Solutions to Math 42 First Exam — January 27, 2011

1. (12 points) Evaluate each of the following integrals, showing all of your reasoning.

   (a) $\int_1^e t^3 \ln t \, dt$

   (6 points) Consider

$$u = \ln t \quad , \quad dv = t^3\,dt$$
$$du = (1/t)\,dt \quad , \quad v = t^4/4$$

Integrating by parts, we find

$$\begin{aligned}
\int_1^e t^3 \ln t\,dt &= \left[\frac{t^4}{4}\cdot \ln t\right]_1^e - \int_1^e \frac{t^4}{4}\cdot\frac{1}{t}dt \\
&= \left(\frac{e^4}{4} - 0\right) - \frac{1}{4}\int_1^e t^3 dt \\
&= \frac{e^4}{4} - \frac{1}{4}\cdot\left[\frac{t^4}{4}\right]_1^e = \frac{e^4}{4} - \frac{e^4}{16} + \frac{1}{16} = \boxed{\frac{3e^4+1}{16}}
\end{aligned}$$

   (b) $\int_0^1 \frac{dx}{(1+\sqrt{x})^5}$

   (6 points) Let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}\,dx$, hence $dx = 2u\,du$. Also, if $x = 0$, then $u = 0$, and if $x = 1$, then $u = 1$. It follows that

$$\int_0^1 \frac{dx}{(1+\sqrt{x})^5} = \int_0^1 \frac{2u\,du}{(1+u)^5}$$

Let $v = u + 1$. Then $dv = du$; also, if $u = 0$, then $v = 1$, and if $u = 1$, then $v = 2$. The integral above becomes

$$\begin{aligned}
\int_1^2 \frac{2(v-1)dv}{v^5} &= -\int_1^2 \frac{2}{v^5}dv + \int_1^2 \frac{2}{v^4}dv \\
&= -2\cdot\left[\frac{v^{-4}}{-4}\right]_1^2 + 2\cdot\left[\frac{v^{-3}}{-3}\right]_1^2 = \frac{1}{2}\cdot\left(\frac{1}{2^4}-1\right) - \frac{2}{3}\cdot\left(\frac{1}{2^3}-1\right) = \boxed{\frac{11}{96}}
\end{aligned}$$

2. (13 points) Evaluate each of the following integrals, showing all of your reasoning.

(a) $\int \tan^3 x \sec x \, dx$

(6 points) We recall the trigonometric identity $\tan^2 x = \sec^2 x - 1$, and we use the substitution

$$u = \sec x \quad \Rightarrow \quad du = \sec x \tan x \, dx.$$

This gives

$$\begin{aligned}\int \tan^3 x \sec x \, dx &= \int \tan^2 x \tan x \sec x \, dx \\ &= \int (\sec^2 x - 1) \tan x \sec x \, dx \\ &= \int (u^2 - 1) \, du = \frac{u^3}{3} - u + C = \boxed{\frac{\sec^3 x}{3} - \sec x + C}.\end{aligned}$$

(b) $\int \frac{x}{\sqrt{5 - 4x - x^2}} \, dx$

(7 points) Here we must complete the square, so we have

$$\begin{aligned}5 - 4x - x^2 = 5 - (x^2 + 4x) &= 5 - (x^2 + 4x + 4 - 4) \\ &= 5 + 4 - (x^2 + 4x + 4) = 9 - (x + 2)^2.\end{aligned}$$

If we let $u = x + 2$, so that $du = dx$, we find that

$$\int \frac{x}{\sqrt{5 - 4x - x^2}} \, dx = \int \frac{x}{\sqrt{9 - (x + 2)^2}} \, dx = \int \frac{u - 2}{\sqrt{9 - u^2}} \, du$$

Now we let $u = 3 \sin \theta$, so that $du = 3 \cos \theta \, d\theta$, with $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. This gives the calculation

$$\begin{aligned}\sqrt{9 - u^2} = \sqrt{9 - (3 \sin \theta)^2} &= \sqrt{9(1 - \sin^2 \theta)} \\ &= \sqrt{9 \cos^2 \theta} \\ &= 3|\cos \theta| = 3 \cos \theta \quad (\text{since } \cos \theta \ge 0 \text{ for } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}).\end{aligned}$$

