Math 42: Calculus
Final Exam — March 15, 2010

Name: ______________________________ **SUID#:** ______________

Circle your section:				
Sukhada Fadnavis 02 (10-10:50am) 07 (1:15-2:05pm)	Dan Jerison 03 (11-11:50am) 08 (1:15-2:05pm)	Jason Lo 04 (11-11:50am) 09 (1:15-2:05pm)	Dmitriy Ivanov 05 (11-11:50am) 10 (2:15-3:05pm)	Yu-jong Tzeng ACE

- Complete the following problems. In order to receive full credit, please show all of your work and justify your answers.
- You do not need to simplify your answers unless specifically instructed to do so. You may use any result from class that you like, but if you cite a theorem be sure to verify the hypotheses are satisfied.
- **You have 3 hours.** This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted. If you finish early, you must hand your exam paper to a member of teaching staff.
- Please check that your copy of this exam contains 18 numbered pages and is correctly stapled.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- Please sign the following:

> "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ______________________________

The following boxes are strictly for grading purposes. Please do not mark.

1	15		**8**	15	
2	8		**9**	15	
3	11		**10**	11	
4	10		**11**	13	
5	10		**12**	6	
6	10		**13**	16	
7	10		**Total**	150	

1. (15 points)

   (a) Let R_1 be the region in the xy-plane bounded by the curve $y = \dfrac{1}{x\sqrt{25 - x^2}}$, the x-axis, and the lines $x = 3$ and $x = 4$. Set up, *but do not evaluate*, an integral representing the volume of the solid generated by revolving R_1 about the x-axis. Justify your answer (by citing the method used and labeling a corresponding sketch).

   (b) Let R_2 be the region bounded by the curve $y = \dfrac{1}{x(x^2 - 16)^{3/2}}$, the x-axis, and the lines $x = 5$ and $x = 6$. Set up, *but do not evaluate*, an integral representing the volume of the solid generated by revolving R_2 about the y-axis. Again, justify your answer.

(c) Choose *one* of the integrals from parts (a) and (b), and evaluate it, showing your steps.

2. (8 points) One end of an 18-foot rope weighing 0.4 lb per foot is fixed atop a high cliff. The other end of the rope hangs below the top of the cliff, and at the bottom end there is attached a bag of sand originally weighing 100 lb. The rope and sandbag are hoisted up to the top of the cliff at a constant rate, and as the bag rises, sand leaks out at a constant rate. The sandbag weighs exactly 10 lb when it reaches the top of the cliff.

   (a) How much work is done to hoist the rope and sandbag up the first 9 feet?

   (b) How much work is subsequently done to hoist the rope and sandbag up the remaining 9 feet?

3. (11 points) Determine whether each of the following improper integrals converges. Explain your reasoning completely.

   (a) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2}\,dx$

   (b) $\displaystyle\int_{1}^{\infty} \frac{\ln x}{x^3+1}\,dx$

4. (10 points) For this problem, use the following information:

   - If f is a normal ("bell-shaped" or "Gaussian") probability density function, then f has the general form $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$
   - A partial list of approximate values of the function $P(z) = \displaystyle\int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$ is as follows:

$P(0.5) \approx 0.69$	$P(1.1) \approx 0.86$
$P(0.6) \approx 0.72$	$P(1.2) \approx 0.88$
$P(0.7) \approx 0.76$	$P(1.3) \approx 0.90$
$P(0.8) \approx 0.79$	$P(1.4) \approx 0.92$
$P(0.9) \approx 0.82$	$P(1.5) \approx 0.93$
$P(1.0) \approx 0.84$	

Suppose that the *hitting distances* of baseball player Joe Slugger (that is, the distances traveled by each baseball Joe hits) are normally distributed, with mean 347 feet and standard deviation 20 feet. For the purposes of this problem, we define a *homer* to be any ball traveling 375 feet or more.

(a) What is the probability that a baseball hit by Joe is a homer? Justify your answer by writing an integral expression that represents this probability and showing how to find its value.

(b) Suppose Joe adjusts his swing so that the mean changes, but the standard deviation remains 20 feet; he now finds that 14 percent of the baseballs he hits are homers. What is Joe's new mean hitting distance? (Again use an integral expression as part of your justification.)

5. (10 points) In each of the problems below, determine whether the series converges or diverges. Indicate clearly what facts you use and how you apply them.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 - n}{n^3 + n + 1}$

(b) $\displaystyle\sum_{n=1}^{\infty} 3^{1/n}$

6. (10 points) Suppose we know that the power series

$$\sum_{n=0}^{\infty} c_n(x+3)^n$$

converges if $x = -7$ and diverges if $x = 2$. We are given no other information about this series. For each of the following statements, circle

- **T** if the statement must be true,
- **F** if the statement must be false, and
- **X** if the statement could be either true or false.

