Solutions to Math 42 Final Exam — March 14, 2011

1. (10 points) Let R be the region in the xy-plane below the curve $y = \dfrac{1}{1+x^2}$ and above the line $y = \dfrac{1}{2}$.

   (a) Suppose S_1 is the solid generated by rotating R about the x-axis. Set up two different integrals, each in terms of a single variable, representing the volume of S_1. Cite the method used in each case, but don't evaluate either integral.

   Washer method (3 points):
   First we find the intersection points of the curves: $1/2 = 1/(1+x^2) \iff 2 = 1+x^2 \iff x = \pm 1$. We'll take slices perpendicular to the x-axis and integrate with respect to x from -1 to 1. The volume of such a slice is given by

   $$V_{slice} = \pi(r_{outer}^2 - r_{inner}^2)\Delta x$$

   where $r_{outer} = y = \frac{1}{1+x^2}$ and $r_{inner} = 1/2$.
   Therefore an integral representing the volume of the solid is

   $$\int_{-1}^{1} \pi\left(\left(\frac{1}{1+x^2}\right)^2 - \left(\frac{1}{2}\right)^2\right)\, dx.$$

   Cylindrical Shells method(4 points):
   $y = \frac{1}{1+x^2}$ is increasing from $x = -1$ (where $y = 1/2$) to $x = 0$ (where $y = 1$), and decreasing from $x = 0$ to $x = 1$ (where again $y = 1/2$), so we may take cylindrical shell slices parallel to the x-axis and integrate with respect to y from $1/2$ to 1. The volume of such a slice is given by

   $$V_{slice} = 2\pi r h \Delta y.$$

   We find the height h by solving for x in terms of y: $y = 1/(1+x^2) \implies x^2 = 1/y - 1 \implies x = \pm\sqrt{1/y - 1}$. Since the curve is symmetric about the y axis, we have $h = 2\sqrt{1/y-1}$. Since $r = y$, an integral representing the volume of the solid is

   $$\int_{1/2}^{1} 2\pi y(2\sqrt{(1/y)-1})\, dy.$$

   (b) Suppose S_2 is the solid whose base is R and whose cross-sections perpendicular to the x-axis are all squares. Set up an integral representing the volume of S_2, but don't evaluate the integral.

   (3 points) A square slice perpendicular to the x-axis has side length $y - 1/2 = \frac{1}{1+x^2} - 1/2$, and its area is the square of its side length, so an integral representing the volume of the solid is

   $$\int_{-1}^{1} \left(\frac{1}{1+x^2} - \frac{1}{2}\right)^2\, dx.$$

2. (12 points)

(a) Evaluate $\displaystyle\int_1^e \frac{1}{x\ln x}\,dx$ or explain why its value does not exist; show all reasoning.

(6 points) Note $1/(x\ln x)$ "blows up" at $x = 1$, therefore the integral $\int_1^e 1/(x\ln x)\,dx$ is improper and we have

$$\int_1^e \frac{dx}{x\ln x} = \lim_{t\to 1^+}\int_t^e \frac{dx}{x\ln x}$$

We let $u = \ln x$, then $du = (1/x)dx$. Then, the above integral is equal to,

$$\begin{aligned}\lim_{t\to 1^+}\int_{\ln t}^1 \frac{du}{u} &= \lim_{t\to 1^+}\left[\ln|u|\right]_{\ln t}^1 \\ &= \lim_{t\to 1^+}\left[\ln 1 - \ln|\ln t|\right] \\ &= -\lim_{t\to 1^+}\ln|\ln t|\end{aligned}$$

Since $\ln 1 = 0$. Note that $|\ln t| \to 0$ as $t \to 1^+$, hence $\ln|\ln t| \to -\infty$ as $t \to 1^+$. It follows that the limit above is equal to ∞. Consequently the value of the integral $\int_1^e 1/(x\ln x)\,dx$ does not exist.

(b) Determine all values of p for which the integral $\displaystyle\int_1^e \frac{1}{x(\ln x)^p}\,dx$ converges, and evaluate the integral for those values of p.

(6 points) As before the integral $\int_1^e 1/(x(\ln x)^p))\,dx$ is improper, and we have,

$$\int_1^e \frac{dx}{x(\ln x)^p} = \lim_{t\to 1^+}\int_t^e \frac{dx}{x(\ln x)^p}$$

We let $u = \ln x$, then $du = (1/x)\,dx$. We can assume that $p \neq 1$, since we already know by part a) that the integral is undefined when $p = 1$. So if $p \neq 1$ then the integral in the previous equation is equal to,

$$\begin{aligned}\lim_{t\to 1^+}\int_{\ln t}^1 \frac{du}{u^p} &= \lim_{t\to 1^+}\left[\frac{u^{-p+1}}{-p+1}\right]_{\ln t}^1 \\ &= \lim_{t\to 1^+}\left[\frac{1}{1-p} - \frac{(\ln t)^{1-p}}{1-p}\right] \\ &= \frac{1}{1-p} - \lim_{t\to 1^+}\frac{(\ln t)^{1-p}}{1-p}\end{aligned}$$

Note that $\ln t \to 0$ as $t \to 1^+$. Therefore $(\ln t)^{1-p}/(1-p) \to 0$ as $t \to 1^+$ if the exponent $1-p$ is positive. Conversely if $1 - p < 0$ then $(\ln t)^{1-p}/(1-p)$ is undefined as $t \to 1^+$ (because we have a $1/0$ type situation). We conclude that the integral converges for all $p < 1$ (and only for those!) and has the value $1/(1-p)$ for those exponents.