Using the above calculation gives us

$$\begin{aligned}\int \frac{x}{\sqrt{5 - 4x - x^2}} \, dx = \int \frac{u - 2}{\sqrt{9 - u^2}} \, du &= \int \frac{3 \sin \theta - 2}{3 \cos \theta} 3 \cos \theta \, d\theta \\ &= \int (3 \sin \theta - 2) d\theta \\ &= -3 \cos \theta - 2\theta + C \\ &= \boxed{-3 \cos \arcsin \left(\frac{x + 2}{3}\right) - 2 \arcsin \left(\frac{x + 2}{3}\right) + C}\end{aligned}$$

Where the last line follows from $x + 2 = u = 3 \sin \theta \Rightarrow \theta = \arcsin\left(\frac{x+2}{3}\right)$.

Notes on common mistakes: Observe that we can't use partial fractions here, due to the square root in the denominator. Also notice that $\sqrt{5 - 4x - x^2} \ne -\sqrt{x^2 + 4x - 5}$, because one cannot factor -1 out of the square root.

3. (7 points) Evaluate $\int \frac{x^4+1}{x^3+2x^2}\,dx$, showing all reasoning.

Since the degree of the numerator is at least as great as the degree of the denominator, we first perform a long division of polynomials (2 points):

$$\begin{array}{r|l} & x-2 \\ \hline x^3+2x^2 & x^4 \qquad\qquad +1 \\ & \underline{x^4+2x^3} \\ & \quad -2x^3 \\ & \quad \underline{-2x^3-4x^2} \\ & \qquad\qquad 4x^2+1 \end{array}$$

We conclude that

$$\frac{x^4+1}{x^3+2x^2} = x-2+\frac{4x^2+1}{x^3+2x^2}$$

Now we factor the denominator and perform the necessary partial fraction decomposition:

$$\frac{4x^2+1}{x^3+2x^2} = \frac{4x^2+1}{x^2(x+2)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+2}$$

Multiplying on both sides by $x^2(x+1)$ we obtain

$$\begin{aligned} & 4x^2+1 = Ax(x+2)+B(x+2)+Cx^2 \\ \iff & 4x^2+1 = (A+C)x^2+(2A+B)x+2B \end{aligned}$$

Equating the coefficients of powers of x on each side we get

$$\begin{cases} A+C=4 \\ 2A+B=0 \\ 2B=1 \end{cases} \Leftrightarrow \begin{cases} A+C=4 \\ 2A+\frac{1}{2}=0 \\ B=\frac{1}{2} \end{cases} \Leftrightarrow \begin{cases} -\frac{1}{4}+C=4 \\ A=-\frac{1}{4} \\ B=\frac{1}{2} \end{cases} \Leftrightarrow \begin{cases} C=\frac{17}{4} \\ A=-\frac{1}{4} \\ B=\frac{1}{2} \end{cases}$$

Substituting above we obtain the required partial fraction decomposition (3 points):

$$\frac{4x^2+1}{x^3+2x^2} = \frac{4x^2+1}{x^2(x+2)} = -\frac{1}{4x}+\frac{1}{2x^2}+\frac{17}{4(x+2)}$$

In conclusion

$$\begin{aligned} \frac{x^4+1}{x^3+2x^2} &= x-2+\frac{4x^2+1}{x^3+2x^2} \\ &= x-2-\frac{1}{4x}+\frac{1}{2x^2}+\frac{17}{4(x+2)} \end{aligned}$$

Now we can compute the required antiderivative (2 points):

$$\begin{aligned} \int \frac{x^4+1}{x^3+2x^2}\,dx &= \int \left(x-2-\frac{1}{4x}+\frac{1}{2x^2}+\frac{17}{4(x+2)}\right)dx \\ &= \boxed{\frac{x^2}{2}-2x-\frac{1}{4}\ln|x|-\frac{1}{2x}+\frac{17}{4}\ln|x+2|+C} \end{aligned}$$

Note on the "plus C": More precisely, any other antiderivative differs from the above by a function which is defined on the union of the intervals $(-\infty,-2)$, $(-2,0)$, and $(0,+\infty)$, and is constant on each of those intervals, so we should think of C as what we call a *locally constant function* defined on this domain.

4. (12 points)

(a) Evaluate $\int_0^1 \frac{z}{\sqrt{1-z^2}}\,dz$ or explain why its value does not exist; show all reasoning.

