1. (10 points) A certain Internet search company owns a data center consisting of thousands of computer hard drives, any of which could fail at any time. The engineers have determined that for the drives they use, the probability density function for the lifespan of a random hard drive is given by

$$f(t) = \begin{cases} Ct & \text{if } 0 \le t \le 2, \\ 2Ce^{-(t-2)/3} & \text{if } t > 2, \\ 0 & \text{otherwise,} \end{cases}$$

where t is measured in years, and C is a positive constant.

(a) Find C, using the fact that f is a probability density function.

We require $\int_{-\infty}^{\infty} f(t)\,dt = 1$, so we find $1 = \int_{-\infty}^{\infty} f(t)\,dt = \int_{-\infty}^{0} 0\,dt + \int_0^2 Ct\,dt + \int_2^{\infty} 2Ce^{-(t-2)/3}\,dt$.

It might help before proceeding to calculate $\int e^{-(t-2)/3}\,dt$ using $u = \frac{-(t-2)}{3}$; $du = -\frac{1}{3}dt$:

$$\int e^{-(t-2)/3}\,dt = -3\int e^u\,du = -3e^u + K = \underline{\underline{-3e^{-(t-2)/3} + K}}.$$ (We're already using the letter C.)

Thus, $$1 = 0 + \left[\frac{Ct^2}{2}\right]_0^2 + \lim_{N\to\infty}\left[(2C)(-3)e^{-(t-2)/3}\right]_2^N$$

$$= \left(\frac{4C}{2} - 0\right) + \lim_{N\to\infty}\left[-6Ce^{-(N-2)/3} + 6Ce^0\right]$$

$$= 2C + 6C + \lim_{N\to\infty}\left(-6Ce^{-(N-2)/3}\right) = 2C + 6C + 0 = 8C, \text{ so } \boxed{C = \frac{1}{8}}.$$

(b) Find the mean lifespan of a hard drive used by the company.

We have $\mu = \int_{-\infty}^{\infty} tf(t)\,dt = \int_{-\infty}^{0} 0\,dt + \int_0^2 \frac{1}{8}t^2\,dt + \int_2^{\infty} \frac{t}{4}e^{-(t-2)/3}\,dt$.

It might help before proceeding to compute $\int te^{-(t-2)/3}\,dt$ by parts: using part (a),

$$\left\{\begin{matrix} u = t \\ dv = e^{-(t-2)/3}dt \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} du = dt \\ v = -3e^{-(t-2)/3} \end{matrix}\right\} \Rightarrow \int te^{-(t-2)/3}\,dt = -3te^{-(t-2)/3} + 3\int e^{-(t-2)/3}\,dt$$

(by part (a)) $= -3te^{-(t-2)/3} - 9e^{-(t-2)/3} + C = \underline{\underline{-(3t+9)e^{-(t-2)/3} + C}}.$

Thus, $$\mu = 0 + \left[\frac{1}{8}\cdot\frac{t^3}{3}\right]_0^2 + \lim_{N\to\infty}\left[-\frac{(3t+9)}{4}e^{-(t-2)/3}\right]_2^N$$

$$= 0 + \left(\frac{1}{8}\cdot\frac{8}{3} - 0\right) + \lim_{N\to\infty}\left(-\left(\frac{3N+9}{4}\right)e^{-(N-2)/3} + \frac{15}{4}e^0\right)$$

$$= \frac{1}{3} + \frac{15}{4} + \lim_{N\to\infty}\left(\frac{-(3N+9)}{4e^{(N-2)/3}}\right) = \frac{1}{3} + \frac{15}{4} + 0 = \boxed{\frac{49}{12}\text{ months}},$$ where we have found $\lim_{N\to\infty}\frac{-3N-9}{4e^{(N-2)/3}} = \lim_{N\to\infty}\frac{-3}{4/3\,e^{(N-2)/3}} = 0$ by l'Hôpital's Rule.

2. (9 points) Determine with justification whether each series converges, and if so, find the sum.

