Solutions to Math 42 Second Exam — February 28, 2012

1. (10 points) For this problem, use the following information:

   - If g is a normal ("bell-shaped" or "Gaussian") probability density function, then g has the general form

   $$g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

   - A partial list of approximate values of the function

   $$F(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt \quad \text{is given at right:}$$

$F(0.25) \approx 0.60$	$F(1.75) \approx 0.960$
$F(0.5) \approx 0.69$	$F(2.0) \approx 0.977$
$F(0.75) \approx 0.77$	$F(2.25) \approx 0.988$
$F(1.0) \approx 0.84$	$F(2.5) \approx 0.994$
$F(1.25) \approx 0.89$	$F(2.75) \approx 0.997$
$F(1.5) \approx 0.93$	$F(3.0) \approx 0.999$

Suppose that a manufacturer of voltmeters tests its devices for quality control before shipment, and discovers that the amount of imprecision in a randomly selected voltmeter is approximately normally distributed with mean 0.25 mV and standard deviation 0.5 mV. (Note that by "imprecision" of a device we mean the difference between a "true" voltage and the device's measurement of that voltage; this difference can be either positive or negative.)

In what follows, write X for the random variable that represents the imprecision of a voltmeter; we have that its probability density function is given by $g(x)$ with $\mu = 0.25$(mV) and $\sigma = 0.5$(mV).

(a) Based on the above information, what is the probability that a randomly chosen voltmeter has a positive value of imprecision? Your answer should be a number; justify it by writing an integral expression that represents this probability and showing how to find its value.

(5 points) The probability that the random variable X is positive is given by:

$$\begin{aligned}\text{Prob}(X > 0) &= \int_0^{\infty} g(x)dx \\ &= \int_0^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx\end{aligned}$$

To realize this quantity in terms of F, we first make a "standardizing" substitution. Let

$$t = \frac{x-\mu}{\sigma}, \quad \text{so that} \quad dt = \frac{1}{\sigma}dx$$

and the limits change as follows:

$$x = 0 \Leftrightarrow t = \frac{0 - 0.25}{0.5} = -0.5 \qquad \text{and} \qquad x \to \infty \Leftrightarrow t \to \infty.$$

Hence, we have

$$\text{Prob}(X > 0) = \int_{-0.5}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$$

One way to proceed from here is to use the "symmetry" substitution $u = -t$ (so that $du = -dt$ and the signs of the limits flip); we find

$$\begin{aligned}\text{Prob}(X > 0) &= -\int_{0.5}^{-\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du \\ &= \int_{-\infty}^{0.5} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du \;=\; F(0.5) \approx \boxed{0.69}\end{aligned}$$

Alternatively, we may use the relation $F(-z) = 1 - F(z)$ (also a "symmetry"; see part (b) for more details) to find that

$$\begin{aligned} \text{Prob}(X > 0) &= \int_{-0.5}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt \\ &= 1 - \int_{-\infty}^{-0.5} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt \\ &= 1 - F(-0.5) \\ &= 1 - (1 - F(0.5)) \;=\; F(0.5) \approx \boxed{0.69} \end{aligned}$$

(b) If the imprecision of a voltmeter is less than 1.0 mV in absolute value, then it is shipped out. Otherwise, it is sent back to be recalibrated. Approximately what fraction of the voltmeters are sent back for recalibration? (Again use an integral expression as part of your justification.)

(5 points) A randomly selected voltmeter is sent back for recalibration when $|X| \geq 1$; we'll first express the probability that this occurs in terms of F, and then we'll evaluate the expression.

$$\begin{aligned} \text{Prob}(|X| \geq 1) &= 1 - \text{Prob}(-1 < X < 1) \\ &= 1 - \int_{-1}^{1} g(x)\, dx \\ &= 1 - \int_{-1}^{1} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}\, dx \\ &= 1 - \int_{-2.5}^{1.5} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt \qquad \left[t = \frac{x-\mu}{\sigma},\ dt = \frac{1}{\sigma} dx,\ \text{etc.}\right] \\ &= 1 - (F(1.5) - F(-2.5)) \\ &= 1 - F(1.5) + F(-2.5) \end{aligned}$$