(6 points) This integral is improper (of type II), because the integrand has a vertical asymptote at $z = 1$. So we write:

$$\int_0^1 \frac{z}{\sqrt{1-z^2}}\,dz = \lim_{t\to 1^-} \int_0^t \frac{z}{\sqrt{1-z^2}}\,dz.$$

To evaluate the integral, we perform a substitution $u = 1-z^2$; then $du = -2z\,dz$, so $z\,dz = -\frac{1}{2}du$, and the integral (ignoring the limit for now) becomes

$$\int_1^{1-t^2} -\frac{1}{2\sqrt{u}}\,du = -\sqrt{u}\Big|_1^{1-t^2} = -\sqrt{1-t^2} + 1.$$

Now we take the limit as t approaches 1 from the left, and in fact $\sqrt{1-t^2}$ is continuous as t approaches 1 from the left, with value 0 at $t = 1$. So the answer is

$$\lim_{t\to 1^-} (-\sqrt{1-t^2} + 1) = 0 + 1 = 1,$$

and we see that the improper integral is convergent with value 1.

Note: In this case, if one forgets that the integral is improper and just evaluates it from zero to 1 as a regular integral, taking the antiderivative and plugging in the endpoints actually makes sense and yields the correct answer. However, this shortcut does not work for all improper integrals; therefore, such solutions did not receive full credit.

(b) Determine whether $\int_{-\infty}^{\infty} \frac{\arctan x}{\sqrt{1+x^2}}\,dx$ converges or diverges; give complete reasoning.

(6 points) Our first idea is to examine the behavior of the integrand for large x. Recall $\arctan x$ is a bounded function that approaches $\pi/2$ as x approaches ∞. And for large x, $\sqrt{1+x^2}$ is very similar to $\sqrt{x^2} = x$. This sugests that for large x, the integrand is very similar to $\frac{\pi/2}{x}$. Moreover, as we learned in class, the integral $\int_1^\infty \frac{1}{x}$ is a divergent improper integral. In particular, we see that

$$\int_1^t \frac{1}{x} = \ln(x)|_1^t = \ln t,$$

which goes to infinity as $t \to \infty$; hence this improper integral is divergent. Based on this, we guess that our integral will also be divergent, and look for an appropriate comparison function. Since

$$\int_{-\infty}^{\infty} \frac{\arctan x}{\sqrt{1+x^2}}\,dx = \int_{-\infty}^{1} \frac{\arctan x}{\sqrt{1+x^2}}\,dx + \int_{1}^{\infty} \frac{\arctan x}{\sqrt{1+x^2}}\,dx,$$

it is enough to show that $\int_1^\infty \frac{\arctan x}{\sqrt{1+x^2}}\,dx$ diverges. To do this, we need to find a smaller comparison function that also diverges. Thus we want a smaller numerator and a larger denominator.

We first notice that $\arctan x$ is an increasing function, so

$$\text{for } x \geq 1, \quad \arctan x \geq \arctan 1 = \pi/4.$$

For the denominator, note that for $x \geq 1$, we have $x^2 \geq 1$, which means

$$0 < 1 + x^2 \leq x^2 + x^2 = 2x^2,$$

so

$$0 < \sqrt{1+x^2} \leq \sqrt{2x^2} = x\sqrt{2}$$

and thus we conclude (by taking reciprocals and reversing the inequality) that

$$\frac{1}{\sqrt{1+x^2}} \geq \frac{1}{x\sqrt{2}} > 0.$$

Putting these inequalities together, we have that for $x \geq 1$,

$$\frac{\arctan x}{\sqrt{1+x^2}} \geq \frac{\pi/4}{x\sqrt{2}} > 0.$$

Also,

$$\int_1^\infty \frac{\pi/4}{x\sqrt{2}}\,dx = \frac{\pi}{4\sqrt{2}} \int_1^\infty \frac{1}{x}\,dx,$$

which diverges. So by the Comparison Theorem, $\int_1^\infty \frac{\arctan x}{\sqrt{1+x^2}}\,dx$ diverges, and therefore the integral from $-\infty$ to ∞ also diverges.

Notes on common mistakes: Several people correctly noticed that the integrand was an odd function, and then incorrectly concluded that the integral must be zero. This odd-ness argument doesn't work with improper integrals; we have to break them up first, and then we see that each piece actually diverges. We can't say that $\infty - \infty = 0$.

