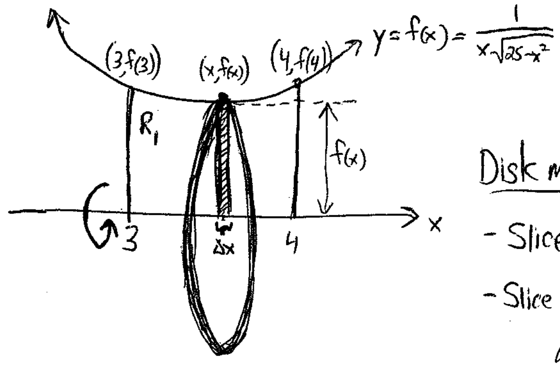
SOLUTIONS

1. (15 points)

(a) Let R_1 be the region in the xy-plane bounded by the curve $y = \dfrac{1}{x\sqrt{25-x^2}}$, the x-axis, and the lines $x = 3$ and $x = 4$. Set up, *but do not evaluate*, an integral representing the volume of the solid generated by revolving R_1 about the x-axis. Justify your answer (by citing the method used and labeling a corresponding sketch).

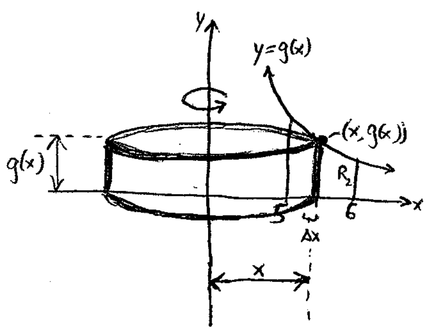


Disk method (i.e. volume by cross sections): slicing is perp to rotational axis.

- Slice R_1 vertically, i.e. perp. to x-axis; slices have thickness Δx.
- Slice at coordinate x (for $3 \le x \le 4$) forms a disk of approximate radius $f(x) = \frac{1}{x\sqrt{25-x^2}}$ when rotated (ht. of slice).
- Disk's volume is thus approx. (area)(thickness) $= \pi (f(x))^2 \Delta x$.
- In the limit as $\Delta x \to 0$ (#slices $\to \infty$),

$$\text{Volume} = \int_3^4 \pi (f(x))^2 dx = \boxed{\pi \int_3^4 \frac{1}{x^2(25-x^2)}\, dx}.$$

(b) Let R_2 be the region bounded by the curve $y = \dfrac{1}{x(x^2-16)^{3/2}}$, the x-axis, and the lines $x = 5$ and $x = 6$. Set up, *but do not evaluate*, an integral representing the volume of the solid generated by revolving R_2 about the y-axis. Again, justify your answer.



Method of cylindrical shells: slicing is parallel to rotational axis.

- Slice R_2 vertically, i.e. perp to x-axis; slices have thickness Δx.
- Slice at coord. x (for $5 \le x \le 6$) forms a cylindrical shell of approximate radius x & height $g(x) = \frac{1}{x(x^2-16)^{3/2}}$ when rotated.
- Shell's volume is thus approx. (surface area)(thickness) $= 2\pi$(radius)(ht.)(thickness) $= 2\pi x g(x) \Delta x$.
- In the limit as $\Delta x \to 0$ (#slices $\to \infty$),

$$\text{Volume} = \int_5^6 2\pi x g(x) dx = \boxed{2\pi \int_5^6 \frac{1}{(x^2-16)^{3/2}}\, dx}.$$

(c) Choose *one* of the integrals from parts (a) and (b), and evaluate it, showing your steps.

Part (a) integral: Requires partial fraction decomposition. (Note the repeated linear factor x !)

Setup: $$\frac{1}{x^2(25-x^2)} = \frac{1}{x\cdot x\cdot(5-x)(5+x)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{5-x}+\frac{D}{5+x}$$

$\left(\text{mult by } x^2(25-x^2)\right)$ $$\Rightarrow\ 1 = Ax(5-x)(5+x)+B(5-x)(5+x)+Cx^2(5+x)+Dx^2(5-x)$$
$$= Ax(25-x^2)+B(25-x^2)+C(5x^2+x^3)+D(5x^2-x^3).$$

Let $x=5$; we obtain $1=C\cdot 25\cdot 10$, so $C=\frac{1}{250}$.

Let $x=-5$; we obtain $1=D\cdot 25\cdot 10$, so $D=\frac{1}{250}$.

Let $x=0$; we obtain $1=B\cdot 25$, so $B=\frac{1}{25}$.

We still don't have A, so plug in B,C,D to above to find

$$1 = Ax(25-x^2)+\frac{1}{25}(25-x^2)+\frac{1}{250}(5x^2+x^3)+\frac{1}{250}(5x^2-x^3)$$
$$= Ax(25-x^2)+1-\frac{x^2}{25}+\frac{10x^2}{250}$$
$$= Ax(25-x^2)+1$$

$\Rightarrow Ax(25-x^2)=0$, so we must have $\underline{A=0}$.

