Solutions to Math 42 Second Exam — February 17, 2011

1. (10 points) In a certain California town, the probability density function for a random day's total rainfall is given by

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ A(x+1)^{-3/2} & \text{if } x \geq 0 \end{cases}$$

where x is measured in millimeters, and A is a positive constant.

(a) Find A, given that f is a probability density function.

(5 points) The problem asks that we find the value of the positive real number A for which the function f defined by

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ A(x+1)^{-\frac{3}{2}} & \text{if } x \geq 0 \end{cases}$$

is a probability density function, thus verifying

$$\int_{-\infty}^{+\infty} f(x)\, dx = 1$$

Observe that

$$\begin{aligned} \int_{-\infty}^{+\infty} f(x)\, dx &= \int_{-\infty}^{0} f(x)\, dx + \int_{0}^{+\infty} f(x)\, dx \\ &= \int_{-\infty}^{0} 0\, dx + \int_{0}^{+\infty} A(x+1)^{-\frac{3}{2}}\, dx \\ &= 0 + \int_{0}^{+\infty} A(x+1)^{-\frac{3}{2}}\, dx \\ &= \int_{0}^{+\infty} A(x+1)^{-\frac{3}{2}}\, dx \end{aligned}$$

as long as the last improper integral converges. In order to compute this improper integral, observe that an anti-derivative of the integrand is

$$\int A(x+1)^{-\frac{3}{2}}\, dx = -2A(x+1)^{-\frac{1}{2}} \tag{1}$$

Therefore

$$\begin{aligned} \int_{-\infty}^{+\infty} f(x)\, dx &= \int_{0}^{+\infty} A(x+1)^{-\frac{3}{2}}\, dx \\ &= \lim_{t\to+\infty} \int_{0}^{t} A(x+1)^{-\frac{3}{2}}\, dx \\ &= \lim_{t\to+\infty} \left(-2A(x+1)^{-\frac{1}{2}}\right]_0^t \\ &= \lim_{t\to+\infty} \left(\frac{-2A}{\sqrt{t+1}} + 2A\right) \\ &= 2A - \lim_{t\to+\infty} \frac{2A}{\sqrt{t+1}} \\ &= 2A \end{aligned}$$

In conclusion, the improper integral converges and equals

$$\int_{-\infty}^{+\infty} f(x)\, dx = 2A$$

Since f is a probability density function then

$$\int_{-\infty}^{+\infty} f(x)\, dx = 1 \;\Rightarrow\; 2A = 1 \;\Rightarrow\; A = \frac{1}{2}$$

(b) What is the median amount of daily rainfall in this town?

(5 points) The median for the probability density function f is defined as a real number m such that

$$\int_{-\infty}^{m} f(x)\,dx = \frac{1}{2}$$

First we observe that for any negative real number $a < 0$

$$\int_{-\infty}^{a} f(x)\,dx = \int_{-\infty}^{a} 0\,dx = 0$$

and therefore we must have $m \geq 0$. Therefore

$$\begin{aligned}\int_{-\infty}^{m} f(x)\,dx &= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{m} f(x)\,dx \\ &= \int_{-\infty}^{0} 0\,dx + \int_{0}^{m} A(x+1)^{-\frac{3}{2}}\,dx \\ &= \int_{0}^{m} A(x+1)^{-\frac{3}{2}}\,dx\end{aligned}$$

similarly to before. We continue the calculation using the anti-derivative (1)

$$\begin{aligned}\int_{-\infty}^{m} f(x)\,dx &= \int_{0}^{m} A(x+1)^{-\frac{3}{2}}\,dx \\ &= \left(-2A(x+1)^{-\frac{1}{2}}\right]_{0}^{m} \\ &= 2A - \frac{2A}{\sqrt{m+1}} \\ &= 1 - \frac{1}{\sqrt{m+1}}\end{aligned}$$

where we used the calculation of the value of A from part (a). We can now obtain the final answer:

$$\begin{aligned}\int_{-\infty}^{m} f(x)\,dx = \frac{1}{2} &\Leftrightarrow 1 - \frac{1}{\sqrt{m+1}} = \frac{1}{2} \\ &\Leftrightarrow \frac{1}{\sqrt{m+1}} = \frac{1}{2} \\ &\Leftrightarrow \sqrt{m+1} = 2 \\ &\Leftrightarrow m+1 = 4 \\ &\Leftrightarrow m = 3\end{aligned}$$

In conclusion, the median amount of daily rainfall in the town from the problem is 3 mm.