The limits above are changed because $x = 1$ if and only if $t = \frac{1-\mu}{\sigma} = \frac{1-0.25}{0.5} = 1.5$, and similarly $x = -1$ if and only if $t = -2.5$. (Equivalently, using the formula $\text{Prob}(X < a) = F(\frac{a-\mu}{\sigma})$ yields:

$$\begin{aligned} \text{Prob}(|X| \geq 1) &= \text{Prob}(X < -1 \text{ or } X > 1) \\ &= \text{Prob}(X < -1) + \text{Prob}(X > 1) \\ &= \text{Prob}(X < -1) + (1 - \text{Prob}(X < 1)) \\ &= F\left(\frac{-1 - 0.25}{0.5}\right) + 1 - F\left(\frac{1 - 0.25}{0.5}\right) = F(-2.5) + 1 - F(1.5), \text{ as before.)} \end{aligned}$$

Now in evaluating this expression, we use the fact that $F(-2.5) = 1 - F(2.5)$, which can be seen by another "symmetric" substitution:

$$\begin{aligned} F(-2.5) = \int_{-\infty}^{-2.5} e^{-t^2/2}\, dt &= -\int_{\infty}^{2.5} e^{-u^2/2}\, du \qquad [u = -t,\ du = -dt,\ \text{etc.}] \\ &= \int_{2.5}^{\infty} e^{-u^2/2}\, du \\ &= 1 - \int_{-\infty}^{2.5} e^{-u^2/2}\, du = 1 - F(2.5) \end{aligned}$$

Thus,

$$P(|X| \geq 1) = 1 - F(1.5) + 1 - F(2.5) \approx 1 - 0.93 + 1 - 0.994 = \boxed{0.076}.$$

So, the fraction of voltmeters sent back for recalibration is $0.076 = \frac{76}{1000}$ (or 7.6 percent).

2. (10 points) Determine, with justification, whether each series converges. *If the series converges, find its sum.*

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{5^{n-1}}{3^{2n}}$

(5 points)

$$\sum_{n=1}^{\infty} \frac{5^{n-1}}{3^{2n}} = \sum_{n=1}^{\infty} \frac{5^{n-1}}{9^n} = \sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{5}{9}\right)^{n-1}$$

This is a geometric series with common ratio $\frac{5}{9}$. It converges because $\left|\frac{5}{9}\right| < 1$. It converges to

$$\frac{\frac{1}{9}}{1-\frac{5}{9}} = \boxed{\frac{1}{4}}$$

The Ratio Test can also be used to show that the series converges:

$$\lim_{n\to\infty} \left|\frac{5^n/3^{2n+2}}{5^{n-1}/3^{2n}}\right| = \lim_{n\to\infty} \frac{5^n 3^{2n}}{5^{n-1}3^{2n+2}} = \lim_{n\to\infty} \frac{5}{3^2} = \frac{5}{9} < 1$$

So the series converges. But the Ratio Test cannot be used to calculate the sum of the series.

(b) $\displaystyle\sum_{n=1}^{\infty} \ln\left(1+\frac{2}{n}\right)$

(5 points) The Test for Divergence is inconclusive. All the terms are positive, so we can use the Limit Comparison Test to compare $\sum_{n=1}^{\infty} \ln\left(1+\frac{2}{n}\right)$ with $\sum_{n=1}^{\infty} \frac{1}{n}$:

$$\begin{aligned}
\lim_{n\to\infty} \frac{\ln\left(1+\frac{2}{n}\right)}{\frac{1}{n}} &= \lim_{n\to\infty} \frac{\frac{d}{dn}\ln\left(1+\frac{2}{n}\right)}{\frac{d}{dn}\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{\left(1+\frac{2}{n}\right)^{-1}\left(\frac{-2}{n^2}\right)}{\frac{-1}{n^2}} \\
&= \lim_{n\to\infty} \left(1+\frac{2}{n}\right)^{-1}(2) \\
&= \lim_{n\to\infty} \frac{2}{1+\frac{2}{n}} \\
&= 2 \neq 0
\end{aligned}$$

so the series $\sum_{n=1}^{\infty} \ln\left(1-\frac{2}{n}\right)$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ either both converge or both diverge. The latter is the harmonic series, which diverges because it is a p-series with $p = 1 \leq 1$. So $\sum_{n=1}^{\infty} \ln\left(1+\frac{2}{n}\right)$ diverges.