Thus, $$\pi\int_3^4\frac{1}{x^2(25-x^2)}dx = \pi\int_3^4\left(\frac{1/25}{x^2}+\frac{1/250}{5-x}+\frac{1/250}{5+x}\right)dx = \pi\left[-\frac{1/25}{x}-\frac{1}{250}\ln|5-x|+\frac{1}{250}\ln|5+x|\right]_3^4$$
$$=\boxed{\pi\left(\frac{1}{25}\left(\frac{1}{3}-\frac{1}{4}\right)+\frac{1}{250}\left(\ln 9-\ln 8+\ln 2-\ln 1\right)\right)}.$$

Part (b) integral: Requires the trig substitution $\begin{cases} x=4\sec\theta \iff \theta=\operatorname{arcsec}(x/4) \\ dx=4\sec\theta\tan\theta\,d\theta. \end{cases}$

We have $$\int\frac{dx}{(x^2-16)^{3/2}} = \int\frac{4\sec\theta\tan\theta\,d\theta}{(16\sec^2\theta-16)^{3/2}} = \int\frac{4\sec\theta\tan\theta\,d\theta}{(16\tan^2\theta)^{3/2}} = \int\frac{4\sec\theta\tan\theta\,d\theta}{64\tan^3\theta} = \int\frac{\sec\theta\,d\theta}{16\tan^2\theta} =$$
$$=\frac{1}{16}\int\frac{\left(\frac{1}{\cos\theta}\right)}{\left(\frac{\sin^2\theta}{\cos^2\theta}\right)}d\theta = \frac{1}{16}\int\frac{\cos\theta}{\sin^2\theta}d\theta.$$ Now let $\begin{Bmatrix} u=\sin\theta \\ du=\cos\theta\,d\theta \end{Bmatrix}$, so the previous integral is equal to $$\frac{1}{16}\int u^{-2}du = \frac{1}{16}\cdot\frac{-1}{u}+C = -\frac{1}{16}\cdot\frac{1}{\sin\theta}+C = -\frac{1}{16}\csc\theta+C = -\frac{1}{16}\csc\left(\operatorname{arcsec}\left(\frac{x}{4}\right)\right)+C.$$

Thus, $$2\pi\int_5^6\frac{dx}{(x^2-16)^{3/2}} = 2\pi\left[-\frac{1}{16}\csc\left(\operatorname{arcsec}\left(\frac{x}{4}\right)\right)\right]_5^6 = \boxed{\frac{\pi}{8}\left(\csc\left(\operatorname{arcsec}\left(\frac{5}{4}\right)\right)-\csc\left(\operatorname{arcsec}\left(\frac{6}{4}\right)\right)\right)}.$$

2. (8 points) One end of an 18-foot rope weighing 0.4 lb per foot is fixed atop a high cliff. The other end of the rope hangs below the top of the cliff, and at the bottom end there is attached a bag of sand originally weighing 100 lb. The rope and sandbag are hoisted up to the top of the cliff at a constant rate, and as the bag rises, sand leaks out at a constant rate. The sandbag weighs exactly 10 lb when it reaches the top of the cliff.

Note that the bag loses 5 pounds for every foot hoisted; thus, the weight of the bag after it has been hoisted x feet upwards is $F_{\text{bag}}(x) = 100 - 5x$ pounds.

(a) How much work is done to hoist the rope and sandbag up the first 9 feet?

We can approach the problem in the most unified way (sandbag and rope together) by thinking of the task as exerting a *variable force* (i.e., the load's weight) *along a line*. Suppose x ($0 \leq x \leq 18$) is the number of feet of rope already hoisted; then if the remaining weight of the load, $F(x)$, is hoisted a tiny amount Δx, this requires work $\Delta W \approx F(x)\Delta x$. Thus, in the limit $\Delta x \to 0$, the *total* work for the task approaches the integral $W = \int_a^b F(x)\,dx$ where a and b are the starting and ending values for x. (In part (a), we have $0 \leq x \leq 9$, and for part (b), $9 \leq x \leq 18$.)

It remains to express the weight of the load $F(x)$ (in pounds) as a function of the amount x (in feet) already hoisted. But $F(x) = F_{\text{bag}}(x) + F_{\text{rope}}(x)$, where $F_{\text{bag}}(x)$ was found above, and where $F_{\text{rope}}(x) = (0.4)(18 - x)$ lb. Thus, for part (a),

$$W = \int_0^9 (F_{\text{bag}}(x) + F_{\text{rope}}(x))\,dx = \int_0^9 (107.2 - 5.4x)\,dx = (107.2)x - (2.7)x^2\Big|_0^9 = \boxed{(107.2)(9) - (2.7)(81)} \text{ ft-lb}$$

(This simplifies to 746.1 ft-lb.)

(b) How much work is subsequently done to hoist the rope and sandbag up the remaining 9 feet?

Based on the approach of part (a), the quick answer is

$$W = \int_9^{18} (F_{\text{bag}}(x)+F_{\text{rope}}(x))\,dx = \int_9^{18} (107.2-5.4x)\,dx = (107.2)x-(2.7)x^2\Big|_9^{18} = \boxed{(107.2)(9) - (2.7)(18^2 - 9^2)} \text{ ft-lb}$$

(This simplifies to 308.7 ft-lb.) But to showcase an alternative approach, let's compute the work to hoist the rope portion in a different way; this can then be added to the work to hoist the sandbag, which remains $W_{\text{bag}} = \int_9^{18} F_{\text{bag}}(x)\,dx$.

Suppose we introduce a y-axis oriented vertically along the 9-foot-long hanging portion of rope, with $y = 0$ at the top edge and $y = 9$ (feet) at the bottom end of the hanging rope.

Now break the rope into tiny segments of length Δy feet; notice that each segment weights $(0.4)\Delta y$ pounds, and that the segment located at coordinate y (for $0 \leq y \leq 9$) is lifted an approximately uniform distance of y feet. Hence, the work to lift the small segment at coordinate y satisfies $\Delta W \approx 0.4y\Delta y$. It follows that in the limit $\Delta y \to 0$, the *total* work to hoist all the rope (without sandbag) approaches the integral $W_{\text{rope}} = \int_0^9 0.4y\,dy$.

Thus, with this approach, the total work to lift both sandbag and rope is

$$\begin{aligned} W = W_{\text{bag}} + W_{\text{rope}} &= \int_9^{18} (100 - 5x)\,dx + \int_0^9 0.4y\,dy = \left(100x - (2.5)x^2\right)\Big|_9^{18} + (0.2)y^2\Big|_0^9 \\ &= \boxed{(100)(9) - (2.5)(18^2 - 9^2) + (0.2)(81)} \text{ ft-lb} \end{aligned}$$

which simplifies to 308.7 ft-lb, as before.