Note also that it is not possible to solve this integral in a closed form by hand. However, it is possible, by a trigonometric substitution, to convert this into an integral of the form $\int \theta \sec\theta\, d\theta$. Several people did this, and it can indeed help if you then apply the comparison theorem; however, one must notice that the limits of integration change, and in fact the problem becomes a Type II improper integral.

5. (6 points) The linear density of a rod of length 6 cm is given by $d(x)$, in grams per centimeter, where x is measured in centimeters from one end of the rod. Here is a table of values of $d(x)$ measured at one-centimeter intervals:

x (cm)	0	1	2	3	4	5	6
$d(x)$ (g/cm)	1	3	7	11	14	16	17

(a) Use the Trapezoidal Rule to estimate the total mass of the rod. Use all the data in the table above, and do not simplify your answer.

(3 points) Applying the trapezoidal rule — on the interval $[0, 6]$ — with six subintervals to the data supplied gives:

$$T_6 = \frac{\Delta x}{2}\left(d(0) + 2d(1) + 2d(2) + 2d(3) + 2d(4) + 2d(5) + d(6)\right)$$

where Δx is the length of the subintervals, i.e. $\Delta x = (6-0)/6 = 1$. Therefore, the desired approximation to the mass of the rod is

$$\boxed{T_6 = \frac{1}{2}(1 + 2\cdot 3 + 2\cdot 7 + 2\cdot 11 + 2\cdot 14 + 2\cdot 16 + 17)}$$

(Although simplification is not required, we would find that the total mass of the rod is approximately 60 g, as obtained via the trapezoidal rule on the interval $[0, 6]$ with six subintervals.)

(b) Use Simpson's Rule to estimate the total mass of the rod. Use all the data in the table above, and do not simplify your answer.

(3 points) Applying Simpson's rule — on the interval $[0, 6]$ — with six subintervals to the data supplied gives:

$$S_6 = \frac{\Delta x}{3}\left(d(0) + 4d(1) + 2d(2) + 4d(3) + 2d(4) + 4d(5) + d(6)\right)$$

where Δx is the length of the subintervals, i.e. $\Delta x = (6-0)/6 = 1$. Therefore, the desired approximation to the mass of the rod is

$$\boxed{S_6 = \frac{1}{3}(1 + 4\cdot 3 + 2\cdot 7 + 4\cdot 11 + 2\cdot 14 + 4\cdot 16 + 17)}$$

(Although simplification is not required, we would find that the total mass of the rod is approximately 60 g, as obtained via Simpson's rule with six subintervals on the interval $[0, 6]$.)

6. (12 points) Let $f(x) = \dfrac{6}{\sqrt{1-x^2}}$. In this problem, we study approximations of π using the identity:

$$\pi = \int_0^{\frac{1}{2}} \frac{6}{\sqrt{1-x^2}}\,dx$$

and the Midpoint Rule. (By the way, you do *not* have to prove the above identity.)

(a) Write an algebraic expression involving only numbers that approximates π using the Midpoint Rule with 5 subintervals. You do *not* have to simplify this expression.

(3 points) We have $\Delta x = \frac{1}{5}(\frac{1}{2} - 0) = \frac{1}{10}$, and for each $i = 0, 1, \ldots, 5$,

$$x_i = 0 + i\Delta x = \frac{i}{10}, \quad \text{so that} \quad \overline{x}_i = \frac{x_{i-1} + x_i}{2} = \frac{2i-1}{20};$$

that is, $\overline{x}_1 = \frac{1}{20}$, $\overline{x}_2 = \frac{3}{20}$, etc. The approximation is:

$$\pi \approx M_5 = \boxed{\frac{1}{10}\left[\frac{6}{\sqrt{1-(\frac{1}{20})^2}} + \frac{6}{\sqrt{1-(\frac{3}{20})^2}} + \frac{6}{\sqrt{1-(\frac{5}{20})^2}} + \frac{6}{\sqrt{1-(\frac{7}{20})^2}} + \frac{6}{\sqrt{1-(\frac{9}{20})^2}}\right]}.$$

(b) Show that the above approximation is accurate to within $\frac{1}{200}$; and explain whether the approximation gives an overestimate or underestimate of π (or whether it is impossible to tell). You may make use of the fact that $f''(x) = \dfrac{6(1+2x^2)}{(1-x^2)^{5/2}}$.

