Solutions to Math 42 First Exam — February 2, 2012

1. (12 points) Evaluate each of the following integrals, showing all of your reasoning.

   (a) $\displaystyle\int_0^{\pi/4} \cos^4\theta\, d\theta$

   (6 points) We use the half-angle formula twice:

$$\begin{aligned}
\int_0^{\pi/4} \cos^4\theta\, d\theta &= \int_0^{\pi/4} (\cos^2\theta)^2\, d\theta \\
&= \int_0^{\pi/4} \left(\frac{1}{2}(1+\cos(2\theta))\right)^2 d\theta \\
&= \frac{1}{4}\int_0^{\pi/4} 1 + 2\cos(2\theta) + \cos^2(2\theta)\, d\theta \\
&= \frac{1}{4}\int_0^{\pi/4} 1 + 2\cos(2\theta) + \frac{1}{2}(1+\cos(4\theta))\, d\theta \\
&= \frac{1}{8}\int_0^{\pi/4} 3 + 4\cos(2\theta) + \cos(4\theta)\, d\theta \\
&= \frac{1}{8}\left[3\theta + 2\sin(2\theta) + \frac{1}{4}\sin(4\theta)\right]_0^{\pi/4} \\
&= \frac{1}{8}\left(3\frac{\pi}{4} + 2\sin\left(\frac{\pi}{2}\right) + \frac{1}{4}\sin(\pi)\right) - \frac{1}{8}\left(3\cdot 0 + 2\sin(0) + \frac{1}{4}\sin(0)\right) \\
&= \frac{1}{8}\left(\frac{3\pi}{4} + 2 + 0\right) - (0) = \boxed{\frac{3\pi}{32} + \frac{1}{4}}
\end{aligned}$$

   (b) $\displaystyle\int \frac{\sqrt{x^2-1}}{x}\, dx$

   (6 points) We can use the trigonometric substitution $x = \sec\theta,\ dx = \sec\theta\tan\theta\, d\theta$ (where either $0 \le \theta \le \frac{\pi}{2}$ or $\pi \le \theta \le \frac{3\pi}{2}$ so that $\tan\theta$ is positive; thus $\sqrt{\tan^2\theta} = \tan\theta$). Then we get

$$\begin{aligned}
\int \frac{\sqrt{x^2-1}}{x}\, dx &= \int \frac{\sqrt{\sec^2\theta - 1}}{\sec\theta}\sec\theta\tan\theta\, d\theta \\
&= \int \frac{\sqrt{\tan^2\theta}}{\sec\theta}\sec\theta\tan\theta\, d\theta \\
&= \int \frac{\tan\theta}{\sec\theta}\sec\theta\tan\theta\, d\theta \\
&= \int \tan^2\theta\, d\theta \\
&= \int (\sec^2\theta - 1)\, d\theta = \tan\theta - \theta + C
\end{aligned}$$

   Since $x = \sec\theta = \frac{1}{\cos\theta}$, solving for θ yields $\theta = \arccos\frac{1}{x}$. Note also that we also know that $\sqrt{x^2-1} = \tan\theta$. Thus,

$$\int \frac{\sqrt{x^2-1}}{x}\, dx = \boxed{\sqrt{x^2-1} - \arccos\frac{1}{x} + C}$$

Alternate solution to 1(b): under the assumption $x > 0$, we could instead attack this problem with the following u-substitution:

$$\begin{aligned} u = \sqrt{x^2-1}, &\quad du = \frac{x}{\sqrt{x^2-1}}\,dx \\ \implies x^2 = u^2+1, &\quad dx = \frac{\sqrt{x^2+1}}{x}\,du = \frac{u}{x}\,du \end{aligned}$$

So

$$\begin{aligned} \int \frac{\sqrt{x^2-1}}{x}\,dx &= \int \frac{u}{x}\frac{u}{x}\,du \\ &= \int \frac{u^2}{x^2}\,du \\ &= \int \frac{u^2}{u^2+1}\,du \end{aligned}$$

Then we need to do polynomial long division. Or in this case we can use the following shortcut version of long division:

$$\begin{aligned} \int \frac{u^2}{u^2+1}\,du &= \int \frac{u^2+1-1}{u^2+1}\,du \\ &= \int \frac{u^2+1}{u^2+1}\,du - \int \frac{1}{u^2+1}\,du \\ &= \int du - \int \frac{1}{u^2+1}\,du \\ &= u - \arctan u + C \\ &= \boxed{\sqrt{x^2-1} - \arctan\sqrt{x^2-1} + C} \end{aligned}$$