3. (11 points) Determine whether each of the following improper integrals converges. Explain your reasoning completely.

(a) $\int_{-\infty}^{\infty} \frac{1}{x^2}\,dx$ Note that this must be split into a sum of improper integrals; in fact, due to the infinities and the discontinuity at $x=0$, we have to split into more than just two integrals!

So $\int_{-\infty}^{\infty} \frac{1}{x^2}dx = \int_{-\infty}^{-1} \frac{1}{x^2}dx + \int_{-1}^{0} \frac{1}{x^2}dx + \int_{0}^{1} \frac{1}{x^2}dx + \int_{1}^{\infty} \frac{1}{x^2}dx$ (for example),

where each of the above four integrals can be evaluated by a limit of definite integrals and must converge for the overall sum to converge.

But $\int_0^1 \frac{1}{x^2}dx = \lim_{a\to 0^+} \int_a^1 \frac{1}{x^2}dx = \lim_{a\to 0^+} \left[-\frac{1}{x}\right]_a^1 = \lim_{a\to 0^+} \left(\frac{1}{a}-1\right) = \infty$,

so the integral we're considering will diverge.

(b) $\int_1^{\infty} \frac{\ln x}{x^3+1}\,dx$ We'll use a comparison theorem for improper integrals here.

We note that $0 \le \ln x < x$ for all $x \ge 1$. (Think about the graphs of the basic curves $y=\ln x$ and $y=x$; the fact can be proved by noting that the function $x-\ln x$ is positive at $x=1$ and increasing for $x>1$, so will be positive for all $x \ge 1$.)

Thus, $0 \le \frac{\ln x}{x^3+1} < \frac{\ln x}{x^3} < \frac{x}{x^3} = \frac{1}{x^2}$ for $x \ge 1$, since $0 < \frac{1}{x^3+1} < \frac{1}{x^3}$ on this domain.

But the integral $\int_1^{\infty} \frac{1}{x^2}dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}dx = \lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t = \lim_{t\to\infty}\left(1-\frac{1}{t}\right) = 1$, so it converges. Thus, the comparison theorem for improper integrals tells us that $\int_1^{\infty} \frac{\ln x}{x^3+1}dx$ converges as well.

Alternate solution, also using comparison (sketch): You can compute $\int \frac{\ln x}{x^3}dx = -\frac{1}{4x^2} - \frac{\ln x}{2x^2} + C$ by integration-by-parts, and then use l'Hôpital's Rule to find that $\int_1^{\infty} \frac{\ln x}{x^3}dx$ converges. Since $\frac{\ln x}{x^3+1} < \frac{\ln x}{x^3}$ for $x \ge 1$, the comparison theorem for integrals again gives us convergence.

4. (10 points) For this problem, use the following information:

- If f is a normal ("bell-shaped" or "Gaussian") probability density function, then f has the general form $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$
- A partial list of approximate values of the function $P(z) = \displaystyle\int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$ is as follows:

$P(0.5) \approx 0.69$	$P(1.1) \approx 0.86$
$P(0.6) \approx 0.72$	$P(1.2) \approx 0.88$
$P(0.7) \approx 0.76$	$P(1.3) \approx 0.90$
$P(0.8) \approx 0.79$	$P(1.4) \approx 0.92$
$P(0.9) \approx 0.82$	$P(1.5) \approx 0.93$
$P(1.0) \approx 0.84$	

Suppose that the *hitting distances* of baseball player Joe Slugger (that is, the distances traveled by each baseball Joe hits) are normally distributed, with mean 347 feet and standard deviation 20 feet. For the purposes of this problem, we define a *homer* to be any ball traveling 375 feet or more.

(a) What is the probability that a baseball hit by Joe is a homer? Justify your answer by writing an integral expression that represents this probability and showing how to find its value.

For $\mu = 347$ and $\sigma = 20$, the probability is

$$\mathrm{Prob}(X \geq 375) = \int_{375}^{\infty} f(x)\,dx = 1 - \int_{-\infty}^{375} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}\,dx$$

let $\begin{cases} t = \frac{x-\mu}{\sigma} = \frac{x-347}{20} \\ dt = \frac{1}{\sigma}dx \\ x = 375 \Leftrightarrow t = \frac{375-347}{20} = 1.4 \end{cases}$

$$= 1 - \int_{-\infty}^{1.4} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt$$

$$= 1 - P(1.4) \approx 1 - 0.92 = \boxed{0.08}$$

(or 8%)

(b) Suppose Joe adjusts his swing so that the mean changes, but the standard deviation remains 20 feet; he now finds that 14 percent of the baseballs he hits are homers. What is Joe's new mean hitting distance? (Again use an integral expression as part of your justification.)

With μ unknown and $\sigma = 20$, we have $\mathrm{Prob}(X \geq 375) = 1 - \mathrm{Prob}(X \leq 375) = 0.14$,

so $\mathrm{Prob}(X \leq 375) = 1 - 0.14 = 0.86 \approx P(1.1)$, according to the above list.

But $$\mathrm{Prob}(X \leq 375) = \int_{-\infty}^{375} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}\,dx$$

let $\begin{cases} t = \frac{x-\mu}{\sigma} = \frac{x-\mu}{20} \\ dt = \frac{1}{\sigma}dx \\ x = 375 \Leftrightarrow t = \frac{375-\mu}{20} \end{cases}$

$$= \int_{-\infty}^{\frac{375-\mu}{20}} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt$$

$$= P\left(\frac{375-\mu}{20}\right),$$ which means

we have $\frac{375-\mu}{20} \approx 1.1$, so that $\mu \approx 375 - (1.1)(20) = 375 - 22 = \boxed{353 \text{ feet}}$.