(5 points) We must show that $|E_M| \leq \frac{1}{200}$, where E_M is the error in the midpoint rule approximation to the integral. To do this, we first note that

$$|E_M| \leq \frac{K_2(\frac{1}{2} - 0)^3}{24n^2}$$

where K_2 is such that

$$|f''(x)| \leq K_2 \text{ for all } x \text{ in } [0, \tfrac{1}{2}].$$

Note that $f''(x) = \frac{6(1+2x^2)}{(1-x^2)^{5/2}}$ is positive and increasing on $[0, \frac{1}{2}]$; this follows because on this interval the numerator is positive and increasing, and the denominator is positive and decreasing (or you could compute f''' and show that it is positive on this interval). Thus, we require $K_2 \geq f''(\frac{1}{2})$. Now

$$f''(\tfrac{1}{2}) = \frac{6(1+2(\frac{1}{2})^2)}{(1-(\frac{1}{2})^2)^{5/2}} = \frac{32}{\sqrt{3}} \leq \frac{36}{1.5} = 24$$

(since $\sqrt{3} > 1.5$), so we may take $K_2 = 24$. In our case, we also have $n = 5$, so

$$|E_M| \leq \frac{24(\frac{1}{2})^3}{24(5)^2} = \frac{1}{8(25)} = \frac{1}{200}, \text{ as desired.}$$

Since $f''(x)$ is positive on $[0, \frac{1}{2}]$, the function f is concave upward on this interval, and we have seen that the midpoint rule approximation always gives an $\boxed{\text{underestimate}}$ in this case.

(c) Again using the Midpoint Rule, how many subintervals n would guarantee an approximation of π that is accurate to within 10^{-12}? Your final answer should give a valid n in simplified form, and be fully justified, but it need not be optimal in any sense.

(4 points) We want to find an n such that $|E_M| \leq 10^{-12}$. Now, with $K_2 = 24$ as in part (b), we have that

$$|E_M| \leq \frac{K_2(b-a)^3}{24n^2} = \frac{24(\frac{1}{2})^3}{24n^2} = \frac{1}{8n^2}.$$

Thus, since we want $|E_M| \leq 10^{-12}$, we conclude that we need find n such that

$$\frac{1}{8n^2} \leq 10^{-12},$$

for then we would have

$$|E_M| \leq \frac{1}{8n^2} \leq 10^{-12} \quad \text{(by transitivity)},$$

as desired. Solving, we find

$$\frac{1}{8n^2} \leq 10^{-12} \implies n \geq \frac{10^6}{\sqrt{8}}.$$

Since $\sqrt{8} > 2$, the choice $\boxed{n = \frac{10^6}{2} = 500000}$ is a simple, quick answer that clearly satisfies the above requirement. (However, since $\sqrt{8} > 2.5$, then taking any $n \geq \frac{10^6}{2.5} = 400000$ is also sufficient.)

Notes for parts (b) and (c): In fact we may take any $K_2 \geq \frac{32}{\sqrt{3}} = 18.475\ldots$, but since we are doing this without a calculator, it's much easier to find a slightly-too-big whole number as we did. Naturally, different choices for K_2 lead to slightly different conditions on the n of part (c); but even the minimum possible choice for K_2 leads to a requirement that $n \geq 310202$, a fact we'd likely need a calculator to see. (If you calculated K_2 incorrectly in part (b), you were not penalized for that calculation again in part (c).)

Most people had confusing/incorrect inequalities throughout their solutions; it was graded with lenience this time, in the future it won't be.

7. (10 points) Let R be the region in the xy-plane lying below the curve $y = \sqrt{x}\cos x$ and above that portion of the x-axis with $0 \le x \le \frac{\pi}{2}$.

(a) Set up, but do not yet evaluate, an integral in terms of a single variable that represents the volume of the solid obtained by rotating R about the x-axis. Justify your answer by drawing a picture, labeling a sample slice, and citing the method used.