(a) $\sum_{n=1}^{\infty} \frac{3^{n/2}}{2^{n+1}} = \sum_{n=1}^{\infty} \frac{(3^{1/2})^n}{2 \cdot 2^n} = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^n = \frac{\sqrt{3}}{4} + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^3 + \cdots$

This is a geometric series with initial term $a = \frac{\sqrt{3}}{4}$ and common ratio $r = \frac{\sqrt{3}}{2}$; since $|r| = \frac{\sqrt{3}}{2} < 1$, this series converges. Its sum is $\frac{a}{1-r} = \boxed{\frac{\sqrt{3}/4}{1-\sqrt{3}/2}} \left(= \frac{\sqrt{3}}{4-2\sqrt{3}} \right)$.

(b) $\sum_{n=1}^{\infty} \left(\frac{5}{8^n} + \ln\left(\frac{2n+3}{2n+1}\right) \right)$ Let $a_n = \frac{5}{8^n}$ and $b_n = \ln\left(\frac{2n+3}{2n+1}\right) = \ln(2n+3) - \ln(2n+1)$;

it's best to analyze these series separately since they are so different.

We look at the first few partial sums of the series $\sum_{n=1}^{\infty} b_n$:

$$s_1 = b_1 = \ln(5) - \ln(3),$$
$$s_2 = b_1 + b_2 = (\ln(5) - \ln(3)) + \ln(7) - \ln(5) = \ln(7) - \ln(3),$$
$$s_3 = b_1 + b_2 + b_3 = (\ln(7) - \ln(3)) + \ln(9) - \ln(7) = \ln(9) - \ln(3),$$
$$s_4 = b_1 + b_2 + b_3 + b_4 = (\ln(9) - \ln(3)) + \ln(11) - \ln(9) = \ln(11) - \ln(3),$$

so we find that $s_n = \ln(2n+3) - \ln(3)$, and so $\sum_{n=1}^{\infty} b_n = \lim_{n\to\infty} s_n = \infty$, i.e. the series diverges, since $\lim_{n\to\infty} \ln(2n+3) = \infty$.

Now we notice that $0 \leq b_n < a_n + b_n$ because a_n and b_n are both positive for each n; thus, $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty}\left(\frac{5}{8^n} + \ln\left(\frac{2n+3}{2n+1}\right)\right)$ diverges as well by Comparison Test.

3. (10 points) Suppose that the series $\sum_{n=1}^{\infty} a_n$ converges, for *positive* numbers a_n.

Decide which of the following series must converge, must diverge, or may either converge or diverge (inconclusive). Circle your answer. You do not need to justify your answers.

The key observation in the above is that $\lim_{n\to\infty} a_n = 0$. —(This is used in parts a, b, c below, and indirectly in part e.)

[The positivity of a_n also allows us to think about Comparison Tests.]

(a) $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ Converges Diverges Inconclusive

Since $a_n > 0$, we have $1+a_n > 1$, so that $0 < \frac{a_n}{1+a_n} < a_n$ for all n.
Thus, by the Comparison Test, since $\sum a_n$ converges, so does $\sum \frac{a_n}{1+a_n}$.
(Limit Comparison Test using a_n and $\frac{a_n}{1+a_n}$ also leads to a conclusive result.)

(b) $\sum_{n=1}^{\infty} \frac{e^{a_n}-1}{a_n}$ Converges Diverges Inconclusive

By l'Hôpital's Rule, $\lim_{x\to 0} \frac{e^x-1}{x} = \lim_{x\to 0} \frac{e^x}{1} = \frac{e^0}{1} = 1$; as a consequence, $\lim_{n\to\infty} \frac{e^{a_n}-1}{a_n} = 1$ because $a_n \to 0$, and so the Test for Divergence says $\sum \frac{e^{a_n}-1}{a_n}$ diverges.