5. (10 points) In each of the problems below, determine whether the series converges or diverges. Indicate clearly what facts you use and how you apply them.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 - n}{n^3 + n + 1}$

Let $a_n = \frac{n^2-n}{n^3+n+1} = \frac{n(n-1)}{n^3+n+1}$; note that all factors in numerator & denominator are positive for $n \geq 2$.

Let $b_n = \frac{1}{n}$; note that $b_n > 0$ for $n \geq 2$ and that $\sum_{n=1}^{\infty} b_n$ diverges (it's a p-series with $p=1$, otherwise known as the harmonic series).

We have
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{n^2-n}{n^3+n+1}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{n^3-n^2}{n^3+n+1} = \lim_{n\to\infty} \frac{(n^3-n^2)(1/n^3)}{(n^3+n+1)(1/n^3)}$$
$$= \lim_{n\to\infty} \frac{1-1/n}{1+1/n^2+1/n^3} = \frac{1-0}{1+0+0} = 1,$$

since this limit is nonzero & finite, the Limit Comparison Test implies that either both $\sum a_n$ and $\sum b_n$ converge or both series diverge. But $\sum_{n=1}^{\infty} b_n$ diverges as noted above; thus, $\sum_{n=1}^{\infty} a_n$ <u>diverges</u> as well.

(b) $\sum_{n=1}^{\infty} 3^{1/n}$

We have $\lim_{n\to\infty} 3^{1/n} = 3^0 = 1 \neq 0$, so the Test for Divergence implies that $\sum_{n=1}^{\infty} 3^{1/n}$ <u>diverges</u>.

6. (10 points) Suppose we know that the power series

$$\sum_{n=0}^{\infty} c_n(x+3)^n$$

converges if $x = -7$ and diverges if $x = 2$. We are given no other information about this series. For each of the following statements, circle

- **T** if the statement must be true,
- **F** if the statement must be false, and
- **X** if the statement could be either true or false.

You do not need to justify your answers.

Center is $x=-3$; the facts above imply that if R is the radius of convergence of the power series, then $\underline{4 \leq R \leq 5}$.

T **F** **X** The series converges for $x = 0$.

$x=0$ is 3 units from center, so definitely inside radius.

T **F** **X** The series converges for $x = -1$.

$x=-1$ is 2 units from center, so definitely inside radius.

T **F** **X** The series diverges for $x = -8$.

$x=-8$ is 5 units from center; may or may not be outside radius.

T **F** **X** The series diverges for $x < -8$.

Any $x<-8$ is more than 5 units from center, so definitely outside radius.

T **F** **X** The series $\sum_{n=0}^{\infty}(-1)^n c_n$ satisfies $c_{n+1} \leq c_n$ for all $n \geq 0$.

The series in question is the "value" of the power series at $x=-4$, which is definitely inside the radius of convergence. However, all this tells us is that $\sum_{n=0}^{\infty}(-1)^n c_n$ converges.

There is no particular requirement that c_n be a decreasing sequence (we don't even know if c_n is strictly positive; it could be something like $c_n = \left(\frac{-1}{5}\right)^n$). We don't even know whether $|c_n|$ is decreasing, as it could be something like $c_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \left(\frac{1}{5}\right)^n & \text{if } n \text{ is odd} \end{cases}$ (or many other possibilities).

7. (10 points) In this problem, we make use of the fact that for all x, $\cos(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$, which you do not have to prove.

(a) Find, with complete justification, a range of values of x for which $\cos x \approx 1 - \frac{x^2}{2}$ with an error of no more than ± 0.01.

Sol#1 (alternating series perspective): Since $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$, using $1-\frac{x^2}{2}$ in place of $\cos x$ amounts to using a partial sum of an alternating series as an approximation to the infinite series. If the conditions of the Alternating Series Test are satisfied, then we'd be able to say that the remainder $\cos x - (1-\frac{x^2}{2})$ satisfies the inequality $\left|\cos x - (1-\frac{x^2}{2})\right| \le \left|\frac{x^4}{4!}\right| = \frac{|x|^4}{24}$; in such a case, if we insisted that $\frac{|x|^4}{24} \le 0.01$, then in fact we'd have $\left|\cos x - (1-\frac{x^2}{2})\right| \le \frac{|x|^4}{24} \le 0.01$, as desired. This would be true if $|x|^4 \le (0.01)(24) = 0.24$, i.e. $\boxed{-\sqrt[4]{0.24} \le x \le \sqrt[4]{0.24}}$.
But we must also check that the AST's conditions are satisfied: (i) Clearly $\frac{x^{2n}}{(2n)!} > 0$ for $n \ge 0$, since $2n$ is even; (ii) Decreasing terms? Yes, because $\frac{x^{2n+2}}{(2n+2)!} \Big/ \frac{x^{2n}}{(2n)!} = \frac{x^2}{(2n+1)(2n+2)} < 1$ for $x \le 1$ and $n \ge 0$; (iii) Does $\lim_{n\to\infty} \frac{x^{2n}}{(2n)!} = 0$? Yes, because we are given that the series converges. Thus, our reasoning by alternating series thm. is sound.