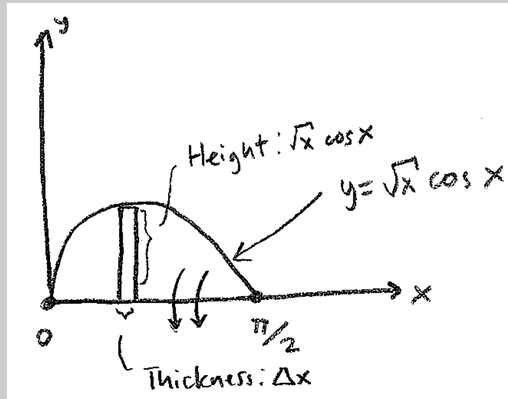
(5 points) We use the disk, or washer, method. We slice perpendicular to the x-axis, which is the axis of rotation. A sample slice of R located at coordinate x rotates to become a thin disk with thickness Δx and radius equal to the height of the slice, which is $\sqrt{x}\cos x$.

Therefore, the area of this cross-section of the solid is approximately

$$\pi(\sqrt{x}\cos x)^2 = \pi x \cos^2 x.$$

The limits of integration are from $x = 0$ to $x = \pi/2$, so we get

$$\boxed{V = \int_0^{\pi/2} \pi x \cos^2 x\, dx}.$$



Note: The method of cylindrical shells doesn't work, because we can't solve for x in the equation of the curve, and we'd need to do that in order to measure the height of each cylindrical shell.

(b) Evaluate the integral of part (a), showing all your steps.

(5 points) First we use the half-angle formula to write:

$$V = \int_0^{\pi/2} \pi x \cos^2 x\, dx = \frac{\pi}{2}\int_0^{\pi/2} x(1+\cos 2x)\, dx$$
$$= \frac{\pi}{2}\left(\int_0^{\pi/2} x\, dx + \int_0^{\pi/2} x\cos 2x\, dx\right)$$

An antiderivative of x is $x^2/2$. To do $\int x\cos 2x\ dx$, we integrate by parts with

$$u = x,\ dv = \cos 2x\, dx \quad \to \quad v = \frac{1}{2}\sin 2x,\ du = dx.$$

Ignoring limits for now, we see that:

$$\int x\cos 2x\, dx = \frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x\, dx$$
$$= \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$$

Now we combine these to see that our integral is:

$$V = \frac{\pi}{2}\left(\frac{1}{2}x^2 + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\right)\Big|_0^{\pi/2}$$
$$= \frac{\pi}{2}\left(\left(\frac{\pi^2}{8} + 0 - \frac{1}{4}\right) - \left(0 + 0 + \frac{1}{4}\right)\right) = \boxed{\frac{\pi^3}{16} - \frac{\pi}{4}}.$$

8. (10 points) Consider the region R in the xy-plane bounded by the curves

$$x = 1 - y^2 \qquad \text{and} \qquad x = y^4 - 1$$

(a) Set up, but do not evaluate, an integral in terms of a single variable that represents the area of R. Justify your answer by drawing a picture and labeling a sample slice.

(5 points) First we find the intersection points of the two curves:

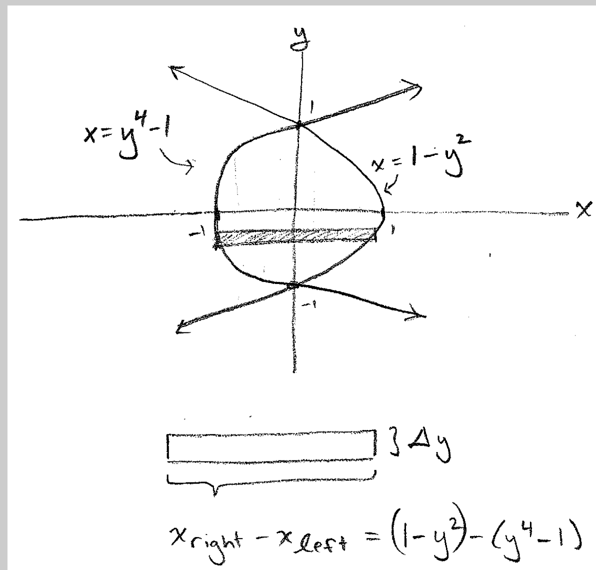
$$1 - y^2 = y^4 - 1 \iff y^4 + y^2 - 2 = 0 \iff (y^2+2)(y^2-1) = 0$$
$$\iff y^2 - 1 = 0 \text{ and } y \text{ is real} \iff y = \pm 1.$$

(We have used the fact that the quadratic y^2+2 has no real roots.) Plugging each of these y-values back into the curve equations, we can find that the intersection points are $(0, 1)$ and $(0, -1)$.