3. (12 points) For this problem, use the following information:

- If f is a normal ("bell-shaped" or "Gaussian") probability density function, then f has the general form

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

- A partial list of approximate values of the function

$$P(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt \quad \text{is given at right:}$$

$P(0.25) \approx 0.60$	$P(1.75) \approx 0.960$
$P(0.5) \approx 0.69$	$P(2.0) \approx 0.977$
$P(0.75) \approx 0.77$	$P(2.25) \approx 0.988$
$P(1.0) \approx 0.84$	$P(2.5) \approx 0.994$
$P(1.25) \approx 0.89$	$P(2.75) \approx 0.997$
$P(1.5) \approx 0.93$	$P(3.0) \approx 0.999$

(a) A subatomic particle is in the ground state of a quantum harmonic oscillator; its position x along a line is given by the normal probability density function

$$f_0(x) = \frac{1}{2\sqrt{2\pi}} e^{-x^2/8}$$

What is the approximate probability that the particle's position is in the interval $[-1, 0]$? Your answer should be a number; justify it by writing an integral expression that represents this probability and showing how to find its value.

(4 points)

$$\begin{aligned}
\Pr(-1 \le X \le 0) &= \int_{-1}^{0} \frac{1}{2\sqrt{2\pi}} e^{-x^2/8}\,dx \\
&= \int_{-1/2}^{0} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz && (z = \tfrac{x}{2}) \\
&= \int_{0}^{1/2} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz && \text{(even integrand)} \\
&= P(1/2) - P(0) \\
&\approx 0.69 - 0.5 \\
&= 0.19.
\end{aligned}$$

Comments: Question was generally well done. Points were deducted for not justifying steps using integrals.

(b) Use integration by parts to show that

$$\int x^2 e^{-x^2/8}\,dx = -4xe^{-x^2/8} + \int 4e^{-x^2/8}\,dx$$

(4 points) Set

$$du = xe^{-x^2/8} \implies u = -4e^{-x^2/8}$$
$$v = x \implies dv = 1.$$

Then

$$\int (x)(xe^{-x^2/8})\,dx = (-4e^{-x^2/8})(x) - \int (-4e^{-x^2/8})(1)\,dx,$$

as required.

Comments: Most people didn't choose u and v in a way that would lead to a solution. There were also frequent assertions that if $dv = e^{-x^2/8}$, then $v = -\frac{4}{x}e^{-x^2/8}$, but this is false by the Chain Rule.

(c) The particle absorbs a photon and transitions to the next energy level. The probability density function for the particle's position now becomes

$$f_1(x) = \frac{x^2}{8\sqrt{2\pi}}e^{-x^2/8}$$

Use the information provided in part (b) and in the preceding chart to compute the approximate probability that the particle is now in the interval $[0, 1.5]$. (Give your answer as a expression involving only numbers, but do not simplify your answer.)

(4 points)

$$\begin{aligned}\int_0^{1.5} \tfrac{x^2}{8\sqrt{2\pi}}e^{-x^2/8}\,dx &= \tfrac{1}{8\sqrt{2\pi}}[-4xe^{-x^2/8}]_0^{1.5} + \tfrac{1}{8\sqrt{2\pi}}\int_0^{1.5} 4e^{-x^2/8}\,dx \\ &= -\frac{3}{4\sqrt{2\pi}}e^{-9/32} + \int_0^{1.5} \tfrac{1}{2\sqrt{2\pi}}e^{-x^2/8}\,dx \\ &= -\frac{3}{4\sqrt{2\pi}}e^{-9/32} + \int_0^{0.75} \tfrac{1}{\sqrt{2\pi}}e^{-z^2/2}\,dz \\ &= -\frac{3}{4\sqrt{2\pi}}e^{-9/32} + P(0.75) - P(0) \\ &\approx -\frac{3}{4\sqrt{2\pi}}e^{-9/32} + 0.27.\end{aligned}$$

Comments: Many people had trouble relating this integral to the one in (b).

4. (10 points) Determine whether each of the series below converges or diverges. Indicate clearly which tests you use and how you apply them.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n - 2^n}$

(5 points) This converges; many tests work.