This looks like a different answer than the one we gave above, but in fact $\arccos 1/x$ and $\arctan\sqrt{x^2-1}$ are the same for positive x. Here's a quick proof: Suppose

$$y = \arctan\sqrt{x^2-1}$$

so $0 \le y \le \pi/2$. Then

$$\begin{aligned} \sqrt{x^2-1} &= \tan y \\ \implies x^2 - 1 &= \tan^2 y \\ &= \sec^2 y - 1 \\ \implies x^2 &= \sec^2 y = \frac{1}{\cos^2 y} \\ \implies x = |x| &= \frac{1}{|\cos y|} = \frac{1}{\cos y} \qquad (x, \cos y > 0) \\ \implies y &= \arccos\frac{1}{x} \end{aligned}$$

2. (13 points) Evaluate each of the following integrals, showing all of your reasoning.

(a) $\int x^2 \arctan x \, dx$

(6 points) We use integration by parts, with

$$u = \arctan x \qquad du = \frac{1}{x^2+1}\,dx$$
$$v = \frac{x^3}{3} \qquad dv = x^2\,dx$$

$$uv - \int v\,du = \frac{x^3}{3}\arctan x - \frac{1}{3}\int \frac{x^3}{x^2+1}\,dx$$

One way to attack this integral is by polynomial long division:

$$\begin{array}{r|l} & x \\ \hline x^2+1 & x^3 \\ & -x^3 - x \\ \hline & -x \end{array}$$

So we have

$$\begin{aligned}\int x^2 \arctan x\,dx &= \frac{x^3}{3}\arctan x - \frac{1}{3}\int x - \frac{x}{x^2+1}\,dx \\ &= \frac{x^3}{3}\arctan x - \frac{1}{3}\int x\,dx + \frac{1}{3}\int \frac{x}{x^2+1}\,dx\end{aligned}$$

The last integral here can be computed using u-substitution, with $u = x^2+1$.

$$= \frac{x^3}{3}\arctan x - \frac{x^2}{6} + \frac{1}{6}\ln|x^2+1| + C$$

We can get rid of the absolute value signs because x^2+1 is always positive:

$$= \frac{x^3}{3}\arctan x - \frac{x^2}{6} + \frac{1}{6}\ln(x^2+1) + C$$

(b) $\displaystyle\int \frac{dx}{(3-2x-x^2)^{3/2}}$

(7 points) First complete the square in the denominator:

$$\begin{aligned}3-2x-x^2 &= -(x^2+2x-3) \\ &= -(x^2+2x+1-1-3) \\ &= -((x+1)^2-4) \\ &= 4-(x+1)^2\end{aligned}$$

So

$$\int \frac{dx}{(3-2x-x^2)^{3/2}} = \int \frac{dx}{(2^2-(x+1)^2)^{3/2}}$$

Use the substitution

$$\begin{aligned} u &= x+1 \\ du &= dx \end{aligned}$$

Then

$$\begin{aligned} \int \frac{dx}{(2^2-(x+1)^2)^{3/2}} &= \int \frac{du}{(2^2-u^2)^{3/2}} \\ &= \int \frac{du}{\left(\sqrt{2^2-u^2}\right)^3} \end{aligned}$$

Now we can use the trigonometric substitution

$$\begin{aligned} u &= 2\sin\theta \\ du &= 2\cos\theta\, d\theta \\ \sqrt{2^2-u^2} &= 2\cos\theta \\ -\frac{\pi}{2} &\le \theta \le \frac{\pi}{2} \end{aligned}$$

And we get

$$\begin{aligned} \int \frac{du}{\left(\sqrt{2^2-u^2}\right)^3} &= \int \frac{2\cos\theta\, d\theta}{(2\cos\theta)^3} \\ &= \int \frac{d\theta}{4\cos^2\theta} \\ &= \frac{1}{4}\int \sec^2\theta\, d\theta \\ &= \frac{1}{4}\tan\theta + C \\ &= \frac{\sin\theta}{4\cos\theta} + C \\ &= \frac{2\sin\theta}{4(2\cos\theta)} + C \\ &= \frac{u}{4\sqrt{4-u^2}} + C \\ &= \frac{x+1}{4\sqrt{4-(x+1)^2}} + C \\ &= \frac{x+1}{4\sqrt{3-2x-x^2}} + C \end{aligned}$$

3. (7 points) Evaluate $\displaystyle\int \frac{x^4+x}{x^4-1}\,dx$, showing all reasoning.