(c) $\sum_{n=1}^{\infty} \sin(a_n)$ Converges Diverges Inconclusive

By l'Hôpital's Rule, $\lim_{x\to 0} \frac{\sin x}{x} = 1$; as a consequence, $\lim_{n\to\infty} \frac{\sin(a_n)}{a_n} = 1$ because $a_n \to 0$.
Next note that $\sin(a_n) > 0$ whenever $0 < a_n < \pi$; this is eventually true for all sufficiently large n because $a_n \to 0$. Now apply the Limit Comparison Test to the series $\sum a_n$ and $\sum \sin(a_n)$ to get the result. (Comparison Test also could work if $0 < \sin(a_n) < a_n$ for sufficiently large n; that's true but tricky to prove.)

(d) $\sum_{n=1}^{\infty} \sqrt{n}\, a_n$ Converges Diverges Inconclusive

If $a_n = \frac{1}{n^3}$, then $\sum \sqrt{n}\, a_n = \sum \frac{1}{n^{5/2}}$, which converges, but
if $a_n = \frac{1}{n^{3/2}}$, then $\sum \sqrt{n}\, a_n = \sum \frac{1}{n}$, which diverges.

(e) $\sum_{n=1}^{\infty} (n\, a_n - 1)$ Converges Diverges Inconclusive

<u>Can't</u> converge, because if so, we'd have to have $\lim_{n\to\infty} (na_n - 1) = 0$.
But then we'd have $1 = \lim_{n\to\infty} na_n = \lim_{n\to\infty} \frac{a_n}{(1/n)}$, meaning that the Limit Comparison Test would say that $\sum a_n$ and $\sum \frac{1}{n}$ either both converge or both diverge. That's impossible – we know $\sum \frac{1}{n}$ diverges and $\sum a_n$ converges.

4. (9 points) For this problem, we consider the series $s = \sum_{n=1}^{\infty} \frac{1}{(n+1)\cdot(\ln(n+1))^2}$.

(a) Explain why this is a convergent series; that is, explain why the number s is defined.

Let $f(x) = \frac{1}{(x+1)(\ln(x+1))^2}$; note that f is continuous, positive and decreasing for $x \geq 1$ (because the factors in the denominator are positive, increasing, continuous functions). By the Integral Test, $\sum_{n=1}^{\infty} f(n)$ converges if and only if $\int_1^{\infty} f(x)dx$ does; we have

$$\int_1^{\infty} f(x)dx = \lim_{N\to\infty} \int_1^N \frac{1}{(x+1)(\ln(x+1))^2}dx \underset{\uparrow}{=} \lim_{N\to\infty} \int_{\ln 2}^{\ln(N+1)} \frac{1}{u^2}du = \lim_{N\to\infty}\left[-\frac{1}{u}\right]_{u=\ln 2}^{u=\ln(N+1)}$$

$\left\{\begin{matrix} u=\ln(x+1) \\ du = \frac{1}{x+1}dx \end{matrix}\right\}$
$(x=1 \Leftrightarrow u=\ln 2, \text{etc.})$

$$= \lim_{N\to\infty}\left(\frac{1}{\ln 2} - \frac{1}{\ln(N+1)}\right) = \frac{1}{\ln 2} - 0 = \frac{1}{\ln 2},$$

which converges, so $\sum_{n=1}^{\infty} f(n)$ converges too.

(b) Let s_{100} stand for the sum of the first 100 terms of the series. Determine the accuracy of using s_{100} as an approximation for s. State your conclusion in a complete sentence, and be as quantitatively precise as you can.