One could use the direct Comparison Test instead of the Limit Comparison Test by proving that for large enough n,

$$\ln\left(1+\frac{2}{n}\right) > \frac{1}{n}$$

(But it is not true that $\ln\left(1+\frac{2}{n}\right) > \frac{2}{n}$.)

Another way to tackle the problem is the telescoping sum approach:

$$\begin{aligned}
\sum_{n=1}^{\infty} \ln\left(1+\frac{2}{n}\right) &= \sum_{n=1}^{\infty} \ln\left(\frac{n+2}{n}\right) \\
&= \sum_{n=1}^{\infty} (\ln(n+2) - \ln(n)) \\
&= (\ln(3)-\ln(1)) + (\ln(4)-\ln(2)) + (\ln(5)-\ln(3)) + (\ln(6)-\ln(4)) + \cdots
\end{aligned}$$

Notice that every positive term cancels with the negative term five terms later, and every negative term cancels with the positive term five terms earlier. So most of the terms in the n-th partial sum s_n cancel each other. If n is at least 2, then only four terms remain:

$$s_n = -\ln(1) - \ln(2) + \ln(n+1) + \ln(n+2)$$

So we can calculate the sum directly by taking the limit of the partial sums:

$$\begin{aligned}
\sum_{n=1}^{\infty} \ln\left(1+\frac{2}{n}\right) &= \lim_{n\to\infty} s_n \\
&= \lim_{n\to\infty} \Big(-\ln(1) - \ln(2) + \ln(n+1) + \ln(n+2)\Big) \\
&= \infty
\end{aligned}$$

Therefore the sum diverges.

3. (12 points) In each of the following parts, give a formula for a_n so that the series $\sum_{n=1}^{\infty} a_n$ has the specified property or properties, or state that such a series cannot exist. *You do not need to justify your answers.* (Please treat each question as independent from the others; properties do not carry over from part (a) to part (b), etc.)

(a) The series $\sum_{n=1}^{\infty} a_n$ diverges and $\lim_{n\to\infty} a_n = 0$.

(3 points) $a_n = \frac{1}{n}$, for example: $\sum_{n=1}^{\infty} \frac{1}{n}$ is the divergent harmonic series (i.e. a p-series with $p = 1$), yet $\lim_{n\to\infty} \frac{1}{n} = 0$.

(b) The series $\sum_{n=1}^{\infty} a_n$ converges and $a_n < a_{n+1}$ for all $n \geq 1$.

(3 points) $a_n = -\frac{1}{n^2}$: since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-Series Rule ($p = 2 > 1$), $\sum_{n=1}^{\infty} -\frac{1}{n^2}$ also converges. Also

$$a_n = -\frac{1}{n^2} < -\frac{1}{(n+1)^2} = a_{n+1}.$$

(c) The series $\sum_{n=1}^{\infty} a_n$ converges and $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$.

(3 points) $a_n = \frac{1}{n^2}$: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges again by the p-Series Rule ($p = 2 > 1$), yet

$$\lim_{n\to\infty} \left|\frac{1/(n+1)^2}{1/n^2}\right| = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1.$$

(d) The series $\sum_{n=1}^{\infty} a_n$ converges absolutely and the series $\sum_{n=1}^{\infty} (a_n)^2$ diverges.

(3 points) Such a series cannot exist: if $\sum_{n=1}^{\infty} |a_n|$ converges, we have $\lim_{n\to\infty} |a_n| = 0$. After a sufficiently large n, we have $|a_n| < 1$. Hence

$$0 \leq (a_n)^2 < |a_n| \qquad \text{after a sufficient large } n.$$

By the Comparison Test, $\sum_{n=1}^{\infty} (a_n)^2$ converges.