By long division, we have

$$\frac{x^4+x}{x^4-1} = 1 + \frac{x+1}{x^4-1}.$$

Factorizing x^4-1 gives $x^4-1=(x+1)(x-1)(x^2+1)$. Hence, we have

$$\frac{x^4+x}{x^4-1} = 1 + \frac{x+1}{(x+1)(x-1)(x^2+1)} = 1 + \frac{1}{(x-1)(x^2+1)}.$$

Use method of partial fractions to decompose $\frac{1}{(x-1)(x^2+1)}$: Let

$$\begin{aligned}\frac{1}{(x-1)(x^2+1)} &= \frac{A}{x-1} + \frac{Bx+C}{x^2+1},\\ 1 &= A(x^2+1) + (Bx+C)(x-1).\end{aligned}$$

- When $x=1$, we have $1 = 2A + (Bx+C)\cdot 0 \Rightarrow A = \frac{1}{2}$.
- When $x=0$, we have $1 = \frac{1}{2} + (B\cdot 0 + C)(-1) \Rightarrow C = -\frac{1}{2}$.
- When $x=-1$, we have $1 = 2(\frac{1}{2}) + (-B-\frac{1}{2})(-2) \Rightarrow B = -\frac{1}{2}$.

Hence we have

$$\frac{x^4+x}{x^4-1} = 1 + \frac{1}{2(x-1)} - \frac{x+1}{2(x^2+1)},$$

and so

$$\begin{aligned}\int \frac{x^4+x}{x^4-1}dx &= \int 1dx + \frac{1}{2}\int\frac{1}{x-1}dx - \frac{1}{2}\int\frac{x}{x^2+1}dx - \frac{1}{2}\int\frac{1}{x^2+1}dx\\ &= x + \frac{1}{2}\ln|x-1| - \frac{1}{4}\ln(x^2+1) - \frac{1}{2}\arctan x + C.\end{aligned}$$

4. (12 points)

(a) Determine whether $\displaystyle\int_1^\infty \frac{2+\cos x}{x\ln x}\,dx$ converges or diverges; give complete reasoning.

(6 points) First note that $-1 \le \cos x \le 1$ and hence $1 \le 2+\cos x \le 3$. Therefore, we have

$$0 < \frac{1}{x\ln x} \le \frac{2+\cos x}{x\ln x} \le \frac{3}{x\ln x} \qquad \text{for } x > 1.$$

With the Comparison Theorem in mind, this suggests that the fate of our integral is closely tied to that of $\displaystyle\int_1^\infty \frac{1}{x\ln x}dx$. To compute the latter, which is improper for two reasons, we split:

$$\int_1^\infty \frac{1}{x\ln x}dx = \int_1^2 \frac{1}{x\ln x}dx + \int_2^\infty \frac{1}{x\ln x}dx.$$

Let $u = \ln x$, then $du = \frac{1}{x}dx$. Since $u \to \infty$ as $x \to \infty$, we have

$$\int_1^2 \frac{1}{x\ln x}dx + \int_2^\infty \frac{1}{x\ln x}dx = \int_0^{\ln 2} \frac{du}{u} + \int_{\ln 2}^\infty \frac{du}{u}.$$

In fact each of the latter two integrals diverges (so we can stop after computing one of them):

$$\lim_{t\to 0^+} \int_t^{\ln 2} \frac{du}{u} = \lim_{t\to 0^+} [\ln|u|]_t^{\ln 2} = \lim_{t\to 0^+} (\ln\ln 2 - \ln t) = -(-\infty)$$
$$\lim_{s\to\infty} \int_{\ln 2}^s \frac{du}{u} = \lim_{s\to\infty} [\ln|u|]_{\ln 2}^s = \lim_{s\to\infty} (\ln s - \ln\ln 2) = \infty.$$

Therefore, $\displaystyle\int_1^\infty \frac{1}{x\ln x}\,dx$ diverges, and we can now apply the Comparison Theorem: since

$$0 < \frac{1}{x\ln x} \le \frac{2+\cos x}{x\ln x} \qquad \text{for } x > 1$$

as we saw above, we can conclude that $\displaystyle\int_1^\infty \frac{2+\cos x}{x\ln x}dx$ also $\boxed{\text{diverges}}$.

(b) Determine whether $\displaystyle\int_0^1 \frac{\sqrt{\sin x}}{x}\,dx$ converges or diverges; give complete reasoning.

(6 points) Use the inequality $0 < \sin x \le x$ for $0 < x \le 1$. We have

$$0 < \sqrt{\sin x} \le \sqrt{x}$$
$$\iff 0 < \frac{\sqrt{\sin x}}{x} \le \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}.$$

Now $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}dx$ converges because

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+} 2\sqrt{x}\Big|_t^1 = \lim_{t\to 0^+} \left(2 - 2\sqrt{t}\right) = 2$$

It now follows by the Comparison Theorem that $\displaystyle\int_0^1 \frac{\sqrt{\sin x}}{x}dx$ also $\boxed{\text{converges}}$.