Ratio Test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{3^n - 2^n}{3^{n+1} - 2^{n+1}}\right| = \lim_{n\to\infty} \frac{3^n - 2^n}{3^{n+1} - 2^{n+1}} \left(\frac{1/3^{n+1}}{1/3^{n+1}}\right)$$

$$= \lim_{n\to\infty} \frac{\frac{1}{3} - \frac{1}{3}\left(\frac{2}{3}\right)^n}{1 - \left(\frac{2}{3}\right)^{n+1}} = \frac{\frac{1}{3} - 0}{1 - 0} = \frac{1}{3}$$

where in the second to last step we have used that $0 < 2/3 < 1$ implies $\lim_{n\to\infty}(2/3)^n = 0$. Since $1/3 < 1$, the Ratio test tells us that the series is convergent.

Limit Comparison Test: The dominant term in the denominator is 3^n, so we compare the series to

$$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = \sum_{n=1}^{\infty} b_n.$$

This is a geometric series with $|r| = 1/3 < 1$, and so it converges. The terms a_n and b_n are positive for all n, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3^n}{3^n - 2^n} \left(\frac{1/3^n}{1/3^n}\right) = \lim_{n\to\infty} \frac{1}{1 - (2/3)^n} = 1$$

where again we have used $0 < 2/3 < 1$ implies $\lim_{n\to\infty}(2/3)^n = 0$. Since 1 is finite and greater than 0, the limit comparison test tells us that because $\sum b_n$ converges, $\sum a_n$ converges also.

Comparison Test: We may compare the series to:

$$\sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=2}^{\infty} c_n.$$

This is a geometric series with $|r| = 1/2 < 1$, and so it converges. For $n \geq 2$, we have

$$0 < \frac{1}{3^n - 2^n} < \frac{1}{2^n},$$

because for $n \geq 2$, we have $3^2 = 9 > 8 = 2 \cdot 2^2$ and $3^{n-2} > 2^{n-2}$ imply

$$\begin{aligned} & (3^2)3^{n-2} > (2 \cdot 2^2)2^{n-2} \\ \Longrightarrow\quad & 3^n > 2 \cdot 2^n = 2^n + 2^n \\ \Longrightarrow\quad & 3^n - 2^n > 2^n. \end{aligned}$$

The terms a_n and c_n are both positive, so by the comparison test, $\sum_{n=2}^{\infty} a_n$ converges. This implies that $\sum_{n=1} a_n$ also converges.

Grading note: It is also true that $1/(3^n - 2^n) < 1/n^2$, so it works to compare $\sum a_n$ to $\sum 1/n^2$, which converges because it is a p-series with $p = 2 > 1$. However in a complete solution to this problem, some justification must be given for the fact that $n^2 < 3^n - 2^n$; it is not enough to check for a few values of n.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(n+1)!}$

(5 points)

$$\sum_{n=1}^{\infty} \frac{n!}{(n+1)!} = \sum_{n=1}^{\infty} \frac{1}{n+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

This is the tail of the harmonic series (i.e. the p-series with $p = 1$, without the first term 1), so it diverges.

5. (12 points) For each of the following questions, give an example of a power series in x with the specified property or properties; for full credit, your answer should be given using sigma notation. *You do not have to justify your answers.* (Please treat each part as independent from the others; properties do not carry over from one part to the next.)

(a) The power series is centered at $x = -1$.

(2 points) For example, $\sum_{n=0}^{\infty}(x+1)^n$.

(b) The power series is centered at $x = 0$ and has radius of convergence ∞.

(3 points) For example, $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ (the MacLaurin series for e^x, which we've seen many times has radius of convergence ∞).

(c) The power series is centered at $x = 2$ and has radius of convergence equal to 3.

(3 points) Any such series converges when $|x-2| < 3$ and diverges when $|x-2| > 3$; put another way, we want convergence when $\left|\frac{x-2}{3}\right| < 1$ and divergence when $\left|\frac{x-2}{3}\right| > 1$. Thus, if we let $L = \frac{x-2}{3}$, it would be sufficient for us to set up a geometric series with common ratio L: for example, $\sum_{n=0}^{\infty}\frac{(x-2)^n}{3^n}$.

(d) The power series has interval of convergence $[2, 6)$.

(4 points) Note that $x \in [2,6)$ if and only if $2 \le x < 6$, if and only if $-1 \le \frac{x-4}{2} < 1$. If we follow the approach of part (c) and let $L = \left|\frac{x-4}{2}\right|$, then we need a series that converges when $L < 1$ and diverges when $L > 1$. However, we now also care about the case $L = 1$: we want convergence when $x = 2$ and divergence when $x = 6$. So the geometric power series $\sum_{n=0}^{\infty}\frac{(x-4)^n}{2^n}$ won't quite work, since $x = 2$ gives $\sum_{n=0}^{\infty}(-1)^n$, which diverges.