Remarks on grading:

1. Since justification is not required, each part is graded on a 0/3-basis. No partial credit is awarded.

2. No points will be deducted (this time) if the first few terms of the a_n's are undefined (such as $a_n = \frac{1}{n^2-1}$; then a_1 is undefined), but students who committed this mistake should be aware of this issue next time.

3. If more than one answers are given but none of them is clearly indicated as the final answer, then it will be at the discretion of the grader(s) to decide which one is the 'first' answer; all the others will not be considered.

4. (10 points) Determine whether each of the series below converges or diverges. Indicate clearly which tests you use and how you apply them.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^3-2}$

(5 points) The series converges. To see this, let $a_n = \dfrac{n-1}{n^3-2}$ and let $b_n = \dfrac{1}{n^2}$. Note that $a_n, b_n > 0$ for $n \geq 2$. Thus, we can apply the Limit Comparison Test. Note that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2(n-1)}{n^3-2} = \lim_{n\to\infty} \frac{(1-1/n)}{1-2/n^3} = 1.$$

Thus, by the Limit Comparison Test both $\sum a_n$ and $\sum b_n$ converge or both diverge. But we know that $\sum \frac{1}{n^2}$ converges by the p-Series Rule, since $p = 2 > 1$. Thus, $\sum_{n=1}^{\infty} \frac{n-1}{n^3-2}$ also converges.

Note: It is also possible to show convergence using the Comparison Test. In both methods full credit was given if the test was explicitly stated and the hypothesis of the tests were shown to hold true (for example, positivity of the terms, inequality or limits, etc.)

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{2^n\, n!(n+1)!}$

(5 points) The series diverges by the Ratio Test. To see this, let $a_n = \dfrac{(2n)!}{2^n n!(n+1)!}$. Then,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(2n+2)!2^n n!(n+1)!}{(2n)!2^{n+1}(n+1)!(n+2)!}\right|$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{2(n+1)(n+2)} = \lim_{n\to\infty} \frac{(2+2/n)(2+1/n)}{2(1+1/n)(1+2/n)} = 2 > 1.$$

Thus, the series diverges by the Ratio Test.

5. (13 points) Find, with complete justification, the interval of convergence of the power series

$$\sum_{n=2}^{\infty} \frac{(3-x)^{3n}}{3^{3n}(\ln n)}$$

The interval of convergence is $[0, 6)$. To see this, we first perform the ratio test:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(3-x)^{3n+3} \cdot 3^{3n} \cdot \ln n}{3^{3n+3}\cdot\ln(n+1)\cdot(3-x)^{3n}}\right| \\
&= \lim_{n\to\infty}\left|\frac{(3-x)^3\cdot\ln(n)}{3^3\cdot\ln(n+1)}\right| \\
&= \frac{|3-x|^3}{27}\lim_{n\to\infty}\left|\frac{\ln(n)}{\ln(n+1)}\right| \qquad \text{form } \frac{\infty}{\infty} \text{ so use l'Hopital} \\
&= \frac{|3-x|^3}{27}\lim_{n\to\infty}\left|\frac{1/n}{1/(n+1)}\right| \\
&= \frac{|3-x|^3}{27}\lim_{n\to\infty}\left|\frac{n+1}{n}\right| \\
&= \frac{|3-x|^3}{27}\cdot 1 \\
&= \frac{|x-3|^3}{27}
\end{aligned}$$

By the ratio test, the sum converges if this value is less than 1, and diverges if if is greater than one. Therefore $\sum \frac{(3-x)^{3n}}{3^{3n}\ln n}$ converges for $|x-3| < 3$, and diverges for $|x-3| > 3$. This tells us that the radius of convergence is $R = 3$; since the power series is centered at $x = 3$, we must check the endpoints 0 and 6.

If $x = 0$, then the sum is

$$\sum_{n=2}^{\infty} \frac{1}{\ln(n)}.$$

For $n > 2$, $0 < \ln(n) < n$, so $0 < 1/n < 1/\ln(n)$. We know that

$$\sum_{n=2}^{\infty} \frac{1}{n}$$

diverges because it is a p-series with $p = 1$, so by the comparison test $\sum 1/\ln(n)$ diverges also. $x = 0$ is not in the interval of convergence.