5. (9 points) A molasses tank has exploded, spreading sticky goo across the ground in all directions. The mass density $\rho(z)$ of molasses (measured in kilograms per square meter) at each point on the ground near the tank is assumed to depend only on the distance z (in meters) from the tank.

   (a) Write an integral involving the function ρ which expresses the total mass of all molasses that lies on the ground within a 60-meter radius of the tank.

   (3 points) Let z be the radial distance to the tank. We slice the region into very thin rings centered at the tank, each of radius z and width Δz (meters), so that the area of a slice is approximately $2\pi z \Delta z$ (square meters) and its mass is approximately $2\pi z \rho(z) \Delta z$ (kilograms). In the limit as $\Delta z \to 0$, the total mass is given by $\boxed{\int_0^{60} 2\pi z \rho(z)\, dz}$.

   (b) The density was measured experimentally at several points on the ground near the tank. The values collected are:

z	(m)	0	10	20	30	40	50	60
$\rho(z)$	(kg/m^2)	3	2.8	2.5	2	1.5	0.8	0.2

   Use the Midpoint Rule to estimate the total mass of molasses as expressed by your integral in (a). Use as much of the data in the table above as possible, and do not simplify your answer.

   (3 points) The largest number of rectangles we can take for this data is $n = 3$ (for example we do not know the values of ρ for $z = 5, 15, 25...$), so that the midpoints of our three intervals are $r = 10, 30, 50$ and $\Delta z = 20$. The midpoint approximation is

$$\int_0^{60} 2\pi z \rho(z)\ dz \approx M_3 = 20\Big(2\pi\Big)\Big((10)\rho(10) + (30)\rho(3) + (50)\rho(50)\Big) \text{ kg}$$
$$= \boxed{40\pi\Big((10)(2.8) + (30)(2) + (50)(.8)\Big) \text{ kg}}$$

   (c) Use Simpson's Rule to estimate the total mass of molasses as expressed by your integral in (a). Use all the data in the table above, and do not simplify your answer.

   (3 points) Here we may take $n = 6$ and $\Delta z = 10$.

$$\int_0^{60} 2\pi z \rho(z)\ dz \approx S_6$$

$$= \boxed{\left(\frac{20}{3}\right)(2\pi)\Bigg(0(3) + 4(10)(2.8) + 2(20)(2.5) + 4(30)(2) + 2(40)(1.5) + 4(50)(0.8) + 60(0.2)\Bigg)}$$

6. (14 points) Let $f(x) = e^{-e^x}$. In this problem, we study approximations of the following integral:

$$\int_0^1 e^{-e^x}\,dx$$

(a) Write an algebraic expression involving only numbers that approximates the above integral using the Trapezoidal Rule with 4 subintervals. You do *not* have to simplify this expression.

(4 points) The length of each subinterval is $\Delta x = \frac{1-0}{4} = \frac{1}{4}$, and the points at which we need to evaluate the function are:

$$x_i = 0 + i\Delta x = \frac{i}{4} \qquad \text{for } 0 \le i \le 4$$

The approximation is now given by

$$\begin{aligned} T_4 &= \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right] \\ &= \frac{\frac{1}{4}}{2}\left(e^{-e^{\frac{0}{4}}} + 2e^{-e^{\frac{1}{4}}} + 2e^{-e^{\frac{2}{4}}} + 2e^{-e^{\frac{3}{4}}} + e^{-e^{\frac{4}{4}}}\right) \\ &= \frac{1}{8}\left(e^{-1} + 2e^{-e^{\frac{1}{4}}} + 2e^{-e^{\frac{1}{2}}} + 2e^{-e^{\frac{3}{4}}} + e^{-e}\right) \end{aligned}$$

(b) Compute $f''(x)$, and show that

$$0 \le f''(x) \le \frac{2}{3} \qquad \text{for all } x \text{ in } [0, 1].$$

(3 points) The first derivative is:

$$f'(x) = -e^x e^{-e^x}$$

and the second derivative is:

$$f''(x) = -e^x e^{-e^x} + e^{2x} e^{-e^x} = \boxed{e^{x-e^x}(e^x - 1)}$$

Next we show that $f''(x) \ge 0$, i.e., that f'' is non-negative on $[0, 1]$. We know that the exponential function only takes positive values and therefore

$$e^{x-e^x} > 0$$

for any real number x. Furthermore, the exponential function is increasing and thus

$$e^x \ge e^0 = 1 \qquad \text{for } x \ge 0$$

and so

$$e^x - 1 \ge 0 \qquad \text{for } x \ge 0$$

We conclude that the product $f''(x) = e^{x-e^x}(e^x - 1)$ is non-negative for $x \ge 0$.