But instead, let's try $\boxed{\sum_{n=0}^{\infty}\frac{1}{n+1}\frac{(x-4)^n}{2^n}}$. In this case, the ratio test still gives

$$L = \lim_{n\to\infty}\left|\frac{\frac{1}{n+2}\frac{(x-4)^{n+1}}{2^{n+1}}}{\frac{1}{n+1}\frac{(x-4)^n}{2^n}}\right| = \lim_{n\to\infty}\frac{n+1}{n+2}\frac{2^n}{2^{n+1}}\left|\frac{(x-4)^{n+1}}{(x-4)^n}\right| = \left|\frac{x-4}{2}\right|\cdot\lim_{n\to\infty}\frac{1+\frac{1}{n}}{1+\frac{2}{n}} = \left|\frac{x-4}{2}\right|,$$

so the radius of convergence is indeed 2 about the center 4. If $x = 6$, we obtain the divergent harmonic series: $\sum_{n=0}^{\infty}\frac{1}{n+1}\frac{(6-4)^n}{2^n} = \sum_{n=0}^{\infty}\frac{1}{n+1}$. And if $x = 2$, we obtain the alternating harmonic series: $\sum_{n=0}^{\infty}\frac{1}{n+1}\frac{(2-4)^n}{2^n} = \sum_{n=0}^{\infty}\frac{1}{n+1}\frac{(-2)^n}{2^n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}$. This series converges by the Alternating Series Test, because the positive values $\frac{1}{n+1}$ are decreasing to 0 as n approaches ∞. Thus, we have convergence if and only if $x = 2$ or $-1 \le \frac{x-4}{2} < 1$, i.e. if and only if $2 \le x < 6$.

6. (12 points) Consider the function $f(x) = e^x \cos x$.

(a) Find $T_3(x)$, the third-degree Taylor polynomial for f centered at 0. Show all the steps of your computation.

(4 points) The third degree Taylor polynomial is given by

$$T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3$$

Computing derivatives gives

$$\begin{cases} f'(x) = e^x \cos(x) - e^x \sin x & \Rightarrow f'(0) = 1 \\ f''(x) = (e^x \cos(x) - e^x \sin(x)) - (e^x \sin(x) + e^x \cos(x)) = -2e^x \sin(x) & \Rightarrow f''(0) = 0 \\ f'''(x) = -2e^x \sin(x) - 2e^x \cos(x) & \Rightarrow f'''(0) = -2 \end{cases}$$

We thus have

$$T_3(x) = 1 + (1)x + \left(\frac{0}{2!}\right)x^2 + \left(\frac{-2}{3!}\right)x^3 = \boxed{1 + x - \frac{x^3}{3}}$$

(b) Use T_3 to obtain an approximation for $e^{0.1}\cos(0.1)$. (You do not need to simplify your answer.)

(2 points) The function $T_3(x)$ evaluated at x values close to 0 gives us an approximation for $f(x)$. Thus

$$e^{0.1}\cos 0.1 = f(0.1) \approx T_3(0.1) = \boxed{1 + (0.1) - \frac{(0.1)^3}{3}}$$

(c) Compute the 4th derivative $f^{(4)}(x)$ of $f(x)$ and show that

$$f^{(4)}(x) = -4f(x) \quad \text{for all } x.$$

(2 points) In (b), we computed the third derivative, so we have

$$\begin{aligned} f'''(x) &= -2e^x \sin(x) - 2e^x \cos(x) \\ \Rightarrow f^{(4)}(x) &= (-2e^x \sin(x) - 2e^x \cos(x)) + (-2e^x \cos(x) + 2e^x \sin(x)) \\ &= -2e^x \cos(x) - 2e^x \cos(x) = -4e^x \cos(x) = -4f(x) \end{aligned}$$

(d) Use the statement of part (c) to show that the estimate of part (b) is accurate to within 5×10^{-5}. Precisely cite the steps of your reasoning. (You may use the fact that $2 \le e \le 3$.)

(4 points) Notice that e^x is an increasing function and $|\cos(x)| \le 1$. Thus, on the interval $[-0.1, 0.1]$ and using (c) gives

$$\begin{aligned} |f^{(4)}(x)| = |-4e^x \cos(x)| = 4e^x|\cos(x)| \le 4e^x(1) = 4e^x &\le 4e^{0.1} \\ &\le 4(3)^{0.1} \le 4(3) = 12 =: M \quad \text{for } x \text{ in } [-0.1, 0.1]. \end{aligned}$$

Hence, we have $|f^{(4)}(x)| \le M = 12$ for x in $[-0.1, 0.1]$, and by Taylor's inequality, we estimate

$$\begin{aligned} |f(0.1) - T_3(0.1)| = |R_3(0.1)| &\le \frac{M}{4!}|0.1|^4 = \frac{12}{4!}(10)^{-4} = \frac{1}{2}(10)^{-4} = 5 \times 10^{-5} \\ \Rightarrow |R_3(0.1)| &\le 5 \times 10^{-5} \end{aligned}$$

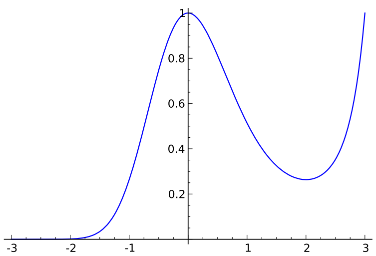
as desired.