Sol#2 (Taylor poly perspective): Using the Taylor series given for $f(x) = \cos x$ above, we recognize $1-\frac{x^2}{2}$ as $T_2(x)$, i.e. the degree 2 Taylor polynomial for f about 0. Noting that $|f'''(x)| = |\sin x| \le 1$ for all x, we can apply Taylor's inequality with $n=2$ and $M=1$ to conclude that for all x, $|R_2(x)| = \left|\cos x - (1-\frac{x^2}{2})\right| = |f(x) - T_2(x)| \le \frac{1}{3!}|x|^3$. If we insist $\frac{|x|^3}{3!} \le 0.01$, then $|R_2(x)| \le \frac{|x|^3}{3!} \le 0.01$ as desired; thus, x should satisfy $|x|^3 \le (0.01)\cdot 3! = 0.06$; i.e. $\boxed{-\sqrt[3]{0.06} \le x \le \sqrt[3]{0.06}}$.
Alternate version: We might also recognize $1-\frac{x^2}{2}$ as $T_3(x)$ since the Taylor series has no x^3 term. Redoing the approach above for Taylor's Inequality with $n=3$ yields an interval for x that matches the one we found in the alternating series perspective.

(b) Let $g(x) = x^2\cos(x^2)$. Find $g^{(2010)}(0)$, the 2010$^{\text{th}}$ derivative of g evaluated at 0.

Taylor series to the rescue! On the one hand, the Taylor series recipe tells us that if $g(x)$ has a power series centered at 0, then its coefficient of x^{2010} is $c_{2010} = \frac{g^{(2010)}(0)}{2010!}$; but on the other hand, we know that g's Taylor series can be found as follows:

$$g(x) = x^2\cos(x^2) = x^2\cdot\left(1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \frac{(x^2)^6}{6!} + \frac{(x^2)^8}{8!} - \cdots\right) = x^2\sum_{n=0}^{\infty}(-1)^n\frac{(x^2)^{2n}}{(2n)!}$$

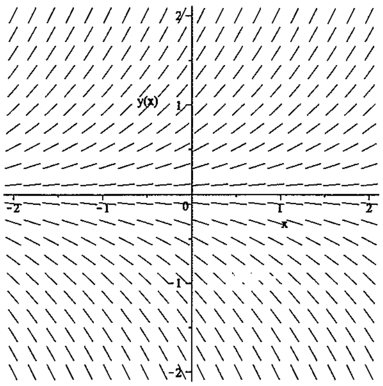
$$= x^2\cdot\left(1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \frac{x^{16}}{8!} - \cdots\right) = x^2\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n}}{(2n)!}$$

$$= x^2 - \frac{x^6}{2!} + \frac{x^{10}}{4!} - \frac{x^{14}}{6!} + \frac{x^{18}}{8!} - \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{(2n)!}$$

The x^{2010}-term of this series corresponds to index $n = \frac{2008}{4} = 502$, so it is $(-1)^{502}\cdot\frac{x^{2010}}{(1004)!}$.

Thus, $\frac{g^{(2010)}(0)}{(2010)!} = c_{2010} = \frac{1}{(1004)!}$, so that $\boxed{g^{(2010)}(0) = \frac{(2010)!}{(1004)!}}$.

8. (15 points) Match the direction fields below with their differential equations. Also indicate which two equations do not have matches.



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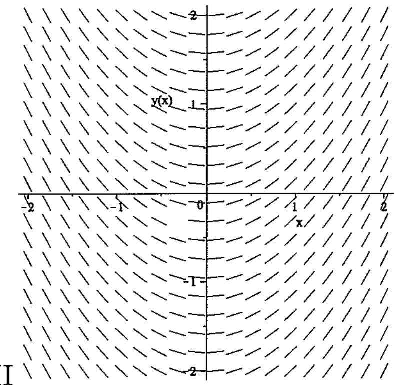
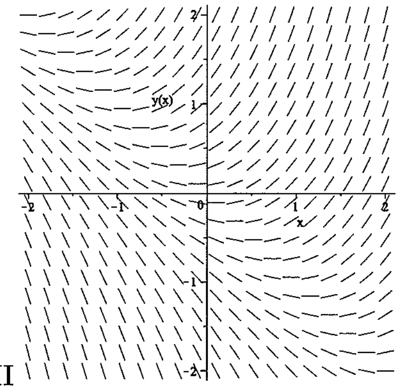
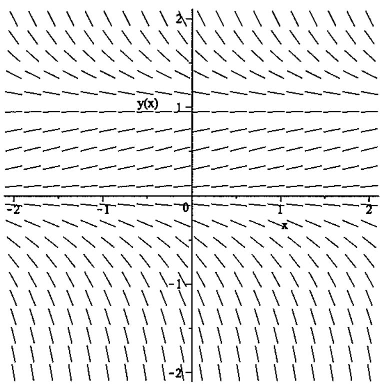


II



III



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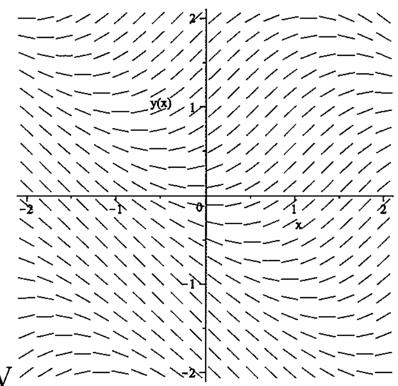
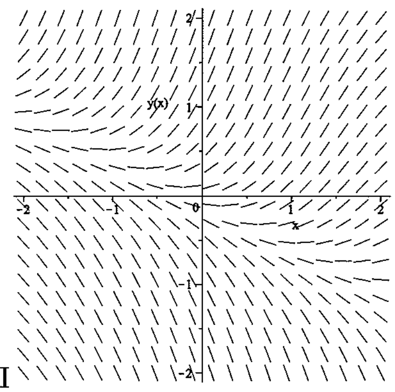


V



VI

Equation	I, II, III, IV, V VI, or "none"	Equation	I, II, III, IV, V VI, or "none"
$y' = y + \sin(x+y)$	VI	$y' = \cos(x+y) - 1$	none
$y' = x^2$	none	$y' = x + y$	III
$y' = y$	I	$y' = x$	II
$y' = \sin(x+y)$	V	$y' = y(1-y)$	IV

9. (15 points) Solve the following initial value problems, showing all your steps.

