphones, headphones, or calculation aids, will be permitted for any reason.
- **You have 3 hours.** Your organizer will signal the times between which you are permitted to be writing, including anything on this cover sheet, and to have the exam booklet open. During these times, the exam and all papers must remain in the testing room. When you are finished, you must hand your exam paper to a member of teaching staff.
- If you need extra room for your answers, use the back sides of each page. If you must use extra paper, use only that provided by teaching staff; make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- Please sign the following:

> "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ______________________________

The following boxes are strictly for grading purposes. Please do not mark.

Question:	1	2	3	4	5	6	7	8	9	10	11	**Total**
Points:	12	10	12	14	15	10	9	13	15	10	12	132
Score:												

1. (12 points) Consider the initial value problem

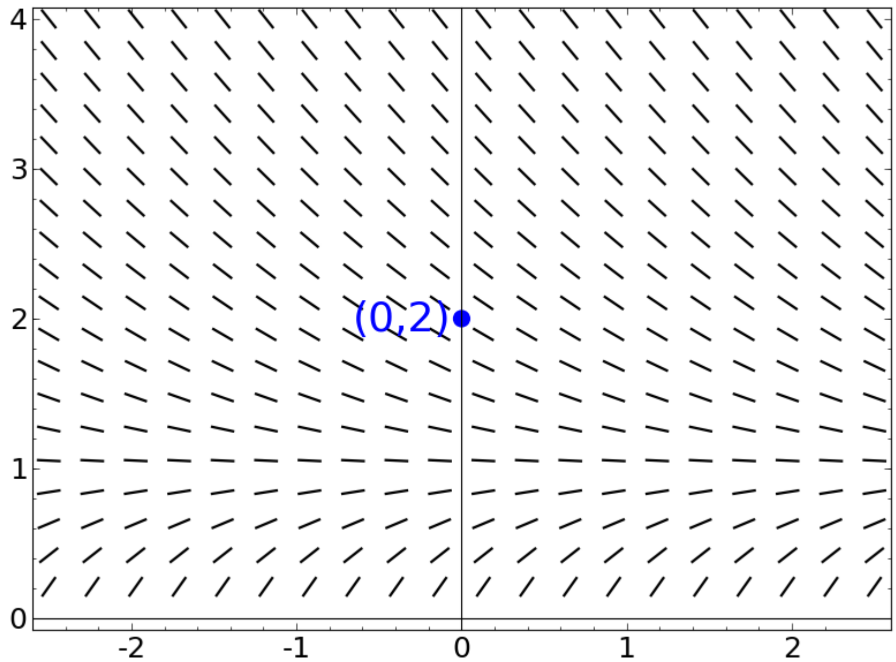
$$\begin{cases} \dfrac{dy}{dx} = -\ln y \\ y(0) = 2 \end{cases}$$

(a) Use Euler's method with step size 0.1 to find an approximation to $y(0.3)$. Show your steps, but you do not need to simplify your answer.

(b) Find the equilibrium solutions of the differential equation $\frac{dy}{dx} = -\ln y$.

(c) The following picture gives a direction field for the differential equation $\frac{dy}{dx} = -\ln y$. Draw on the picture:

- the equilibrium solutions for $\frac{dy}{dx} = -\ln y$, and
- an approximation to the graph of the solution curve having $y(0) = 2$.



(d) What is the behavior of the solution to the above initial value problem as $x \to +\infty$ and as $x \to -\infty$? Briefly explain your reasoning.

2. (10 points) A large cylindrical tank is positioned upright, with its circular base parallel to the ground. The tank is filled with water, which is being drained through a hole at the base of the tank. Torricelli's law states that, in this situation, the volume of water in the tank decreases at a rate proportional to the square root of the height of the water in the tank.

   Initially, at time $t = 0$ seconds, the water occupies a height of 10 meters in the tank. Moreover, it is experimentally determined that at this time, the height is decreasing at a rate of $-\frac{1}{100}$ m/s.