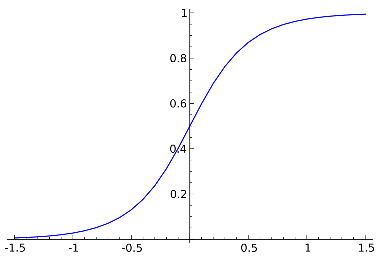
(Important note: In order to apply Taylor's inequality and to obtain an M value, we had to setup an interval that included 0 and 0.1. One cannot just plug in $x = 0.1$ into the function $|f^{(4)}(x)| = 4e^x|\cos(x)|$ to obtain M since Taylor's inequality requires us to overestimate $|f^{(4)}(x)|$ by a *number* M over an *interval* $[0-d, 0+d]$ for some $d > 0$.
Even if you picked your interval to be $[-0.1, 0.1]$, you still cannot, without justification, just plug in $x = 0.1$ into $|f^{(4)}(x)|$ since it's not obvious that this function is increasing on this interval, as $\cos(x)$ actually decreases on $[0, 0.1]$. In this case though, notice that

$$\frac{d}{dx}(e^x \cos(x)) = e^x(\cos(x) - \sin(x)) > 0$$

for x in $[-0.1, 0.1]$, so $|f^{(4)}(x)| = 4e^x \cos(x)$ for x in $[-0.1, 0.1]$ actually is increasing on this interval, and hence, one could pick $M = |f^{(4)}(0.1)| = 4e^{0.1}\cos(0.1)$ after giving this justification.

7. (15 points) Each of the curves below is a solution to exactly one differential equation in the chart at bottom. Match each curve with its equation; no justification is necessary.

I


II


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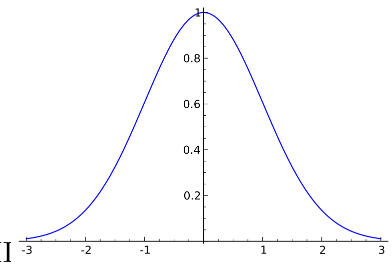
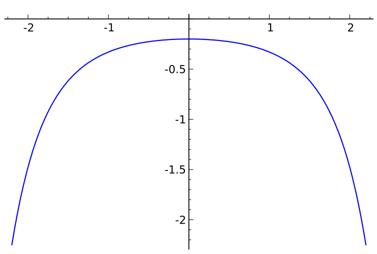


IV


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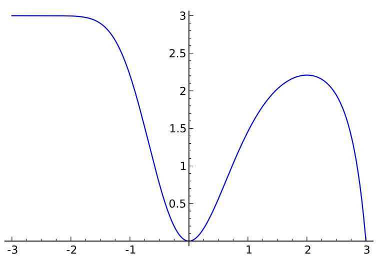

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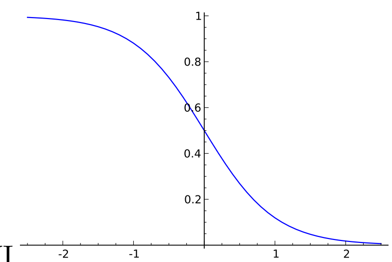
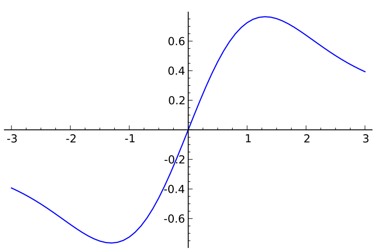
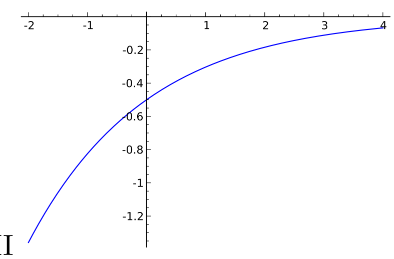


VII


VIII


Equation	I, II, III, IV, V, VI, VII, or VIII	Equation	I, II, III, IV, V, VI, VII, or VIII
$y' = (y-3)x(x-2)$	**V**	$y' = -xy$	**III**
$y' = 1 - xy$	**VII**	$y' = -\dfrac{y}{2}$	**VIII**
$y' = 2y(y-1)$	**VI**	$y' = x(x-2)y$	**I**
$y' = \sin(\pi y)$	**II**	$y' = xy$	**IV**

8. (12 points)