You do not need to justify your answers.

T **F** **X** The series converges for $x = 0$.

T **F** **X** The series converges for $x = -1$.

T **F** **X** The series diverges for $x = -8$.

T **F** **X** The series diverges for $x < -8$.

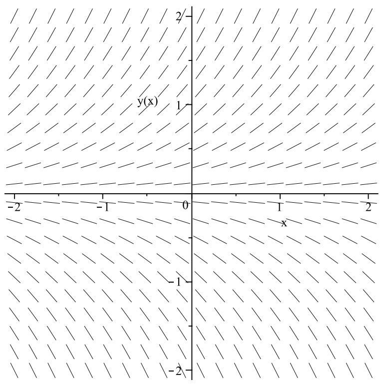
T **F** **X** The series $\sum_{n=0}^{\infty}(-1)^n c_n$ satisfies $c_{n+1} \leq c_n$ for all $n \geq 0$.

7. (10 points) In this problem, we make use of the fact that for all x, $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, which you do not have to prove.

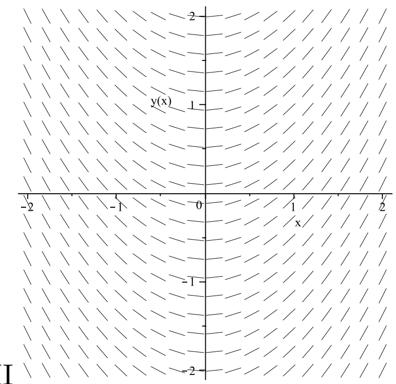
   (a) Find, with complete justification, a range of values of x for which $\cos x \approx 1 - \frac{x^2}{2}$ with an error of no more than ± 0.01.

   (b) Let $g(x) = x^2 \cos(x^2)$. Find $g^{(2010)}(0)$, the 2010^{th} derivative of g evaluated at 0.

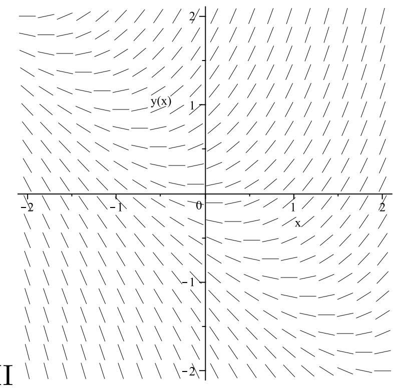
8. (15 points) Match the direction fields below with their differential equations. Also indicate which two equations do not have matches.



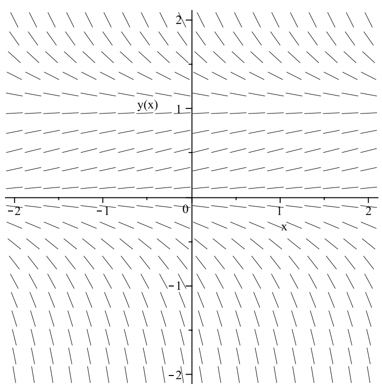
I



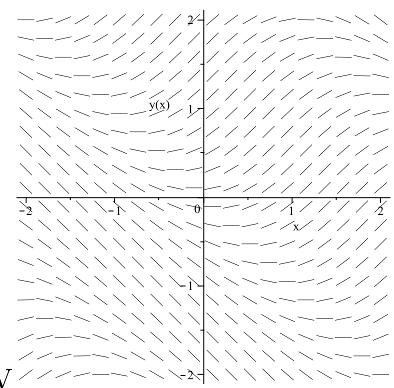
II



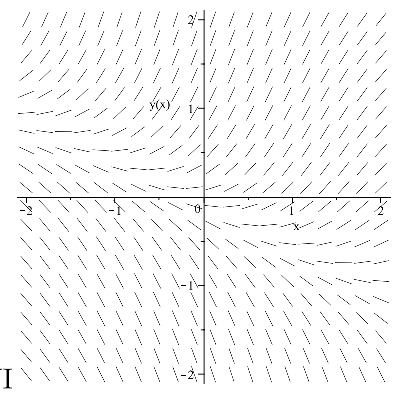
III



IV



V



VI

Equation	I, II, III, IV, V VI, or "none"	Equation	I, II, III, IV, V VI, or "none"
$y' = y + \sin(x+y)$		$y' = \cos(x+y) - 1$	
$y' = x^2$		$y' = x + y$	
$y' = y$		$y' = x$	
$y' = \sin(x+y)$		$y' = y(1-y)$	

9. (15 points) Solve the following initial value problems, showing all your steps.