Finally, we prove that $f''(x) \le \frac{2}{3}$ for $x \in [0,1]$. First observe that

$$\frac{d}{dx}(x - e^x) = 1 - e^x$$

is negative for $x > 0$. Thus $x - e^x$ is decreasing for $x \ge 0$. Since the exponential function is increasing, it follows that e^{x-e^x} is decreasing for $x \ge 0$. In particular

$$e^{0-e^0} \ge e^{x-e^x} \iff e^{-1} \ge e^{x-e^x}$$

for $x \ge 0$. Moreover, we have already seen $e^{x-e^x} > 0$. Putting both inequalities together, we conclude:

$$0 < e^{x-e^x} \le e^{-1} \qquad \text{for } x \ge 0$$

Furthermore, note that since the exponential function is increasing we have that

$$e^0 - 1 \le e^x - 1 \le e^1 - 1 \iff 0 \le e^x - 1 \le e - 1$$

for $0 \le x \le 1$. Putting together the last two inequalities, we conclude about the second derivative of f:

$$f''(x) = e^{x-e^x}(e^x - 1) \le e^{-1}(e - 1) = 1 - \frac{1}{e} \le 1 - \frac{1}{3} = \frac{2}{3}$$

where we have used that $e \le 3$.

Grading remarks:

- Finding a correct expression for f'' was worth 0.5 points.
- Proving that the second derivative of f is non-negative was worth 1 point.
- Most people attempted to prove the required inequalities by simply calculating $f''(0)$ and $f''(1)$ and assuming those values were the minimum and maximum of f'' on $[0,1]$. That would be relevant and true if f'' were monotonic (increasing or decreasing) on $[0,1]$, which it is not.

(c) Using the fact stated in part (b), show that your Trapezoidal Rule approximation in part (a) is accurate to within $\frac{1}{250}$. (You may cite the fact of part (b) even if you did not prove it.) In addition, explain whether the approximation of part (a) gives an overestimate or underestimate of the integral, or whether it is impossible to tell.

(4 points) We know that the error of the approximation found in part (a) using the Trapezoidal Rule with 4 subintervals

$$E_T = \int_0^1 f(x)\,dx - T_4$$

is bounded by:

$$|E_T| \le \frac{K_2(1-0)^3}{12 \cdot 4^2}$$

where K_2 is any constant such that $|f''(x)| \le K_2$ for all $x \in [0,1]$. By (b), we can take $K_2 = \frac{2}{3}$, and so

$$|E_T| \le \frac{\frac{2}{3}}{12 \cdot 4^2} = \frac{1}{288} < \frac{1}{250}$$

Finally, since $f''(x) \ge 0$ for $0 \le x \le 1$, the function f is concave up on $[0,1]$. We conclude that the Trapezoidal Rule gives an overestimate of the integral $\int_0^1 f(x)\,dx$.

(d) Find a value of n which guarantees that a Trapezoidal Rule approximation of the above integral using n subintervals is accurate to within 10^{-10}. Your final answer should give a valid n in simplified form, and be fully justified, but it need not be optimal in any sense. (You may again apply the fact of part (b), even if you did not prove it.)

(3 points) The error of the Trapezoidal Rule approximation to the integral $\int_0^1 f(x)\,dx$ using n subintervals has error

$$E_T = \int_0^1 f(x)\,dx - T_n$$

which is bounded by

$$|E_T| \le \frac{K_2(1-0)^3}{12n^2} = \frac{K_2}{12n^2}$$

where we can take $K_2 = \frac{2}{3}$ as seen in (c). To guarantee that $|E_T| \le 10^{-10}$, we can demand

$$\frac{K_2}{12n^2} \le 10^{-10}$$

Solving for $n > 0$ we obtain:

$$\begin{aligned} \frac{\frac{2}{3}}{12n^2} \le 10^{-10} &\iff \frac{1}{18n^2} \le 10^{-10} \\ &\iff n^2 \ge \frac{10^{10}}{18} \\ &\iff n \ge \sqrt{\frac{10^{10}}{18}} = \frac{10^5}{\sqrt{18}} \end{aligned}$$

Since $\sqrt{18} > 4$, one value that satisfies the above condition is $\dfrac{10^5}{4} > \dfrac{10^5}{\sqrt{18}}$. Thus any value for n greater than or equal to $\frac{10^5}{4} = 25000$, for example $\boxed{n = 25000}$, gives an approximation to $\int_0^1 f(x)\,dx$ accurate to within 10^{-10}.

Grading remarks: As stated in the question, a complete answer required a fully simplified positive integer value for n.