(a) Solve the initial value problem

$$\frac{dy}{dx} = x^2 e^{y+x}, \quad y(0) = 0.$$

(6 points) Any equilibrium solution would have to be identically zero (because of the $y(0) = 0$ requirement). However $y \equiv 0$ does not satisfy the equation,

$$\frac{dy}{dx} = x^2 \cdot e^{y+x}$$

because it leads to $0 = x^2 \cdot e^x$ for all x, which is impossible. Thus there are no equilibrium solutions, and we can solve the above differential equation by separation of variables. Separating the variables we obtain,

$$\frac{dy}{e^y} = x^2 \cdot e^x \, dx$$

Hence,

$$\int \frac{dy}{e^y} = \int x^2 \cdot e^x \, dx$$

We now deal with each side of the equation separately. On the one hand,

$$\int \frac{dy}{e^y} = -e^{-y} + C$$

On the other hand, integrating by parts (with $v = x^2,\ dv = 2x,\ du = e^x,\ u = e^x$),

$$\int x^2 \cdot e^x \, dx = x^2 \cdot e^x + C - 2 \int x \cdot e^x \, dx$$

Integrating by parts again (with $v = x,\ dv = 1,\ du = e^x, u = e^x$), we find

$$\int x \cdot e^x \, dx = xe^x + C - \int e^x \, dx = (x-1)e^x + C$$

Therefore, combining the previous two equations we obtain,

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - 2(x-1)e^x + C \\ &= x^2 e^x - 2xe^x + 2e^x + C \end{aligned}$$

Thus the equation,

$$\int \frac{dy}{e^y} = \int x^2 \cdot e^x \, dx$$

can be re-written as

$$-e^{-y} = x^2 e^x - 2xe^x + 2e^x + C$$

The condition $y(0) = 0$ gives $-1 = 0 - 2 \cdot 0 + 2 + C$, hence $C = -3$. Multiplying by -1 in the above equation and then taking the logarithm we obtain,

$$-y = \ln\left(-x^2 e^x + 2xe^x - 2e^x + 3\right)$$

Hence $y = -\ln(-x^2 e^x + 2xe^x - 2e^x + 3)$.

(b) Solve the initial value problem

$$y\,\frac{dy}{dx} + 2x = 0, \quad y(0) = -3.$$

(6 points) Again, an equilibrium solution would have to be identically equal to -3 (because of the condition $y(0) = -3$). However if $y \equiv -3$ is a solution to the differential equation then $-3 \cdot 0 + 2x = 0$ for all x which is clearly impossible. Therefore there are no equilibrium solutions and we can carry on with separation of variables. Separating the variables we obtain,

$$y\,dy = -2x\,dx$$

Integrating this gives,

$$\int y\,dy = -2\int x\,dx$$

Hence,

$$\frac{y^2}{2} = -2\frac{x^2}{2} + C$$

which after simplification becomes,

$$y = \pm\sqrt{-2x^2 + 2C}$$

Since $y(0) = -3$ this means that in $\pm$ we must have the $-$ sign. In addition this means that $-3 = -\sqrt{-2 \cdot 0^2 + 2C}$, hence $2C = 9$. We conclude that,

$$y = -\sqrt{9 - 2x^2}$$

is the solution to the differential equation.

9. (10 points) The air in a room with volume $500\,\mathrm{m}^3$ is permeated with a toxic substance. Let $x(t)$ denote the concentration, in $\mathrm{mg/m}^3$, of the toxic substance in the room after t minutes, and suppose that $x(0) = 10^{-4}\,\mathrm{mg/m}^3$. A scrubbing system removes air from the room at a rate of $2\,\mathrm{m}^3/\mathrm{min}$, *partially* cleans it, and quickly reintroduces it into the room at the same rate. Assume that at any time t, the concentration of the toxic substance in the cleaner air that is being reintroduced into the room is equal to $x(t)^{3/2}$.

(a) Write a differential equation for x.

(4 points) Let A(t) be the amount of toxic substance at time t in mg. Then

$$A'(t) = (\text{rate in}) - (\text{rate out}) = 2x(t)^{3/2} - 2x(t).$$

Since $A(t) = 500x(t)$, then we conclude that

$$x'(t) = \frac{1}{250}(x(t)^{3/2} - x(t)).$$

Comments: Very few individuals identified the difference between a concentration and an amount.

(b) Determine the equilibrium solutions of the differential equation from part (a).

(2 points) Equilibrium solutions satisfy $x'(t) = 0$. Using the differential equation in (a), this immediately implies that $x(t) = 0,\ 1$.

Comments: Question was done very well. Mistakes from (a) were carried.

(c) Use Euler's method with a step size of 1 minute to estimate $x(2)$, the concentration after 2 minutes. Show your steps, but you do *not* need to simplify your answer.