(a) $\dfrac{dy}{dx} = ky(1+\ln x)$, $y(1) = 8$ (here k is a fixed positive constant)

By separation of variables, we find that

$$\int \frac{1}{y}\,dy = \int k(1+\ln x)\,dx,$$

so that

$$\begin{aligned} \ln|y| &= k\int (1+\ln x)\,dx \\ &= k\left[x(1+\ln x) - \int x\cdot\tfrac{1}{x}\,dx\right] \\ &= kx(1+\ln x) - k\int dx \\ &= kx + kx\ln x - kx + C \\ &= kx\ln x + C, \quad \text{for any } C, \end{aligned}$$

int by parts $\left\{ \begin{array}{ll} u = 1+\ln x, & du = \frac{1}{x}dx \\ dv = dx, & v = x \end{array} \right\}$

meaning

$$|y| = e^{kx\ln x + C} = e^{kx\ln x}\cdot e^{C},$$

and

$$\begin{aligned} y &= \pm e^{C}\cdot e^{kx\ln x} \\ &= Ae^{kx\ln x}, \quad \text{for any nonzero } A. \end{aligned}$$

If $x=1$, then $y=8$, so $8 = Ae^{k\cdot\ln 1} = Ae^{0} = A$, and thus

$$\boxed{y = 8e^{kx\ln x}}$$

or equivalently $y = 8e^{\ln(x^{kx})} = 8x^{(kx)}$.

(b) $\dfrac{dy}{dx} = y^{-1} - y^{-2}, \quad y(3) = 4$

(Leave your answer as an implicit equation in x and y; don't try to solve for y.)

We have $\dfrac{dy}{dx} = y^{-1} - y^{-2} = \dfrac{1}{y} - \dfrac{1}{y^2} = \dfrac{y-1}{y^2}$.

By separation of variables, we proceed as follows: $\displaystyle\int \frac{y^2}{y-1}\,dy = \int dx = x + C$;

the left-hand integral may be computed either after long-division $\left(\dfrac{y^2}{y-1} = y+1+\dfrac{1}{y-1}\right)$ or via the substitution $\left\{\begin{array}{l} u = y-1 \iff y = u+1 \\ du = dy \end{array}\right\}$, which we'll do here:

$$\begin{aligned} x + C &= \int \frac{y^2}{y-1}\,dy \\ &= \int \frac{(u+1)^2}{u}\,du = \int \frac{u^2+2u+1}{u}\,du \\ &= \int \left(u + 2 + \frac{1}{u}\right) du \\ &= \frac{u^2}{2} + 2u + \ln|u| \\ &= \frac{(y-1)^2}{2} + 2(y-1) + \ln|y-1|. \end{aligned}$$

Setting $(x,y) = (3,4)$, we find

$$3 + C = \frac{3^2}{2} + 2\cdot 3 + \ln 3$$

$$\Rightarrow C = \frac{15}{2} + \ln 3,$$

so the solution curve is given by the equation

$$\boxed{x + \frac{15}{2} + \ln 3 = \frac{(y-1)^2}{2} + 2(y-1) + \ln|y-1|}.$$

This turns out to be equivalent to the curve $\boxed{x = \frac{y^2}{2} + y + \ln\left|\frac{y-1}{3}\right| - 9}$ after some simplifying.

10. (11 points) Alice has an 80 gallon fish tank. The water in the tank has 2 lbs of chlorine in it, which is too much to be safe for the fish. Beginning at noon, Alice runs water containing $\frac{1}{100}$ lb of chlorine per gallon into the tank at a rate of 2 gallons per minute, while also draining off the well-mixed water from the tank at the same rate.

   (a) Write down a differential equation for $c(t)$, the amount of chlorine in the tank after t minutes. Be sure to state your initial condition, including the units involved.

The initial condition is: $\boxed{c(0) = 2 \text{ lb}}$. Meanwhile, we note that since $c(t)$ is measured in lbs, we have that $c'(t) =$ net rate of change of amt. of chlorine, in lb/min; furthermore, at any time t, the current concentration of chlorine in the tank is $\frac{c(t)}{80}$, measured in lb/gal. We have that

$$c'(t) = \text{net rate of chg of salt} = \left(\begin{smallmatrix}Cl\\ \text{rate in}\end{smallmatrix}\right) - \left(\begin{smallmatrix}Cl\\ \text{rate out}\end{smallmatrix}\right) = \left(\begin{smallmatrix}\text{incoming}\\ Cl \text{ concentration}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{flow}\\ \text{rate in}\end{smallmatrix}\right) - \left(\begin{smallmatrix}\text{outgoing } Cl\\ \text{concentration}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{flow}\\ \text{rate out}\end{smallmatrix}\right)$$

$$= \left(\frac{1}{100} \ \text{lb/gal}\right)\left(2 \ \text{gal/min}\right) - \left(\frac{c(t)}{80} \ \frac{\text{lb}}{\text{gal}}\right)\left(2 \ \text{gal/min}\right);$$

thus, $\boxed{c'(t) = \frac{1}{50} - \frac{c(t)}{40}}$.