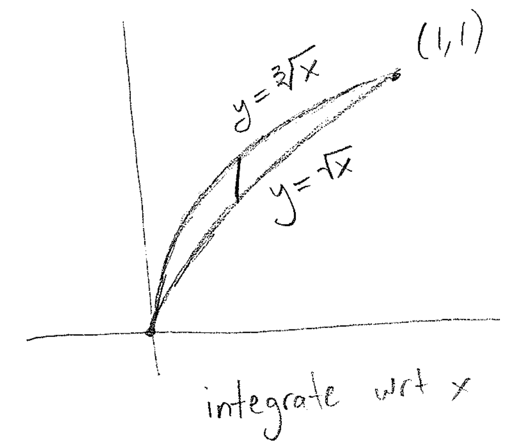
7. (10 points) Let R be the bounded region enclosed by the curves $y = \sqrt{x}$ and $y = x^{1/3}$ in the first quadrant.

   (a) Set up two distinct integrals, each in terms of a single variable, representing the area of R. For each, justify your answer by drawing a picture and marking a sample slice. Don't evaluate either integral.

(5 points) The points of intersection of the two curves are $(0,0)$ and $(1,1)$.

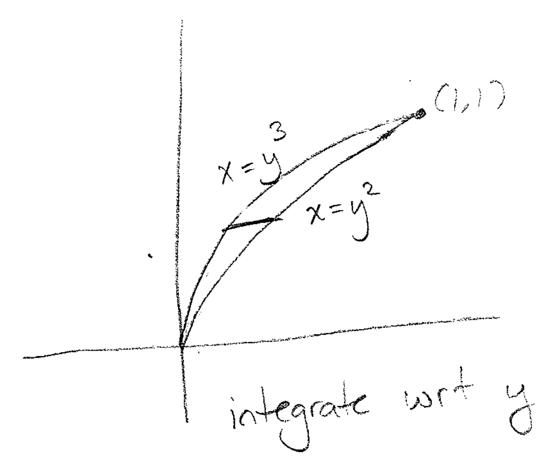
For $0 \le x \le 1$, we have $x^{1/3} \ge x^{1/2}$. Taking slices perpendicular to the x-axis gives that the area is equal to

$$A = \int_0^1 \left(x^{1/3} - x^{1/2}\right)\, dx.$$



We solve for x in terms of y; note that for $0 \le y \le 1$, we have $y^2 \ge y^3$. Taking slices perpendicular to the y-axis gives that the area is equal to

$$A = \int_0^1 \left(y^2 - y^3\right)\, dy.$$



(b) Set up two distinct integrals, each in terms of a single variable, which represent the volume of the solid obtained by rotating R about the line $x = 1$. Justify your answer by drawing pictures, labeling sample slices, and citing the methods used. Don't evaluate either integral.

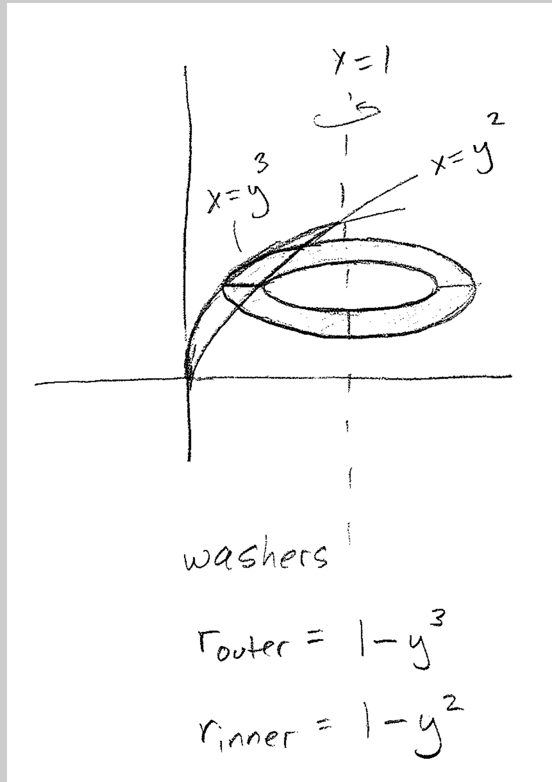
(5 points)