(4 points)

$$x(0) = 10^{-4}$$

$$x'(0) = \frac{1}{250}(x(0)^{3/2} - x(0)) = \frac{1}{250}(10^{-6} - 10^{-4})$$

$$x(1) = x(0) + x'(0) = 10^{-4} + \frac{1}{250}(10^{-6} - 10^{-4})$$

$$x'(1) = \frac{1}{250}(x(1)^{3/2} - x(1)) = \frac{1}{250}\left\{\left[10^{-4} + \frac{1}{250}(10^{-6} - 10^{-4})\right]^{3/2} - \left[10^{-4} + \frac{1}{250}(10^{-6} - 10^{-4})\right]\right\}$$

$$\begin{aligned} x(2) &= x(1) + x'(1) \\ &= 10^{-4} + \frac{1}{250}(10^{-6} - 10^{-4}) + \frac{1}{250}\left\{\left[10^{-4} + \frac{1}{250}(10^{-6} - 10^{-4})\right]^{3/2} - \left[10^{-4} + \frac{1}{250}(10^{-6} - 10^{-4})\right]\right\}. \end{aligned}$$

Comments: This problem was done very well. Mistakes were carried from (a).

10. (13 points) A population $P(t)$ (measured in thousands of beings) is roughly described at time t by a logistic model with proportionality constant $k = 1$ and carrying capacity 2.

(a) Write a differential equation for P.

(2 points) The population P verifies a logistic equation with relative growth rate 1 and carrying capacity 2:

$$P' = P\left(1 - \frac{P}{2}\right)$$

(b) Determine the equilibrium solutions of the equation in (a).

(2 points) The equilibrium solutions of the autonomous differential equation

$$P' = P\left(1 - \frac{P}{2}\right)$$

are the constant solutions. These nullify the right hand side:

$$\begin{aligned} &P\left(1 - \frac{P}{2}\right) = 0 \\ \Leftrightarrow\ &P = 0 \quad \text{or} \quad 1 - \frac{P}{2} = 0 \\ \Leftrightarrow\ &P = 0 \quad \text{or} \quad P = 2 \end{aligned}$$

Thus the equilibrium solutions are $P = 0$ and $P = 2$.

(c) Solve the initial value problem given by the initial condition $P(0) = 1$. What is the long-term behavior of the population?

(2 points) We want to solve the initial value problem

$$\begin{cases} P' = P\left(1 - \frac{P}{2}\right) \\ P(0) = 1 \end{cases} \tag{1}$$

From the classification of the solutions of the logistic equation, we know that the solution to (1) must be

$$P(t) = \frac{2}{1 + Ae^{-t}}$$

for some real number A — we have used that $P(0) = 1$ to exclude the constant solution $P = 0$. It remains to determine the value of A using the initial condition from (1):

$$\begin{aligned} &P(0) = 1 \\ \Leftrightarrow\ &\frac{2}{1 + Ae^{-0}} = 1 \\ \Leftrightarrow\ &2 = 1 + A \\ \Leftrightarrow\ &A = 1 \end{aligned}$$

In conclusion, the solution to the initial value problem (1) is

$$P(t) = \frac{2}{1 + e^{-t}} \qquad \text{for all } t$$

Regarding the long-term behavior of the population:

$$\lim_{t \to +\infty} P(t) = \lim_{t \to +\infty} \frac{2}{1 + e^{-t}} = \frac{2}{1 + 0} = 2$$

Thus the size of the population is expected to converge to 2000 individuals.

(d) A severe disease appears among the population. The mortality due to the disease is approximately constant and equal to 0.5 thousands of beings per unit time. Modify the differential equation from (a) to account for the increased mortality.

(2 points) The required modified logistic equation is

$$P' = P\left(1 - \frac{P}{2}\right) - \frac{1}{2} \tag{2}$$

where we have added the term $-\frac{1}{2}$ to account for the constant mortality due to the disease. Let us now observe that

$$\begin{aligned} P\left(1 - \frac{P}{2}\right) - \frac{1}{2} &= P - \frac{P^2}{2} - \frac{1}{2} \\ &= -\frac{1}{2}(P^2 - 2P + 1) \\ &= -\frac{1}{2}(P-1)^2 \end{aligned}$$

Thus equation (2) becomes:

$$P' = -\frac{1}{2}(P-1)^2 \tag{3}$$

(e) Solve the differential equation from (d), subject to the initial condition $P(0) = 2$. What is the long-term behavior of the population subject to the disease?