Now see the figure at right, where we have sliced perpendicular to the y-axis, from $y = -1$ to $y = 1$, into pieces of thickness Δy:

The area of a slice is given by $\left((1-y^2) - (y^4-1)\right)\Delta y$, and so the area between the two curves is given by

$$\boxed{\int_{-1}^{1} (1-y^2) - (y^4-1)\, dy}.$$



(b) Set up, but do not evaluate, an integral in terms of a single variable that represents the volume of the solid obtained by rotating R around the line $y = 1$. Justify your answer by citing the method used, drawing a picture and labeling a sample slice.

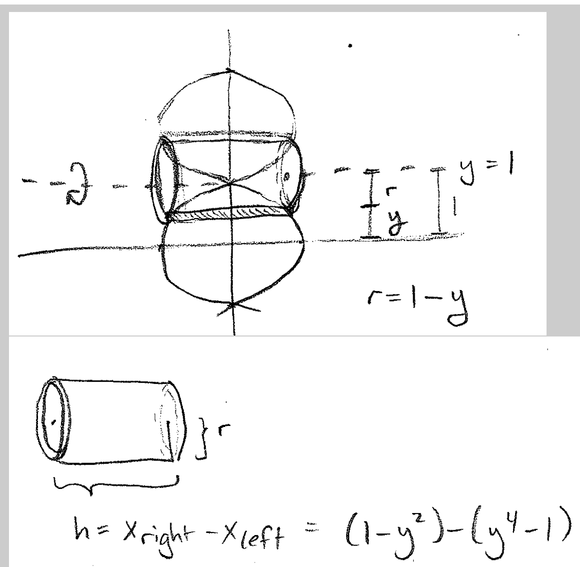
(5 points) The simplest method is by using cylindrical shells, since we slice perpendicular to the y-axis (again from $y = -1$ to $y = 1$).

The slice located at coordinate y, when rotated, creates a shell of volume

$$\Delta V = 2\pi r h \Delta y$$

and as shown in the picture, we have $r = 1 - y$ and $h = (1-y^2) - (y^4-1)$. Therefore the volume of the solid is given by

$$\boxed{\int_{-1}^{1} 2\pi(1-y)\left((1-y^2) - (y^4-1)\right)\, dy}.$$



9. (6 points) A spherical planet has radius R kilometers. The planet's atmosphere, also spherical in shape, extends from its surface up to an altitude of A kilometers above the surface. The density of the atmosphere at an altitude h kilometers above the surface of the planet can be given approximately by

$$\rho(h) = A - h \quad (\text{kg/km}^3)$$

Set up, but do not evaluate, an integral in terms of a single variable that represents the total mass of the planet's atmosphere (in kg). Justify your answer by describing how to slice up the atmosphere into thin pieces, each of approximately uniform density.

Remark. Some formulas that may or may not be useful: the volume V and surface area S of a sphere of radius r are given, respectively, by

$$V = \frac{4}{3}\pi r^3, \qquad S = 4\pi r^2.$$

Since the air density is approximately constant at a fixed height h above the surface of the planet, we will slice the atmosphere into very thin spherical slices, each sharing its center with the center of the planet and having thickness Δh. For any h in the interval $\{0 \leq h \leq A\}$, the slice located at height h above the surface has radius $(R + h)$, and since Δh is very small, the volume of this slice (measured in km^3) is approximately

$$\begin{aligned}\Delta V &\approx (\text{surface area}) \cdot (\text{thickness}) \\ &= 4\pi(R+h)^2 \Delta h.\end{aligned}$$

The density of the atmosphere at height h is $\rho(h) = (A - h)$ kg/km^3, and so the mass of the slice at height h (measured in kg) is given by

$$\begin{aligned}\Delta M &\approx (\text{density}) \cdot (\text{volume}) \\ &\approx (A-h)4\pi(R+h)^2 \Delta h\end{aligned}$$

As a stylistic Riemann sum, the total mass of the atmosphere is therefore approximately

$$\sum (A-h)4\pi(R+h)^2 \Delta h$$

kilograms, which comes from adding the contributions from each spherical shell. In the limit as $\Delta h \to 0$, the integral corresponding to this Riemann sum gives the total mass of the atmosphere (in kilograms):

$$\boxed{\text{total mass} = \int_0^A (A-h)4\pi(R+h)^2 \, dh}$$