   (a) $\dfrac{dy}{dx} = ky(1 + \ln x), \quad y(1) = 8$ (here k is a fixed positive constant)

(b) $\dfrac{dy}{dx} = y^{-1} - y^{-2}, \quad y(3) = 4$

(Leave your answer as an implicit equation in x and y; don't try to solve for y.)

10. (11 points) Alice has an 80 gallon fish tank. The water in the tank has 2 lbs of chlorine in it, which is too much to be safe for the fish. Beginning at noon, Alice runs water containing $\frac{1}{100}$ lb of chlorine per gallon into the tank at a rate of 2 gallons per minute, while also draining off the well-mixed water from the tank at the same rate.

    (a) Write down a differential equation for $c(t)$, the amount of chlorine in the tank after t minutes. Be sure to state your initial condition, including the units involved.

    (b) By solving the differential equation, find the amount of chlorine in the tank after 30 minutes.

11. (13 points) A remote Transylvanian village experiences an outbreak of vampire conversion, in which residents turn into vampires upon contact with the beasts. Legend has it that at any given time, the rate of vampire conversion is jointly proportional to the number of residents that have become vampires and the number of residents that have not yet become vampires; that is, it is proportional to the product of these two quantities.

    Suppose the total population of the village at any given time, including vampires plus those not yet converted, is 2000. (So there are no other factors affecting the size of the village, like births, deaths, or migration.)

    Let $y(t)$ be the total number of residents that have become vampires by time t, which is measured in days. Suppose also that at the moment when there are 500 vampires, the growth rate is 75 vampires per day.

    (a) Write a differential equation that is satisfied by y, according to the above information.

    (b) For this and the subsequent parts, suppose $y(0) = 100$. Use Euler's method with $h = 5$ to estimate the number of residents that have become vampires after 10 days.

(c) Solve the differential equation, using any method.

(d) How long will it take before 1500 residents have become vampires?

12. (6 points) For this problem, no justification is necessary; simply circle your answers.

(a) Suppose that you wish to model a population with a differential equation of the form $dP/dt = f(P)$, where $P(t)$ is the population at time t. Experiments have been performed on the population that give the following information:

- The population $P = 0$ is an equilibrium solution.
- A population of $0 < P < 20$ will decrease.
- A population of $P = 20$ does not change.
- A population of $20 < P < 100$ increases.
- A population of $P > 100$ will decrease.

Which of the following differential equations best models this population? Circle one answer.

(i) $\dfrac{dP}{dt} = (P - 20)(P - 100)$

(iii) $\dfrac{dP}{dt} = P(20 - P)(P - 100)$

(ii) $\dfrac{dP}{dt} = P(20 - P)(100 - P)$

(iv) $\dfrac{dP}{dt} = (20 - P)(P - 100)$

(b) Which of the pairs of equations below represents the following predator-prey system: "Cattle eat blades of grass. A cow needs to eat thousands of blades of grass every day to survive." Circle one answer.

(i) $$\frac{dy}{dt} = -y + 0.00001xy \qquad \frac{dx}{dt} = x - 5000xy$$

(iii) $$\frac{dy}{dt} = y - 0.00001xy \qquad \frac{dx}{dt} = -x - 5000xy$$

(ii) $$\frac{dy}{dt} = y - 0.00001xy \qquad \frac{dx}{dt} = -x + 5000xy$$

(iv) $$\frac{dy}{dt} = -y + 0.00001xy \qquad \frac{dx}{dt} = x + 5000xy$$

13. (16 points) In a certain closed ecosystem, let functions $p(t)$ and $q(t)$ represent the population sizes (in thousands of beings) of two species, P and Q, respectively; here the time t is measured in months. Suppose further that the population sizes are modeled by the equations

$$\frac{dp}{dt} = \frac{p}{3} - \frac{pq}{12}$$

$$\frac{dq}{dt} = \frac{q}{4} - \frac{pq}{12}$$

(a) Describe the nature of the relationship between the two species: is it one of competition, cooperation, or predator and prey, and how can you tell? (If the relationship is predator and prey, make sure to explain how to tell which species is which.)

(b) For each species, describe what happens if the other is not present.

(c) Find all equilibrium solutions for this system.

For quick reference, here again is the system:

$$\frac{dp}{dt} = \frac{p}{3} - \frac{pq}{12}$$
$$\frac{dq}{dt} = \frac{q}{4} - \frac{pq}{12}$$

(d) Suppose that at time $t = 0$ months, we have $p(0) = 2$ and $q(0) = 5$. Use the differential equations to predict the sizes of the two populations in one month's time; be as mathematically precise as possible.

(e) For the initial conditions of part (d), consider the *signs* of dp/dt and dq/dt at $t = 0$. Based the prediction you made in part (d), make a further prediction about whether dp/dt or dq/dt will change sign at some point *after* the first month. Explain fully how you are able to tell.