With f as in part (a), we see that the Integral Test Remainder Estimate (with $n=100$) implies

$$\int_{101}^{\infty} f(x)dx \leq s - s_{100} \leq \int_{100}^{\infty} f(x)dx.$$

We can copy the calculation in (a) to find $\int_{100}^{\infty} f(x)dx = \lim_{N\to\infty}\int_{100}^{N} \frac{1}{(x+1)(\ln(x+1))^2}dx = \lim_{N\to\infty}\int_{\ln(101)}^{\ln(N+1)} \frac{1}{u^2}du$ (for $u=\ln(x+1)$ etc)

$$= \lim_{N\to\infty}\left[-\frac{1}{u}\right]_{u=\ln(101)}^{u=\ln(N+1)} = \lim_{N\to\infty}\left(\frac{1}{\ln 101} - \frac{1}{\ln(N+1)}\right) = \frac{1}{\ln(101)},$$

and similarly that $\int_{101}^{\infty} f(x)dx = \frac{1}{\ln(102)}$. Thus, $\underline{\frac{1}{\ln(102)} \leq s - s_{100} \leq \frac{1}{\ln(101)}}$; alternatively, in plain English we could say that s_{100} underestimates s by an amount that is no less than $\frac{1}{\ln(102)}$, but no more than $\frac{1}{\ln(101)}$.

(c) It turns out that $s_{100} = 1.8933...$ Based on your reasoning from part (b), find a more accurate approximation for s, *without* having to consider any more terms from the series, and precisely state the accuracy of your new approximation. Your answers do not need to be simplified.

The above implies that $s_{100} + \frac{1}{\ln(102)} \leq s \leq s_{100} + \frac{1}{\ln(101)}$. One reasonable guess for s could be the midpoint of this interval, namely $\boxed{s_{100} + \frac{1}{2}\left(\frac{1}{\ln(102)} + \frac{1}{\ln(101)}\right)}$; this estimate could be either too large or too small, but in no case different from s by more than $\frac{1}{2}\left(\frac{1}{\ln(101)} - \frac{1}{\ln(102)}\right)$, half the length of the above interval.

5. (20 points) Determine whether each of the series below converges or diverges. Indicate clearly which tests you use and how you apply them.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{5^n + 3n}$

Let $b_n = \frac{3^n}{5^n} = \left(\frac{3}{5}\right)^n$. Note that $\sum_{n=1}^{\infty} b_n$ converges because it's a geometric series with $r = \frac{3}{5}$, and $\left|\frac{3}{5}\right| < 1$.

Then we have $0 < \frac{3^n}{5^n+3n} < \frac{3^n}{5^n} = b_n$ for $n \geq 1$,

So $\sum_{n=1}^{\infty} \frac{3^n}{5^n+3n}$ converges by the Comparison Test because $\sum_{n=1}^{\infty} b_n$ converges.

(There are also valid ways to reason using either the Limit Comparison Test or the Ratio Test; these methods require evaluating an indeterminate limit, e.g. with l'Hôpital's Rule.)

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{(-2)^n}$

We have $a_n = \dfrac{n^2}{(-2)^n}$; we'll use the Ratio Test.

We find $$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2}{(-2)^{n+1}} \cdot \frac{(-2)^n}{n^2} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-2)^n}{(-2)^{n+1}} \right| \cdot \left(\frac{n+1}{n} \right)^2$$

$$= \frac{1}{2} \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^2$$

$$= \frac{1}{2} \lim_{n\to\infty} \left(1 + \frac{1}{n} \right)^2 = \frac{1}{2}(1+0)^2 = \frac{1}{2} < 1,$$

so the series <u>converges</u>.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \arctan n}{n^3}$

We note that for positive x, $\arctan(x)$ is positive, and in fact $\arctan(x) < \frac{\pi}{2}$.

Thus, for $a_n = \frac{(-1)^n \arctan n}{n^3}$, we have

$$0 < |a_n| = \frac{\arctan n}{n^3} < \frac{(\pi/2)}{n^3}.$$

Meanwhile, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (it's a p-series with $p = 3 > 1$), so $\displaystyle\sum_{n=1}^{\infty} \frac{\pi/2}{n^3} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^3}$

converges as well. It follows by the Comparison Test that $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges too.

Now we see that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges by the Absolute Convergence Rule.