2. (10 points)

(a) Determine with justification if the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$$

converges, and if so, find the sum.

(5 points) By the method of partial fractions, we have that for $n \geq 1$:

$$\frac{1}{n(n+3)} = \frac{1}{3n} - \frac{1}{3(n+3)}.$$

Next, for $n \geq 1$ set $a_n = \frac{1}{n(n+3)}$ and $b_n = \frac{1}{3n}$. Hence, $a_n = b_n - b_{n+3}$. So, if $N \geq 4$, then

$$\begin{aligned} s_N &= \sum_{n=1}^{N} a_n \\ &= \sum_{n=1}^{N} (b_n - b_{n+3}) \\ &= b_1 + b_2 + b_3 - b_{N+1} - b_{N+2} - b_{N+3} \\ &= \frac{1}{3} + \frac{1}{6} + \frac{1}{9} - \frac{1}{3(N+1)} - \frac{1}{3(N+2)} - \frac{1}{3(N+3)} \\ &= \frac{11}{18} - -\frac{1}{3(N+1)} - \frac{1}{3(N+2)} - \frac{1}{3(N+3)} \\ &\xrightarrow{N\to\infty} \frac{11}{18}. \end{aligned}$$

So, we conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+3)} = \frac{11}{18},$$

and in particular, the series is convergent.

(b) Express $2.0\overline{134} = 2.0134134134134\ldots$ as a ratio of integers.

(5 points)

$$\begin{aligned}
2.0\overline{134} &= 2 + \frac{134}{10^4} + \frac{134}{10^7} + \frac{134}{10^{10}} + \ldots \\
&= 2 + \frac{134}{10^4} \sum_{n=0}^{\infty} 10^{-3n} \\
&= 2 + \left(\frac{134}{10^4}\right)\left(\frac{1}{1 - 10^{-3}}\right) \qquad \text{(sum of a geometric series with common ratio } 10^{-3}\text{)} \\
&= 2 + \frac{134}{10^4 - 10} \\
&= 2 + \frac{134}{9990} \\
&= \frac{9990 + 67}{4995} \\
&= \frac{10057}{4995}.
\end{aligned}$$

3. (10 points) Determine whether each of the series below converges or diverges. Indicate clearly which tests you use and how you apply them.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-4)^{2n}}{n^3\, 5^n}$

   (5 points) The presence of n in the exponents indicates that the Ratio Test is likely appropriate. Applying the Ratio Test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(-4)^{2(n+1)}}{(n+1)^3 5^{n+1}}}{\frac{(-4)^{2n}}{n^3 5^n}}\right| = \lim_{n\to\infty} \frac{4^{2n+2} n^3 5^n}{4^{2n}(n+1)^3 5^{n+1}}$$

$$= \lim_{n\to\infty} \frac{16n^3}{5(n+1)^3} = \frac{16}{5} \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^3} = \frac{16}{5}.$$

   Because the limit exists and is greater than 1, the series diverges by the Ratio Test.

   Alternative solution path: one could use the Test for Divergence (the terms are all positive and increase towards infinity), but that's actually harder in this case.

(b) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+1}\right)$

(5 points) Consider the limit of the terms as n approaches infinity. $\lim_{n\to\infty}(\frac{n}{2n+1}) = \frac{1}{2}$, so $\lim_{n\to\infty}\ln(\frac{n}{2n+1}) = \ln(\frac{1}{2}) \neq 0$. So the terms do not converge to zero. Therefore the series diverges by the Test for Divergence.

Note: Several people tried to turn this into a telescoping sum, writing each term as $\ln n - \ln(2n+1)$ and then noticing that all the odd terms "eventually cancel," leaving only the even terms. It's an ingenious idea, but it doesn't quite work. The problem with this is that the odd terms don't cancel fast enough - there is no clean expression for s_n, the nth partial sum, because there are a growing number of un-canceled odd terms. So our telescoping sum method does not work. And we cannot rearrange all the terms of the series at once to say that "the odd terms all cancel;" that kind of rearrangement does not work unless the series is absolutely convergent, which this one is not. (In fact, if that kind of rearrangement worked, you could prove that $0 = 1$, which would be a problem!)

4. (10 points) Determine whether each of the series below converges or diverges. Indicate clearly which tests you use and how you apply them.