Taking slices perpendicular to the axis of rotation and using the washer method gives that the area of a slice at height y is

$$\begin{aligned} A(y) &= \pi\left(r_{outer}^2 - r_{inner}^2\right) \\ &= \pi\left((1-y^3)^2 - (1-y^2)^2\right) \end{aligned}$$

so that the total volume is given by

$$V = \int_0^1 \pi\left((1-y^3)^2 - (1-y^2)^2\right)\, dy.$$

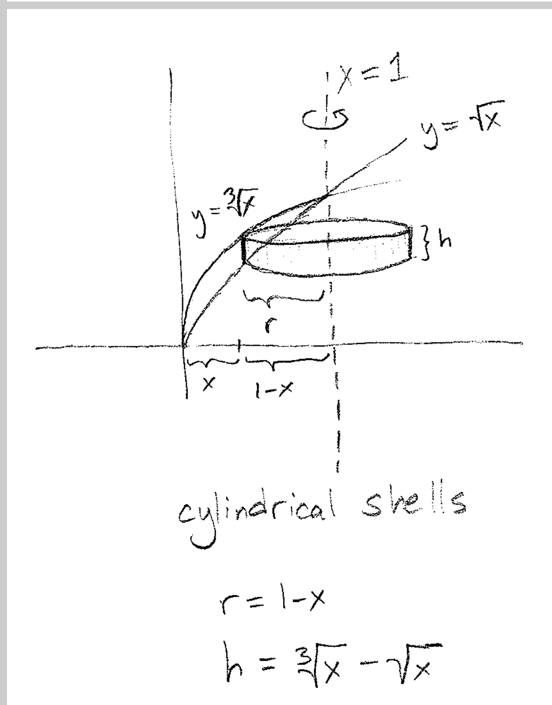


Taking slices parallel to the axis of rotation and using the cylindrical shell method gives that the area of a slice at x is

$$\begin{aligned} A(x) &= 2\pi r h \\ &= 2\pi(1-x)(x^{1/3} - x^{1/2}) \end{aligned}$$

so that the total volume is given by

$$V = \int_0^1 2\pi(1-x)(x^{1/3} - x^{1/2})\, dx.$$

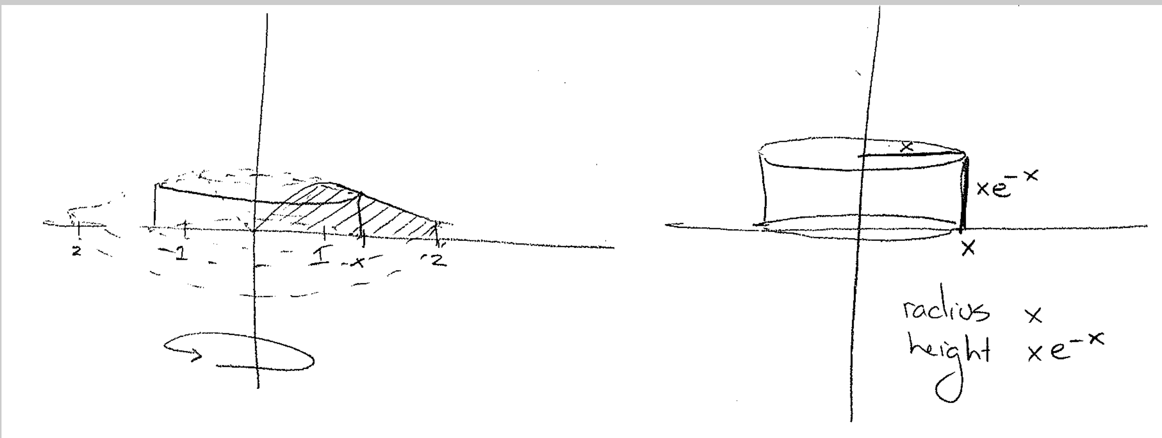


8. (11 points) Consider the region R in the xy-plane below the curve $y = xe^{-x}$ and above the portion of the x-axis with $0 \leq x \leq 2$.

   (a) Set up, but do not yet evaluate, an integral in terms of a single variable which represents the volume of the solid of revolution obtained by rotating R about the y-axis. Justify your answer by drawing a picture, labeling a sample slice, and citing the method used.

   (3 points) We want to rotate about the y-axis and our function is in x, and not obviously rewritable in terms of y. So we must use Shell method with radius x, height $xe^{-x} - 0$. Our volume is

$$2\pi \int_0^2 x(xe^{-x})dx = \boxed{2\pi \int_0^2 x^2 e^{-x}\, dx}.$$



   (b) Evaluate the integral of part (a), showing all your steps.