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n} = \frac{1}{(\ln 2)^2} + \frac{1}{(\ln 3)^3} + \frac{1}{(\ln 4)^4} + \frac{1}{(\ln 5)^5} + \cdots$

[Informal stuff: We notice that both base and exponent in the denominator increase, so we ought to be seriously suspecting convergence; after all, $\sum \frac{1}{b^n}$ converges even for a constant value of b (so long as $|b|>1$). This does not constitute a proof, but sometimes intuition can lead you in a good direction — namely, towards comparison with a geometric series that itself converges.
(On the other hand, blindly applying the Ratio Test to $a_n = \frac{1}{(\ln n)^n}$ is a bad direction, because the algebra is a mess.)]

We note that for $n \geq 3$, we have $1 < \ln 3 \leq \ln n$ (because $e < 3$),

so that $0 < \frac{1}{(\ln n)^n} \leq \frac{1}{(\ln 3)^n}$ for $n \geq 3$.

But $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(\ln 3)^n}$ is a convergent series (it's geometric with $|r| = \frac{1}{\ln 3} < 1$),

so $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(\ln n)^n}$ converges by the Comparison Test. It follows that

$$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n} = \frac{1}{(\ln 2)^2} + \sum_{n=3}^{\infty} \frac{1}{(\ln n)^n} \text{ converges as well.}$$

6. (12 points) Find, with complete justification, the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(5x-3)^n}{n\,5^n}$$

Let $a_n = \frac{(5x-3)^n}{n5^n}$; we'll apply the Ratio Test first. We find

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(5x-3)^{n+1}}{(n+1)\cdot 5^{n+1}} \cdot \frac{n\cdot 5^n}{(5x-3)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(5x-3)^{n+1}}{(5x-3)^n}\cdot\frac{5^n}{5^{n+1}}\cdot\frac{n}{n+1}\right|$$

$$= \frac{|5x-3|}{5}\lim_{n\to\infty}\left|\frac{n}{n+1}\right| = \frac{|5x-3|}{5}\lim_{n\to\infty}\left|\frac{1}{1+1/n}\right| = \frac{|5x-3|}{5}\cdot 1,$$

so we have convergence for $L<1$, which is written as

$$\frac{|5x-3|}{5} < 1 \iff |5x-3| < 5$$

$$\iff -5 < 5x-3 < 5$$

$$\iff -2 < 5x < 8 \iff -2/5 < x < 8/5.$$

For $L>1$, we have divergence, but the endpts where $L=1$, namely $|5x-3|=5$ or $x = 8/5, -2/5$, need to be checked by another test, since the Ratio Test is inconclusive.

Case $x = 8/5$: we have $a_n = \frac{5^n}{n\cdot 5^n} = \frac{1}{n}$, i.e. the series is $\sum_{n=1}^{\infty}\frac{1}{n}$.

This diverges, as it's a p-series with $p=1$ (and it's also known as the harmonic series, which rather famously diverges).

Case $x = -2/5$: we have $a_n = \frac{(-5)^n}{n\cdot 5^n} = \frac{(-1)^n}{n}$, i.e. the series is $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$.

In this case, we may apply the Alternating Series Test, since $b_n = \frac{1}{n}$ is positive, decreasing, and approaches 0 as $n\to\infty$. Thus, the series converges.

Conclusion: the power series converges for x in $\boxed{\{-2/5 \le x < 8/5\} = [-2/5, 8/5)}$.

7. (9 points) Let $f(x) = x^{3/2}$.

(a) Find the degree-2 Taylor polynomial T_2 for f about 9. (That is, $a=9$.)

$$f(x) = x^{3/2} \Rightarrow f(9) = 9^{3/2} = 27$$
$$f'(x) = \frac{3}{2}x^{1/2} \Rightarrow f'(9) = \frac{3}{2}\cdot 9^{1/2} = \frac{9}{2}$$
$$f''(x) = \frac{3}{4}x^{-1/2} \Rightarrow f''(9) = \frac{3}{4}\cdot 9^{-1/2} = \frac{1}{4}$$

Thus, $T_2(x) = f(9) + f'(9)\cdot(x-9) + \frac{f''(9)}{2!}(x-9)^2$

$$= \boxed{27 + \frac{9}{2}(x-9) + \frac{1}{8}(x-9)^2}.$$

(b) Use T_2 to find an approximation for $(9.1)^{3/2}$.