   (a) $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$

(5 points)

First solution: Comparison. We note that $n + \sqrt{n} \leq 2n$. Therefore

$$\frac{1}{2n} \leq \frac{1}{n+\sqrt{n}}$$

It follows that

$$\sum_{n=1}^{\infty} \frac{1}{2n} \leq \sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$$

Since the harmonic series $\sum 1/n$ diverges, it follows by comparison that the series $\sum 1/(n+\sqrt{n})$ also diverges.

Second solution: Limit comparison. Note that,

$$\lim_{n\to\infty} \frac{1/(n+\sqrt{n})}{1/n} = \lim_{n\to\infty} \frac{1}{1+1/\sqrt{n}} = 1 > 0$$

Since the limit is positive, and both functions $1/(n+\sqrt{n})$ and $1/n$ are of constant sign, the limit comparison test applies. Thus, by the limit comparison test the series $\sum 1/n$ and $\sum 1/(n+\sqrt{n})$ either both diverge, or both converge. Since $\sum 1/n$ diverges (it's a p-series with $p = 1$) it follows that $\sum 1/(n+\sqrt{n})$ diverges too.

Third solution: Integral test. The function $1/(n+\sqrt{n})$ is decreasing, and positive. Hence the integral test applies, meaning that $\sum 1/(n+\sqrt{n})$ and $\int_2^{\infty} 1/(x+\sqrt{x})dx$ either both converge or both diverge. Making the u-substitution given by $u = \sqrt{x}$, we find that $du = (1/2\sqrt{x})dx$, hence $dx = 2udu$, and so

$$\int_1^{\infty} \frac{dx}{x+\sqrt{x}} = \int_{\sqrt{1}}^{\infty} \frac{2udu}{u^2+u} = \int_{\sqrt{1}}^{\infty} \frac{2du}{u+1} = 2\left[\ln\left(|u+1|\right)\right]_1^{\infty} = \infty$$

The integral diverges, hence the series $\sum 1/(n+\sqrt{n})$ diverges too.

(b) $\displaystyle\sum_{n=1}^{\infty} \sin^2 \frac{\pi}{n}$

(5 points)

First solution: Comparison. We note that $\sin(\pi/n)$ is positive, for integer n, and that in addition $\sin(x) \le x$. Thus $\sin(\pi/n) \le \pi/n$. Since $\sin(\pi/n)$ is positive, we can square the above relation and obtain $\sin^2(\pi/n) \le (\pi/n)^2$. It follows that

$$\sum_{n=1}^{\infty} \sin^2(\pi/n) \le \sum_{n=1}^{\infty} \frac{\pi^2}{n^2}$$

Since $\sum 1/n^2$ converges (it's a p-series with $p = 2$) it follows by comparison that the series $\sum \sin^2(\pi/n)$ is also convergent.

Second solution: Limit comparison.

We compare $\sin^2(\pi/n)$ with π^2/n^2 . We note that

$$\lim_{n\to\infty} \frac{\sin(\pi/n)}{\pi/n} = 1 \quad \text{since} \quad \lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{\cos(x)}{1} = \cos(0) = 1$$

Therefore, squaring $\lim_{n\to\infty} \sin(\pi/n)/(\pi/n) = 1$ we obtain,

$$\lim_{n\to\infty} \frac{\sin^2(\pi/n)}{\pi^2/n^2} \quad = \quad 1$$

It follows by the limit comparison test (which does apply since the above limit is positive, and both functions are of constant sign) that the series $\sum \pi^2/n^2$ and the series $\sum \sin^2(\pi/n)$ either both converge or both diverge. Since the series $\sum 1/n^2$ converges (it's a p-series with $p = 2 > 1$) it follows that $\sum \sin^2(\pi/n)$ is also convergent.

5. (12 points) Suppose that the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Decide which of the following series must converge, must diverge, or may either converge or diverge (inconclusive). Circle your answer. You do not need to justify your answers.

(2 points each) The given information implies that $\sum_{n=1}^{\infty} |a_n|$ converges, and also that $\lim_{n\to\infty} a_n = 0$.

(a) $\sum_{n=1}^{\infty} \left(a_n + \frac{1}{n^2}\right)$ **[Converges]** Diverges Inconclusive

Since both series are separately convergent (note $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2 > 1$), it follows that the series consisting of the sums of their n-th terms is also convergent.