   (b) By solving the differential equation, find the amount of chlorine in the tank after 30 minutes.

$$\frac{dc}{dt} = \frac{1}{50} - \frac{c}{40} = \frac{1}{40}\left(\frac{4}{5} - c\right).$$

We could solve by separation of variables; as a variant, let's instead use u-substitution, with

$$\left\{\begin{array}{l} u(t) = \frac{4}{5} - c(t) \iff c(t) = \frac{4}{5} - u(t) \\ \frac{du}{dt} = -\frac{dc}{dt} \iff \frac{dc}{dt} = -\frac{du}{dt} \end{array}\right\}:$$

$$\Rightarrow \frac{du}{dt} = \frac{-1}{40} u$$

$$\Rightarrow u = Ae^{-t/40} \quad (\text{any value } A) \quad [\text{By the Law of Natural Growth/Decay}]$$

$$\Rightarrow c(t) = \frac{4}{5} - Ae^{-t/40}$$

Since $c(0) = 2$, we have $2 = \frac{4}{5} - Ae^0 = \frac{4}{5} - A \Rightarrow A = -\frac{6}{5}$;

thus, $c(t) = \frac{4}{5} + \frac{6}{5}e^{-t/40}$,

so $c(30) = \boxed{\frac{4}{5} + \frac{6}{5}e^{-3/4} \text{ lbs}}$.

11. (13 points) A remote Transylvanian village experiences an outbreak of vampire conversion, in which residents turn into vampires upon contact with the beasts. Legend has it that at any given time, the rate of vampire conversion is jointly proportional to the number of residents that have become vampires and the number of residents that have not yet become vampires; that is, it is proportional to the product of these two quantities.

Suppose the total population of the village at any given time, including vampires plus those not yet converted, is 2000. (So there are no other factors affecting the size of the village, like births, deaths, or migration.)

Let $y(t)$ be the total number of residents that have become vampires by time t, which is measured in days. Suppose also that at the moment when there are 500 vampires, the growth rate is 75 vampires per day.

(a) Write a differential equation that is satisfied by y, according to the above information.

We have $\frac{dy}{dt} = ky(2000-y)$ for some proportionality constant k;

we also know that $75 = k \cdot 500(2000-500) = k \cdot 500 \cdot 1500 = k \cdot 750000$,

so $k = \frac{1}{10000}$. Thus, $\boxed{\frac{dy}{dt} = \frac{1}{10000} y(2000-y)}$.

(b) For this and the subsequent parts, suppose $y(0) = 100$. Use Euler's method with $h = 5$ to estimate the number of residents that have become vampires after 10 days.

We'll need $\frac{10}{5} = 2$ steps to estimate $y(10)$. We have $(t_0, y_0) = (0, 100)$.

Thus, $t_1 = t_0 + h = 0 + 5 = 5$ (days) and

$y_1 = y_0 + h \cdot \left(\frac{1}{10000} \cdot 100 \cdot (2000-100)\right) = 100 + 5 \cdot \frac{1900}{100} = 100 + 95 = 195$ (vampires),

so $t_2 = t_1 + h = 5 + 5 = 10$ (days) and

$y_2 = y_1 + h \cdot \left(\frac{1}{10000} \cdot 195 \cdot (2000-195)\right) = 195 + 5 \cdot \frac{195 \cdot (1805)}{10000}$;

thus, $y(10) \approx y_2 = \boxed{195 + \frac{5 \cdot 195 \cdot 1805}{10000}}$ vampires. (This turns out to be approx. 371 vampires.)

(c) Solve the differential equation, using any method.

It's useful to recognize the differential equation as a logistic equation:

$$\frac{dy}{dt} = \frac{1}{10000}y(2000-y) = \frac{2000}{10000}y\left(1-\frac{y}{2000}\right) = \frac{1}{5}y\left(1-\frac{y}{2000}\right).$$

Thus, $y(t) = \dfrac{2000}{1+Ae^{-t/5}}$; since $y(0)=100$ we have $100 = \dfrac{2000}{1+Ae^{0}} = \dfrac{2000}{1+A}$,

So $1+A = \frac{2000}{100} = 20$, and $A=19$; thus $\boxed{y(t) = \dfrac{2000}{1+19e^{-t/5}}}$.

Alternatively, instead of citing the general solution to the logistic equation, we could separate variables and integrate, with the assistance of partial fraction decomp:

$$\int \frac{dt}{10000} = \int \frac{dy}{y(2000-y)} = \int \left(\frac{1/2000}{y} + \frac{1/2000}{2000-y}\right)dy$$

$$\Rightarrow \frac{1}{10000}t + C = \frac{1}{2000}\ln|y| - \frac{1}{2000}\ln|2000-y| = \frac{1}{2000}\ln\left|\frac{y}{2000-y}\right|$$

$$\Rightarrow \ln\left|\frac{y}{2000-y}\right| = \frac{t}{5}+C_1 \quad \Rightarrow \quad \frac{y}{2000-y} = A_1e^{t/5},$$ where we find $A_1 = \frac{1}{19}$.

Solving for y, we find $\dfrac{2000-y}{y} = 19e^{-t/5} \Rightarrow \dfrac{2000}{y} = 1+19e^{-t/5} \Rightarrow y(t) = \dfrac{2000}{1+19e^{-t/5}}$.

(d) How long will it take before 1500 residents have become vampires?

Solving $y(t)=1500$ for t, we find

$$1500 = \frac{2000}{1+19e^{-t/5}} \Rightarrow 1+19e^{-t/5} = \frac{2000}{1500} = \frac{4}{3}$$

$$\Rightarrow 19e^{-t/5} = \frac{4}{3}-1 = \frac{1}{3}$$

$$\Rightarrow e^{-t/5} = \frac{1}{57}$$

$$\Rightarrow t = \boxed{5\ln 57} \text{ days.}$$

12. (6 points) For this problem, no justification is necessary; simply circle your answers.

(a) Suppose that you wish to model a population with a differential equation of the form $dP/dt = f(P)$, where $P(t)$ is the population at time t. Experiments have been performed on the population that give the following information:

- The population $P = 0$ is an equilibrium solution.
- A population of $0 < P < 20$ will decrease.
- A population of $P = 20$ does not change.
- A population of $20 < P < 100$ increases.
- A population of $P > 100$ will decrease.

Which of the following differential equations best models this population? Circle one answer.