If $x = 6$, then the sum is

$$\sum_{n=2}^{\infty} \frac{(-1)^{3n}}{\ln(n)} = \sum_{n=2}^{\infty} \frac{(-1)^{3n}}{\ln(n)}.$$

This is an alternating sum, and since $\ln(n)$ is an increasing sequence, $1/\ln(n)$ is a decreasing sequence; also since $\ln(N)$ goes to infinity as n goes to infinity, we have

$$\lim_{n\to\infty} 1/\ln(n) = 0.$$

By the alternating series test, the sum converges. The interval of convergence is therefore $(0, 6]$.

6. (12 points) Suppose the power series $\sum_{n=0}^{\infty} c_n(x-2)^n$ converges for $x = 7$ but not for $x = -4$; no other information about the values of c_n is given. Decide which of the following series must converge, must diverge, or may either converge or diverge (inconclusive). Circle your answer. You do not need to justify your answers.

(2 points each) Let $f(x) = \sum_{n=0}^{\infty} c_n(x-2)^n$. The information above tells us partial information about the radius of convergence R of f: specifically, we can conclude that $5 \le R \le 6$.

(a) $\sum_{n=0}^{\infty} 2^n c_n$ — **[Converges]** — Diverges — Inconclusive

This is $f(x)$ evaluated at $x = 4$, which lies inside the smallest possible radius of convergence about the center 2, so we know the series converges.

(b) $\sum_{n=0}^{\infty} 7^n c_n$ — Converges — **[Diverges]** — Inconclusive

This is $f(x)$ evaluated at $x = 9$, which lies outside the largest possible radius of convergence about the center 2, so we know the series diverges.

(c) $\sum_{n=1}^{\infty} nc_n$ — **[Converges]** — Diverges — Inconclusive

This is $f'(x)$ evaluated at $x = 3$. Since f' has the same radius of convergence as f, and since 3 lies inside the smallest possible radius of convergence about 2, we know the series converges.

(d) $\sum_{n=0}^{\infty} 6^n \frac{c_n}{n+1}$ — Converges — Diverges — **[Inconclusive]**

This is $\frac{1}{6}\int_2^x f(t)\,dt$ evaluated at $x = 8$; we know that antiderivatives of f have the same radius of convergence as f. In this case 8 lies at a distance of 6 from the center 2, and since we don't know whether $R = 6$ or $R < 6$, we don't know whether the series converges. (In fact, even if we knew $R = 6$, we wouldn't know whether the endpoints — i.e., the points that lie exactly 6 units from the center — lie in the interval of convergence of an antiderivative of f.)

(e) $\sum_{n=0}^{\infty} |c_n|$ — **[Converges]** — Diverges — Inconclusive

Since $\sum_{n=0}^{\infty} 5^n c_n$ converges, we know that $\lim_{n\to\infty} 5^n c_n = 0$, which means that for all sufficiently large n we have that $|5^n c_n| < 1$, so that $0 < |c_n| < \frac{1}{5^n}$. Thus, by the Comparison Test, $\sum_{n=0}^{\infty} |c_n|$ converges because $\sum_{n=0}^{\infty} \frac{1}{5^n}$ does (the latter is a geometric series with common ratio $\frac{1}{5} < 1$).

(f) $\sum_{n=1}^{\infty} \frac{1}{n + (c_n)^4}$ — Converges — **[Diverges]** — Inconclusive

Note that $\lim_{n\to\infty} \frac{1/n}{1/(n+(c_n)^4)} = \lim_{n\to\infty} \frac{n+(c_n)^4}{n} = 1 + \lim_{n\to\infty} \frac{(c_n)^4}{n} = 1 + 0 = 1 \neq 0$, so by the Limit Comparison Test, the divergence of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ implies that our series diverges.

7. (11 points) Let $f(x) = x^{1/3}$.

(a) Find $T_3(x)$, the degree-3 Taylor polynomial for f centered at 8.