(b) $\sum_{n=1}^{\infty} (-1)^n a_n$ **[Converges]** Diverges Inconclusive

As $\sum_{n=1}^{\infty} |(-1)^n a_n| = \sum_{n=1}^{\infty} |a_n|$ converges, the Absolute Convergence Rule implies $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

(c) $\sum_{n=1}^{\infty} \frac{1}{1 + a_n^2}$ Converges **[Diverges]** Inconclusive

Since $\lim_{n\to\infty} \frac{1}{1 + a_n^2} = \frac{1}{1+0} = 1 \neq 0$, the series diverges, by the Test for Divergence.

(d) $\sum_{n=1}^{\infty} \frac{|a_n|}{n}$ **[Converges]** Diverges Inconclusive

We have $0 \le \frac{|a_n|}{n} \le |a_n|$ for $n \ge 1$, and since $\sum_{n=1}^{\infty} |a_n|$ converges, it follows that $\sum_{n=1}^{\infty} \frac{|a_n|}{n}$ also converges by the Comparison Test.

(e) $\sum_{n=1}^{\infty} n^2\, a_n$ Converges Diverges **[Inconclusive]**

By the p-series Rule, we get convergence if $a_n = n^{-4}$, but divergence if $a_n = n^{-3}$.

(f) $\sum_{n=1}^{\infty} n!\, a_n$ Converges Diverges **[Inconclusive]**

We get convergence if $a_n = \frac{1}{(n!)^2}$, but divergence if $a_n = \frac{1}{n^2}$.

6. (13 points) Find, with complete justification, the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}(3x-1)^n$$

We have

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(3x-1)^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{(3x-1)^n}\right| \\
&= \lim_{n\to\infty}|3x-1|\sqrt{\frac{n}{n+1}} \\
&= |3x-1|\left(\lim_{n\to\infty}\frac{n}{n+1}\right)^{1/2} \\
&= |3x-1|\,(1)^{1/2} \\
&= |3x-1|
\end{aligned}$$

(where we have used that the square root function is continuous on its domain). Therefore, by the Ratio test, the power series converges for $|3x-1|<1$ and diverges for $|3x-1|>1$. We have

$$|3x-1|<1 \iff \frac{1}{3}|3x-1|<\frac{1}{3} \iff |x-\frac{1}{3}|<\frac{1}{3}$$

so the radius of convergence is $R=1/3$. The power series is centered at $x=1/3$, so we must check the endpoints $x=1/3-1/3=0$ and $x=1/3+1/3=2/3$.

For $x=0$, the power series becomes

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}(-1)^n = \sum_{n=1}^{\infty}\frac{(-1)^{2n}}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}},$$

which is a p-series with $p=1/2<1$, and hence diverges.

For $x=2/3$, the power series becomes

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}(1)^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$$

This is an alternating series with $b_n=\frac{1}{\sqrt{n}}>0$ for all n, and it satisfies

1. $b_{n+1}=\dfrac{1}{\sqrt{n+1}}<\dfrac{1}{\sqrt{n}}=b_n$ for all n, and
2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\dfrac{1}{\sqrt{n}}=0.$

Therefore by the alternating series test, the series converges. This tells us that the interval of convergence is $(0,\frac{2}{3}]$.

7. (13 points) In each of the parts below, show all the steps in your reasoning.

(a) Write $\frac{1}{1+x^2}$ as a power series about 0, and state the interval of convergence. (Hint: use geometric series.)

(3 points) Notice that $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$ is the sum of a geometric series with radius $r = -x^2$ and $a = 1$. Thus we have

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

The geometric series converges only when $|r| < 1 \Leftrightarrow |-x^2| < 1 \Rightarrow |x|^2 < 1 \Rightarrow \sqrt{|x|^2} < \sqrt{1} \Rightarrow |x| < 1$. Thus, the interval of convergence is $(-1, 1)$.

(*Note* : For those who used the ratio test to find the interval, the ratio test would only tell you the open interval for convergence, (-1, 1), and one would need to check the endpoints $x = -1$ and $x = 1$ explicitly to justify divergence at these points. The point is that our knowledge of geometric series could save us from going through the ratio test and checking endpoints, which was already done when we first investigated geometric series anyway.)

(b) Find a power series for $\arctan x$. What is the radius of convergence?