(2 points) We want to solve the initial value problem

$$\begin{cases} P' = -\frac{1}{2}(P-1)^2 \\ P(0) = 2 \end{cases} \tag{4}$$

The equation

$$P' = -\frac{1}{2}(P-1)^2$$

has the equilibrium solution

$$P = 1$$

Since the initial condition in (4) is $P(0) = 2$, we conclude that the solution to (4) does not equal 1 at any point (by uniqueness of the solution to initial value problems for the differential equation (3)). Thus the solution to (4) verifies:

$$\begin{aligned} & P' = -\frac{1}{2}(P-1)^2 \\ \Rightarrow\ & \frac{-2P'}{(P-1)^2} = 1 \\ \Rightarrow\ & \int \frac{-2P'}{(P-1)^2}\,dt = \int 1\,dt \\ \Rightarrow\ & \int \frac{-2}{(P-1)^2}\,dP = t + C \\ \Rightarrow\ & \frac{2}{P-1} = t + C \\ \Rightarrow\ & P - 1 = \frac{2}{t+C} \\ \Rightarrow\ & P = \frac{2}{t+C} + 1 \end{aligned}$$

We have thus obtained

$$P(t) = \frac{2}{t+C} + 1$$

for some real number C. It remains to find the value of the constant C by using the initial condition from (4):

$$\begin{aligned} & P(0) = 2 \\ \Leftrightarrow\ & \frac{2}{0+C} + 1 = 2 \\ \Leftrightarrow\ & \frac{2}{C} = 1 \\ \Leftrightarrow\ & C = 2 \end{aligned}$$

In conclusion, the solution to the initial value problem (4) is given by

$$P(t) = \frac{2}{t+2} + 1 \qquad \text{for } t > -2$$

Regarding the long-term behavior of the population:

$$\lim_{t\to+\infty} P(t) = \lim_{t\to+\infty} \left(\frac{2}{t+2} + 1 \right) = 0 + 1 = 1$$

Thus the size of the population subject to the disease is expected to converge to 1000 individuals.

11. (14 points) Two populations coexist within a territory, and are approximately modeled by the system of differential equations

$$\begin{cases} x' = 4x - xy \\ y' = -y + \frac{xy}{2} \end{cases}$$

(a) Describe the nature of the relationship between the two species: is it one of competition, cooperation, or predator and prey, and how can you tell? (If the relationship is predator and prey, make sure to explain how to tell which species is which.)

(3 points) Looking at the interaction terms "xy" in each growth rate, we see that an increases in species y decreases the growth rate (x') of species x, while an increase in species x increases the growth rate (y') of species y. This is consistent with a predator/prey model where x is the prey, and species y is the predator.

Important Note: Many mentioned that this is a predator/prey model where x is the prey and y is the predator simply because x grows in the absence of y, while y decays in the absence of x. This argument only works to determine which species is the predator/prey *once we already know that this is a predator/prey relationship* . This argument alone does not reveal a predator/prey relationship. Only the signs of the interaction terms "xy" will tell us what type of relationship this is; in this case, opposite signs as indicated by "$-xy$" and "$+(xy)/2$" tell us that we have a predator/prey relationship.

(b) For each species, describe what happens if the other is not present.

(2 points) In the absence of species y, we have $x' = 4x$, so species x experiences exponential growth. In the absence of species x, we have $y' = -y$ so species y decays exponentially.

(c) Find all equilibrium solutions of this system.

(3 points) For equilibrium, we must have $x' = 0$ and $y' = 0$ at the same time. Hence,

$$\begin{cases} 0 = x' = x\,(4 - y) & \Rightarrow x = 0 \ \text{ or } \ y = 4 \\ 0 = y' = y\left(-1 + \frac{x}{2}\right) & \Rightarrow y = 0 \ \text{ or } \ x = 2 \end{cases}$$

Thus, if $x = 0$, then $x' = 0 \Rightarrow y = 0$ in order that $y' = 0$. For the other case, if $y = 4$ then $x' = 0$ and we need $x = 2$ in order that $y' = 0$. Thus, the equilibrium solutions (x, y) are $(0, 0)$ and $(2, 4)$.

For easy reference, here again is the system:

$$\begin{cases} x' = 4x - xy \\ y' = -y + \frac{xy}{2} \end{cases}$$

(d) Suppose that at time $t = 0$, we have $x(0) = 2$ and $y(0) = 2$. For each of the two populations, determine if it is increasing, decreasing, or not changing size at this moment in time, and explain how you know.

(3 points) For this problem, we must determine $x'(t)$ and $y'(t)$ evaluated at $t = 0$, which means we must calculate $x'(0)$ and $y'(0)$. We have

$$x'(0) = 4x(0) - x(0)y(0) = 4(2) - 2(2) = 4$$
$$y'(0) = -y(0) + (x(0)y(0))/2 = -2 + [(2)(2)]/2 = 0$$

Thus, at this moment in time, species x is increasing as indicated by $x'(0) > 0$, and the growth rate of species y is unchanging as indicated by $y'(0) = 0$.

(e) In the direction field below, mark the equilibrium solutions found in part (c). Furthermore, draw an approximation to the phase trajectory of the solution with initial condition $x(0) = 2$, $y(0) = 2$. Mark with an arrow the direction in which the phase trajectory is traversed.

(3 points) See graph below. Notice that $x'(0) = 4 > 0$ in part (d) indicates that species x starts off increasing at $(x(0), y(0)) = (2, 2)$; this indicates that the phase trajectory is traversed counterclockwise.