(4 points) We have

$$T_3(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

Thus, for $a = 8$ and $f(x) = x^{1/3}$, we need to find $f(8)$, $f'(8)$, $f''(8)$, $f'''(8)$.

$$\begin{aligned} f(x) = x^{1/3} &\implies f'(x) = \frac{1}{3}x^{-2/3} \\ &\implies f''(x) = \frac{1}{3} \cdot \frac{-2}{3} x^{-5/3} \\ &\implies f'''(x) = \frac{1}{3} \cdot \frac{-2}{3} \cdot \frac{-5}{3} x^{-8/3} \end{aligned}$$

Thus, $f(8) = 2$, and $f'(8) = \frac{1}{3} \cdot 2^{-2}$, and $f''(8) = \frac{1}{3} \cdot \frac{-2}{3} \cdot 2^{-5}$, and $f'''(8) = \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{5}{3} \cdot 2^{-8}$.

Therefore, $\boxed{T_3(x) = 2 + \frac{1}{1!} \cdot \frac{1}{3} \cdot \frac{1}{2^2}(x-8) - \frac{1}{2!} \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{2^5}(x-8)^2 + \frac{1}{3!} \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{5}{3} \cdot \frac{1}{2^8}(x-8)^3}$.

(b) Use T_3 to obtain an approximation for the cube root of 7.9. (You do not need to simplify your answer.)

(1 point) We seek $T_3(7.9)$, because $T_3(x)$ approximates $f(x) = \sqrt[3]{x}$ for x near 8. We find

$$\begin{aligned} T_3(7.9) &= 2 + \frac{1}{1!} \cdot \frac{1}{3} \cdot \frac{1}{2^2}(7.9-8) - \frac{1}{2!} \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{2^5}(7.9-8)^2 + \frac{1}{3!} \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{5}{3} \cdot \frac{1}{2^8}(7.9-8)^3 \\ &= 2 - \frac{1}{1!} \cdot \frac{1}{3} \cdot \frac{1}{2^2} \cdot \frac{1}{10} - \frac{1}{2!} \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{2^5} \cdot \frac{1}{10^2} - \frac{1}{3!} \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{5}{3} \cdot \frac{1}{2^8} \cdot \frac{1}{10^3} \end{aligned}$$

(c) Determine the accuracy of your approximation from part (b), explaining the steps of your reasoning, and giving your final conclusion in sentence form.

(6 points) By Taylor's Inequality,

$$|f(x) - T_3(x)| = |R_3(x)| \le \frac{M}{4!}|x-8|^4 \text{ for } x \text{ in } [8-d, 8+d]$$

where $M \ge |f^{(4)}|$ for $x \in [8-d, 8+d]$. We want this to hold specifically for $x = 7.9$. So we choose an interval containing 7.9; the smallest such is $[7.9, 8.1]$, when $d = 0.1$.