(3 points) Using part (a), we can use the theorem that allows us to pass the integral inside the summation of a power series within the interval $-1 < x < 1$:

$$\arctan x = \int \frac{1}{1+x^2} dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} dx = \sum_{n=0}^{\infty} (-1)^n \int x^{2n} dx$$
$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

When $x = 0$, the terms in the summation above are $0 \Rightarrow 0 = \arctan(0) = C + 0$ so $C = 0$. Thus, we have

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

Since this series was obtained by integrating a power series with a radius of convergence of 1, we know by a theorem in the book, that the radius of the integrated series remains the same, so $R = 1$.

(*Note*: Some justification for the radius of convergence is needed here since it's not obvious and we have a special theorem to determine it for integrated power series. Notice that the interval of convergence for the integrated power series is not the same as in (a) and is actually $[-1, 1]$. The fact that the series actually converges to $\arctan(x)$ at $x = 1, -1$ is more subtle to prove. Using the ratio test to obtain the radius also works.)

(c) Express the number $\int_0^{0.1} \frac{\arctan x}{x}\, dx$ as a series.

(4 points) Using part (b), the interval $(0, 0.1)$ is within the radius of convergence for $\frac{\arctan x}{x}$, so we compute

$$\int_0^{0.1} \frac{\arctan x}{x} dx = \int_0^{0.1} \frac{1}{x} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} dx = \int_0^{0.1} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{x(2n+1)} dx = \int_0^{0.1} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n+1} dx$$
$$= \sum_{n=0}^{\infty} (-1)^n \int_0^{0.1} \frac{x^{2n}}{2n+1} dx = \sum_{n=0}^{\infty} (-1)^n \left[\frac{x^{2n+1}}{(2n+1)^2} \right]_0^{0.1}$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{(0.1)^{2n+1}}{(2n+1)^2}$$

(d) Find, with complete justification, a partial sum of the series in part (c) that approximates the value of the integral to within 10^{-8}. (You do not need to simplify the sum.)

(3 points) Here we will use the Alternating Series estimation theorem with $b_n = \frac{(0.1)^{2n+1}}{(2n+1)^2} = \frac{1}{10^{2n+1}(2n+1)^2}$. We can clearly see that $b_n > 0$, $b_{n+1} \leq b_n$ and $\lim_{n\to\infty} b_n = 0$ so the hypothesis of the theorem are fulfilled. Also notice that $b_2 = \frac{1}{10^5(25)} > 10^{-8}$ and

$$b_3 = \frac{1}{10^7(49)} = \frac{1}{10^8(4.9)} < \frac{1}{10^8}.$$

So letting R_n denote the remainder of the n'th partial sum of the series in (c), the theorem tells us that

$$|R_2| \leq b_3 < 10^{-8}$$

using the earlier calculation. Thus the partial sum

$$\sum_{n=0}^{2} (-1)^n \frac{(0.1)^{2n+1}}{(2n+1)^2} = 0.1 - \frac{(0.1)^3}{9} + \frac{(0.1)^5}{25}$$

approximates the series to within 10^{-8}.

8. (10 points)

(a) Find, showing all your steps, the Taylor series for e^x with center 0.

(4 points) Use Taylor's recipe: the Taylor series for $f(x)$ about 0 is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

And if $f(x) = e^x$, then $f^{(n)}(x) = e^x$ for all n, so $f^{(n)}(0) = e^0 = 1$ for all n. So we see that the Taylor series we want is

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

(b) Use series to find $\lim_{x \to 0} \frac{e^{-x^2} - 1}{e^x - x - 1}$. (You may take for granted the fact that the Taylor series for e^x converges to e^x.)

(6 points) From part a) (and, technically, the fact that the Taylor series for e^x converges to e^x), we know that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Replacing x with $-x^2$, we see that:

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \dots.$$

Now that we know power series expansions for everything involved in the limit, plug them in; we see that

$$\begin{aligned}
\lim_{x \to 0} \frac{e^{-x^2} - 1}{e^x - x - 1} &= \lim_{x \to 0} \frac{(1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \dots) - 1}{(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots) - x - 1} \\
&= \lim_{x \to 0} \frac{-\frac{x^2}{1!} + \frac{x^4}{2!} - \dots}{\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots} \\
&= \lim_{x \to 0} \frac{-1 + \frac{x^2}{2!} - \dots}{\frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \dots} \\
&= \frac{-1}{(1/2)} = -2.
\end{aligned}$$

(Notice that we canceled an x^2 from all terms in the numerator and denominator in the third-to-last step).