Note $(9.1)^{3/2} = f(9.1)$, so we use $T_2(9.1)$ to approximate it. (That is, $x=9.1$.)

$$\Rightarrow (9.1)^{3/2} = f(9.1) \approx T_2(9.1) = 27 + \frac{9}{2}(9.1-9) + \frac{1}{8}(9.1-9)^2 = 27 + \frac{9}{2}(0.1) + \frac{1}{8}(0.01)$$
$$= \boxed{27 + \frac{0.9}{2} + \frac{0.01}{8}} \left(= 27 + 0.45 + 0.00125 = 27.45125 \right)$$

(c) Determine the accuracy of your approximation from part (b), explaining the steps of your reasoning, and giving your final conclusion in sentence form.

By Taylor's Inequality, $|f(9.1) - T_2(9.1)| \leq \frac{M}{3!}\cdot|9.1-9|^3 = \frac{M}{6000}$,

where $M \geq |f'''|$ on the interval $[8.9, 9.1]$.

We have $f'''(x) = -\frac{3}{8}x^{-3/2}$, so $|f'''(x)| = \frac{3}{8}x^{-3/2}$, which is a <u>decreasing</u> function for positive x, and thus $|f'''(x)| \leq |f'''(8.9)| = \frac{3}{8}(8.9)^{-3/2}$ on $[8.9, 9.1]$.

Thus, we may take $M = \frac{3}{8}(8.9)^{-3/2}$, whatever that is.

It follows that $|f(9.1) - T_2(9.1)| \leq \frac{(3/8)(8.9)^{-3/2}}{6000} = \frac{1}{(16000)(8.9)^{3/2}}$; in plain English, $T_2(9.1)$ approximates $(9.1)^{3/2}$ to within $\frac{1}{(16000)(8.9)^{3/2}}$ units at worst.

8. (9 points)

(a) Find, showing all your steps, the Taylor series for $\sin x$ with center 0. (So, $f(x) = \sin x$ with $a = 0$.)

n	$f^{(n)}(x)$	$f^{(n)}(0)$	c_n
0	$\sin x$	0	$0/0! = 0$
1	$\cos x$	1	$1/1! = 1$
2	$-\sin x$	0	$0/2! = 0$
3	$-\cos x$	-1	$-1/3!$
4	$\sin x$	0	$0/4! = 0$

The pattern repeats, so

$$c_n = \begin{cases} 1/n! & \text{for } n = 1, 5, 9, \ldots \\ -1/n! & \text{for } n = 3, 7, 11, \ldots \\ 0 & \text{for } n \text{ even} \end{cases}$$

Thus, the Taylor series is $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}$.

(b) Use series to find $\lim_{x \to 0} \left(\frac{1}{x \sin x} - \frac{1}{x^2} \right)$. (You may take for granted the fact that the Taylor series for $\sin x$ converges to $\sin x$.)

$$\lim_{x \to 0} \left(\frac{1}{x \sin x} - \frac{1}{x^2} \right) = \lim_{x \to 0} \left(\frac{x - \sin x}{x^2 \sin x} \right)$$

$$= \lim_{x \to 0} \left(\frac{x - (x - x^3/3! + x^5/5! - x^7/7! + \cdots)}{x^2 (x - x^3/3! + x^5/5! - x^7/7! + \cdots)} \right)$$

$$= \lim_{x \to 0} \left(\frac{x^3/3! - x^5/5! + x^7/7! - \cdots}{x^3 - x^5/3! + x^7/5! - \cdots} \right)$$

$$= \lim_{x \to 0} \frac{x^3 \left(\frac{1}{3!} - \frac{x^2}{5!} + \frac{x^4}{7!} - \cdots \right)}{x^3 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots \right)}$$

$$= \lim_{x \to 0} \frac{(1/3! - x^2/5! + \cdots)}{(1 - x^2/3! + \cdots)} = \frac{1/3! - 0 + 0 - \cdots}{1 - 0 + 0 - \cdots} = \boxed{\frac{1}{6}}.$$

9. (12 points) In each of the parts below, show all the steps in your reasoning.