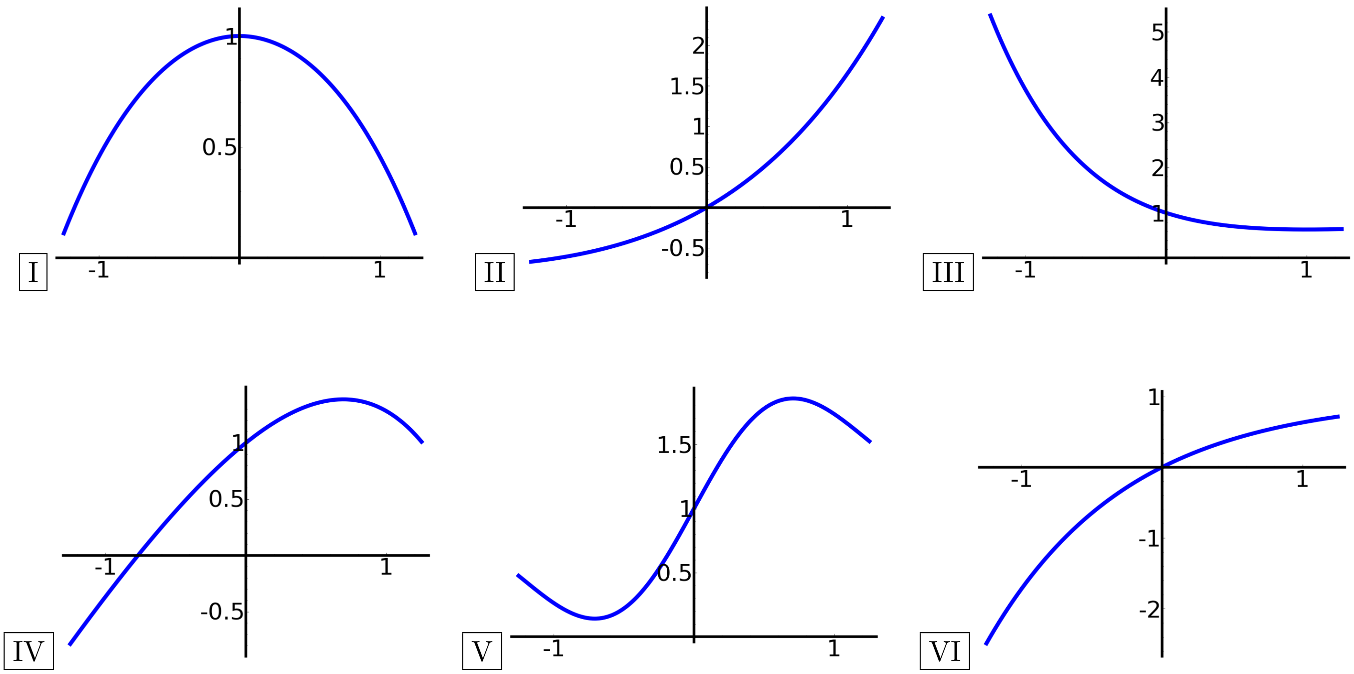
We have $f^{(4)}(x) = \frac{1}{3} \cdot \frac{-2}{3} \cdot \frac{-5}{3} \cdot \frac{-8}{3} x^{-11/3} = -\frac{80}{81}x^{-11/3}$, and so $|f^{(4)}(x)| = \frac{80}{81}x^{-11/3}$. This is a decreasing function, so is maximized on the above interval when $x = 7.9$. So we may choose $M = \frac{80}{81}(7.9)^{-11/3}$. Note we may also choose $M = 1$ as a simpler if not as refined upper bound, since $80/81 < 1$ and 7.9 to a negative power is trivially less than 1 also.

Finally, we conclude

$$|R_3(7.9)| \le \frac{\frac{80}{81}(7.9)^{-11/3}}{4!}(7.9-8)^4.$$

So our estimate in part (b) of the cube root of 7.9 was within $\frac{\frac{80}{81}(7.9)^{-11/3}}{4!}(7.9-8)^4$ of the real value.

8. (10 points) Match each power series below to its graph, chosen from among the six displayed. (Note that each series has a match, but exactly one of the graphs does not correspond to any power series in the list). You do not need to justify your answers.



Series	I, II, III, IV, V, or VI
$f(x) = 1 + 2x - 2x^3 + \dfrac{2x^5}{2!} - \dfrac{2x^7}{3!} + \cdots$	**V**
$f(x) = x - \dfrac{x^2}{2!} + \dfrac{x^3}{3!} - \dfrac{x^4}{4!} + \dfrac{x^5}{5!} - \cdots$	**VI**
$f(x) = 1 - \dfrac{x^2}{2!} - \dfrac{x^4}{4!} - \dfrac{x^6}{6!} - \dfrac{x^8}{8!} - \cdots$	**I**
$f(x) = 1 + x - \dfrac{x^2}{2!} - \dfrac{x^3}{3!} - \dfrac{x^4}{4!} - \cdots$	**IV**
$f(x) = x + \dfrac{x^2}{2} + \dfrac{x^3}{2^2 \cdot 2!} + \dfrac{x^4}{2^3 \cdot 3!} + \dfrac{x^5}{2^4 \cdot 4!} + \cdots$	**II**