(i) $\dfrac{dP}{dt} = (P-20)(P-100)$

(iii) $\dfrac{dP}{dt} = P(20-P)(P-100)$

(ii) $\dfrac{dP}{dt} = P(20-P)(100-P)$

(iv) $\dfrac{dP}{dt} = (20-P)(P-100)$

(b) Which of the pairs of equations below represents the following predator-prey system: "Cattle eat blades of grass. A cow needs to eat thousands of blades of grass every day to survive." Circle one answer.

(i) $\dfrac{dy}{dt} = -y + 0.00001xy$, $\dfrac{dx}{dt} = x - 5000xy$

(iii) $\dfrac{dy}{dt} = y - 0.00001xy$, $\dfrac{dx}{dt} = -x - 5000xy$

(ii) $\dfrac{dy}{dt} = y - 0.00001xy$, $\dfrac{dx}{dt} = -x + 5000xy$

(iv) $\dfrac{dy}{dt} = -y + 0.00001xy$, $\dfrac{dx}{dt} = x + 5000xy$

13. (16 points) In a certain closed ecosystem, let functions $p(t)$ and $q(t)$ represent the population sizes (in thousands of beings) of two species, P and Q, respectively; here the time t is measured in months. Suppose further that the population sizes are modeled by the equations

$$\frac{dp}{dt} = \frac{p}{3} - \frac{pq}{12}$$
$$\frac{dq}{dt} = \frac{q}{4} - \frac{pq}{12}$$

(a) Describe the nature of the relationship between the two species: is it one of competition, cooperation, or predator and prey, and how can you tell? (If the relationship is predator and prey, make sure to explain how to tell which species is which.)

The "$-\frac{pq}{12}$" terms in each equation imply that the presence of one species diminishes the growth rate of the other species; that is, as the size of one species increases, the growth rate of the other decreases. This is a relationship of <u>mutual competition</u>. (<u>Note</u>: it's slightly incorrect to state that as p increases, q decreases; rather, it's the <u>growth rate</u> of species Q that decreases with larger p.)

(b) For each species, describe what happens if the other is not present.

If species Q is not present, i.e. if $q=0$, then $\frac{dp}{dt} = \frac{p}{3}$, and species P experiences natural growth at a relative rate of $\frac{1}{3}$ per month; i.e., p increases exponentially (for positive p).

If species P is not present, i.e. if $p=0$, then $\frac{dq}{dt} = \frac{q}{4}$, and species Q experiences natural growth at a relative rate of $\frac{1}{4}$ per month (or 25% per month); i.e., the population size of Q increases exponentially (for positive q).

(c) Find all equilibrium solutions for this system.

If p and q are both constant, then $\frac{dp}{dt} = \frac{dq}{dt} = 0$. Thus,

$$0 = \frac{p}{3} - \frac{pq}{12} = \frac{1}{12}p(4-q) \quad \text{and}$$
$$0 = \frac{q}{4} - \frac{pq}{12} = \frac{1}{12}q(3-p)$$

The first equation holds if $p=0$ or $q=4$. If $p=0$, then the second equation becomes $0=\frac{1}{12}q\cdot(3-0)$, so $q=0$. If $q=4$, the second equation becomes $0=\frac{1}{12}\cdot 4\cdot(3-p)$, so $p=3$. Thus, the only solutions are $\underline{\underline{(p,q)=(0,0)}}$ and $\underline{\underline{(p,q)=(3,4)}}$ (in thousands of beings).

For quick reference, here again is the system:

$$\frac{dp}{dt} = \frac{p}{3} - \frac{pq}{12} = \frac{1}{12}p(4-q)$$
$$\frac{dq}{dt} = \frac{q}{4} - \frac{pq}{12} = \frac{1}{12}q(3-p)$$

(d) Suppose that at time $t = 0$ months, we have $p(0) = 2$ and $q(0) = 5$. Use the differential equations to predict the sizes of the two populations in one month's time; be as mathematically precise as possible.

From the equations, we find that at $t=0$, $\frac{dp}{dt} = \frac{1}{12}p(4-q) = \frac{2}{12}(4-5) = -\frac{2}{12}$ and $\frac{dq}{dt} = \frac{1}{12}q(3-p) = \frac{5}{12}(3-2) = \frac{5}{12}$. Thus, an approximation (via Euler's method with step size $h=1$ month, or essentially a linear approximation) for the values of $p(1)$ and $q(1)$ are:

$$p(1) \approx p(0) + \frac{dp}{dt}\cdot 1 = 2 - \frac{2}{12} = \frac{11}{6} \text{ thousand beings, and}$$
$$q(1) \approx q(0) + \frac{dq}{dt}\cdot 1 = 5 + \frac{5}{12} = \frac{65}{12} \text{ thousand beings.}$$

(We could try a smaller step size for an even closer approximation, etc.)

(e) For the initial conditions of part (d), consider the *signs* of dp/dt and dq/dt at $t = 0$. Based the prediction you made in part (d), make a further prediction about whether dp/dt or dq/dt will change sign at some point *after* the first month. Explain fully how you are able to tell.

We found that $\frac{dp}{dt} < 0$ and $\frac{dq}{dt} > 0$ at $t=0$, leading to a decrease in the size of species P (from 2 thousand) and an increase in the size of species Q (from 5 thousand).

Having factored the differential equations, for nonzero values of p & q, we see that

$$\frac{dp}{dt} < 0 \text{ if and only if } q > 4, \text{ and}$$
$$\frac{dq}{dt} > 0 \text{ if and only if } 0 < p < 3.$$

Thus, the short-term behavior of species P & Q, which leads to q increasing further from 4 and p decreasing further below 3, in turn leads to further decreases in p and increases in q; this pattern will continue in the long term with no sign changes possible in $\frac{dp}{dt}$ or $\frac{dq}{dt}$.