   (a) *Set up* an initial value problem satisfied by $h(t)$, the height of the water in the tank after t seconds. (A complete answer should include a differential equation that does not depend on unknown constants; but finding a solution is not necessary for part (a).)

   (b) How long does it take for the tank to become empty? Show all reasoning.

3. (12 points)

   (a) Show all steps in solving the initial value problem

$$\begin{cases} \dfrac{dx}{dt} = x + xt - \sqrt{x} - t\sqrt{x} \\ x(0) = \dfrac{1}{4} \end{cases}$$

(b) Show all steps in solving the initial value problem

$$\begin{cases} \dfrac{dx}{dt} = x \ln\left(x^{(3t^2)}\right) \\ x(0) = e \end{cases}$$

4. (14 points) In River Logistyx, there exist many species of fish. Among them are well-known species W and X.

   (a) Species W, measured in thousands of individuals, approximately follows a logistic model with constant of proportionality $k = 2$, and carrying capacity 0.5. The initial population of W is one thousand individuals, i.e. $W(0) = 1$.

   Set up the corresponding initial value problem for W, and give its solution.

   (b) What happens to the population of W in the long term? Explain.

(c) Fish of the species X are considered a very tasty treat by people in Lotka City, and are heavily fished at a rate of one thousand individuals each year. Under these conditions, the population of X, measured in thousands of individuals, approximately satisfies the initial value problem

$$\begin{cases} X' = 2X\left(1 - \dfrac{X}{2}\right) - 1 \\ X(0) = 2 \end{cases}$$

Solve this initial value problem for X, showing all steps.

(d) Use your answer to (c) to explain what happens to the population of X in the long term.

(e) Are there other initial conditions $X(0)$ for which the species eventually goes extinct? Explain completely.

5. (15 points) An environmental act has been passed in Volterra County declaring species X from River Logistyx an endangered species. Consequently, all fishing of this species has been halted. In the meantime, an invasive species Y has been introduced into the river from a nearby industrial aquaculture farm. (*Note:* Despite our storyline, this problem does not depend in any way on Problem 4.)

   The populations of species X and Y, measured in thousands, follow closely the following system of differential equations:

$$\begin{aligned} X' &= 2X\left(1 - \frac{X}{2}\right) - 2XY \\ Y' &= Y - XY \end{aligned}$$

   (a) Describe the nature of the relationship between the species X and Y: is it one of competition, cooperation, or predation? If the relationship is one of predation, indicate which species is the predator and which is the prey. Explain every part of your answer.

   (b) Find all equilibrium solutions of the above system. (You may omit negative X, Y here.)

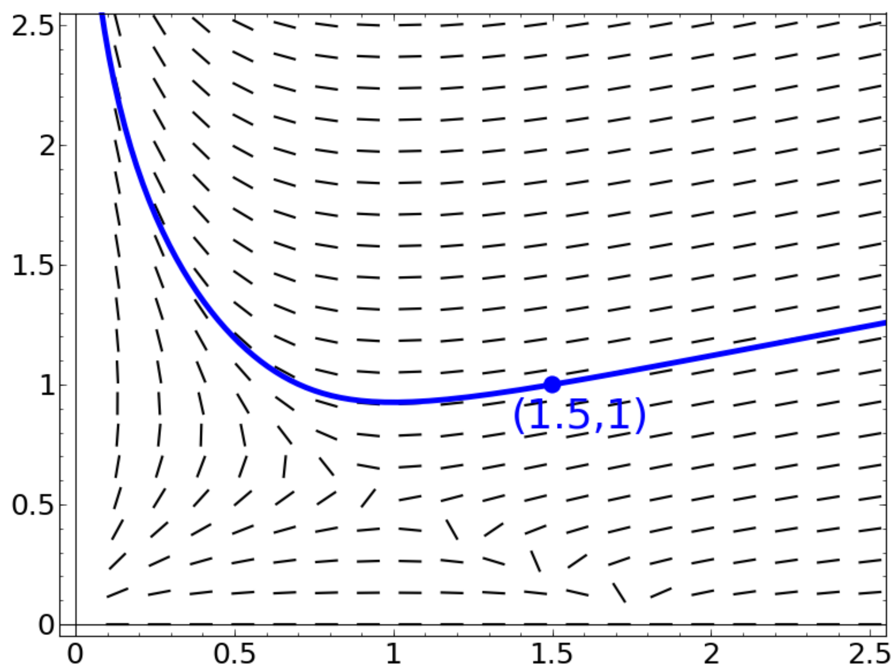
For quick reference, here again is the system:

$$X' = 2X\left(1 - \frac{X}{2}\right) - 2XY$$
$$Y' = Y - XY$$

(c) The fish populations are measured to be $X(0) = 1.5$ and $Y(0) = 1$ at a certain point in time. For each of the two populations, determine if it is increasing, decreasing, or not changing size at that moment. Explain all your reasoning.

(d) Below is a picture of a direction field for the system of differential equations above satisfied by X and Y. On it is drawn the phase trajectory corresponding to the initial condition $X(0) = 1.5$, $Y(0) = 1$ considered in part (c). Make the following marks on the diagram:

- draw the equilibrium solution(s) found in part (b); and
- use part (c) to draw an arrow indicating the direction for the phase trajectory.



(e) Use the above parts to describe the eventual fate of the species X and Y, starting from the initial condition $X(0) = 1.5$, $Y(0) = 1$.

6. (10 points) Let R be the bounded region in the xy-plane enclosed between the curves $y = x^4 - 1$ and $y = 2x^4 - 2$.

   (a) Suppose S_1 is the solid generated by rotating R about the y-axis. Set up an integral in terms of a single variable representing the volume of S_1. Cite the method used. Do not evaluate the integral.

   (b) Suppose S_2 is the solid generated by rotating R about the line $y = -3$. Set up an integral in terms of a single variable representing the volume of S_2. Cite the method used. Do not evaluate the integral.

7. (9 points) A sample of genetic material to be analyzed is placed in a test tube and then centrifuged; assume that the test tube is a cylinder of radius 0.5 cm. At the end of the process, the sample occupies a 1 cm-high portion at the bottom of the tube. The density $\rho(z)$, in grams per cubic centimeter, of the sample at each point is assumed to depend only on the height z measured from the base of the tube.

   (a) Write an integral involving the function ρ which expresses the total mass of the sample in grams.

   (b) If $\rho(z) = e^{-z}\cos z$, evaluate the integral for mass that you found in part (a). Give complete reasoning, but you do not have to simplify your answer.

8. (13 points) The annual rainfall (in meters) in Lotka City approximately follows a probability distribution given by:

$$f(x) = \begin{cases} 0 & x < 0 \\ \dfrac{4}{\pi(x^2+1)^2} & x \geq 0 \end{cases}$$

(You do not have to prove that this is a valid probability density function.)

(a) What is the mean annual rainfall in Lotka City? Show all reasoning.

(b) Find the probability that a given year's rainfall in Lotka City will be greater than 1 meter.

9. (15 points) Determine whether each of the series below converges or diverges. Indicate clearly which tests you use and how you apply them.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{n!+1}{(n+1)!}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{3^{(n^2)}}$

(c) $\displaystyle\sum_{n=0}^{\infty}\left(1-\frac{\sqrt{n}}{\sqrt{n+1}}\right)$

10. (10 points)

(a) Find, showing all your steps, the Taylor series for $\cos x$ with center 0.

(b) Use series to compute $\lim_{x \to 0} \dfrac{1-\cos(x^2)}{x\sqrt{1-\cos(3x^3)}}$.

(You may take for granted the fact that the Taylor series for $\cos x$ converges to $\cos x$.)

11. (12 points) Show all steps in completing the problem below.

(a) Use the fact that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x, a fact which you do *not* have to prove, to find a series that converges to the number

$$\int_0^{1/2} \frac{e^{(-x^2)} - 1}{x}\, dx$$

(b) Write a partial sum of the series of part (a) that estimates the above integral to within 10^{-3}, and completely justify the accuracy of your partial sum.