   (5 points) We'll use integration by parts twice. For the first time we let $u = x^2$, $dv = e^{-x}dx$, so we get $du = 2xdx$, $v = -e^{-x}$:

$$\begin{aligned}2\pi \int_0^2 x^2 e^{-x} dx &= 2\pi \left(\left[x^2(-e^{-x})\right]_0^2 - \int_0^2 (-e^{-x})2xdx \right) \\ &= 2\pi \left(\left[-x^2 e^{-x}\right]_0^2 + \int_0^2 e^{-x}2xdx \right)\end{aligned}$$

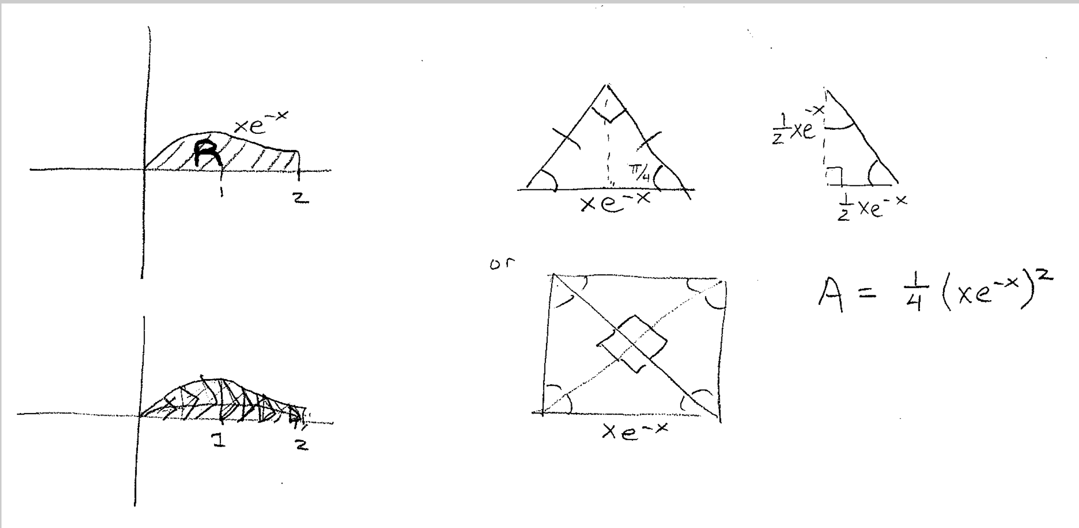
   So we need to use integration by parts again. This time $u = 2x$, $dv = e^{-x}dx$, which means $du = 2dx$, $v = -e^{-x}$:

$$\begin{aligned}2\pi \left(\left[-x^2 e^{-x}\right]_0^2 + \int_0^2 e^{-x}2xdx \right) &= 2\pi \left(\left[-x^2 e^{-x}\right]_0^2 + \left[2x(-e^{-x})\right]_0^2 - \int_0^2 -e^{-x}2dx \right) \\ &= 2\pi \left(\left[-x^2 e^{-x}\right]_0^2 + \left[-2xe^{-x}\right]_0^2 + \int_0^2 2e^{-x}dx \right) \\ &= 2\pi \left(\left[-x^2 e^{-x}\right]_0^2 + \left[-2xe^{-x}\right]_0^2 + \left[-2e^{-x}\right]_0^2 \right) \\ &= 2\pi \left(-4e^{-2} - 4(e^{-2}) - 2e^{-2} + 0 + 0 + 2 \right) \\ &= 2\pi \left(-10e^{-2} + 2 \right) \\ &= 4\pi - 20\pi(e^{-2})\end{aligned}$$

   Note that $e^{-2} < 1/5$, so this answer is in fact positive as we expect since it's a volume.

(c) Suppose a three-dimensional solid V has the following properties: it has R as its base; and each cross-section of V perpendicular to the x-axis is an isosceles right triangle with hypotenuse along the base. Set up, but do not evaluate, an integral that gives the volume of V.

(3 points) We observe that the area of a right isosceles triangle with hypoteneuse y is $\frac{1}{4}y^2$. There are many ways to show this. Notice this is a $45-45-90$ triangle, so the non-hypoteneuse sides are length $y/\sqrt{2}$ in length, so $\frac{1}{2}(b)(h) = \frac{1}{2}(y/\sqrt{2})^2$. Alternatively, spliting the triangle along the altitude perpendicular to the hypoteneuse still gives a $45-45-90$ triangle with sides $y/2$, so the area of the original triangle is $\frac{1}{2}y\left(\frac{y}{2}\right)$. Using trigonometry (since we know the angles) also works to find side lengths. Finally, notice that four identical triangles can be arranged to make a square of side length y, and thus of area y^2.



Now if we slice our solid V (with base R from the original problem statement) perpendicular to the x-axis at coordinate x, we find that the resulting slice of R has length xe^{-x}. This means that the cross-section of V has area $A(x) = \frac{1}{4}(xe^{-x})^2$. So the formula for V is

$$V = \int_0^2 A(x)\, dx = \int_0^2 \frac{1}{4}(xe^{-x})^2\, dx.$$