(a) Write $\frac{1}{1+x}$ as a power series about 0, and state the interval of convergence. (Hint: use geometric series.)

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1+(-x)+(-x)^2+(-x)^3+(-x)^4+\dots$$
$$= 1-x+x^2-x^3+x^4-\dots = \boxed{\sum_{n=0}^{\infty}(-1)^n x^n},$$

using the geometric series rule with $a=1$ and $r=-x$; hence we have convergence if and only if $|r|<1$, i.e. $|-x|<1 \iff -1<x<1$.

That is, the interval of convergence is $\underline{(-1,1)}$.

(b) Find a power series for $\ln(1+x)$. What is the radius of convergence?

If we integrate the series of part (a) with respect to x, we find

$$\ln|1+x| = \int \frac{1}{1+x}dx = \int(1-x+x^2-x^3+x^4-\dots)dx = \int\left(\sum_{n=0}^{\infty}(-1)^n x^n\right)dx$$
$$= C+x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\dots = C+\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1},$$

and plugging in $x=0$ to both the function $\ln|1+x|$ and the series yields

$$\ln|1+0| = 0 = C+0-0+0-\dots \implies \underline{C=0}.$$

By a result from the course, the radius of convergence is the same as that for the expression we started with (i.e. before we integrated), which is $\underline{1}$.

Thus, for $-1<x<1$, we have $\ln|1+x| = \ln(1+x) = \boxed{\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}}$.

(c) Express the number $\int_0^{0.01} \frac{\ln(1+x)}{x}\, dx$ as a series.

We have $\int_0^{0.01} \frac{\ln(1+x)}{x} dx = \int_0^{0.01} \frac{x - x^2/2 + x^3/3 - x^4/4 + x^5/5 - \cdots}{x} dx = \int_0^{0.01} \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{x(n+1)} \right) dx$

$$= \int_0^{0.01} \left(1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} - \cdots \right) dx = \int_0^{0.01} \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1} \right) dx$$

$$= \left(x - \frac{x^2}{4} + \frac{x^3}{9} - \frac{x^4}{16} + \frac{x^5}{25} - \cdots \right) \Big]_0^{0.01} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)^2} \Big]_0^{0.01}$$

$$= 0.01 - \frac{(0.01)^2}{4} + \frac{(0.01)^3}{9} - \frac{(0.01)^4}{16} + \frac{(0.01)^5}{25} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n (0.01)^{n+1}}{(n+1)^2}$$

$$= \frac{1}{100} - \frac{1}{4 \cdot 100^2} + \frac{1}{9 \cdot 100^3} - \frac{1}{16 \cdot 100^4} + \frac{1}{25 \cdot 100^5} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2 (100)^{n+1}}$$

(or equivalent)

(d) How many terms of the series of part (c) are required to estimate the integral to within 0.000005, or 5×10^{-6}? Explain completely.

We can apply the Alternating Series Estimation Theorem as long as the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2 (100)^{n+1}}$ satisfies the conditions of the Alternating Series Test; that is, we require $b_n = \frac{1}{(n+1)^2 (100)^{n+1}}$ to be positive, decreasing, and approaching zero as $n \to \infty$. This holds because the denominator $(n+1)^2 (100)^{n+1}$ is a product of positive, increasing functions of n that grows to ∞ as $n \to \infty$.

Thus, the k^{th} partial sum of the series is known to be an approximation for the integral up to an error whose absolute value is bounded by the absolute value of the $(k+1)$st term in the series.

Since $\frac{1}{9 \cdot 100^3} = \frac{1}{9} \times 10^{-6} < 5 \times 10^{-6}$, we know that the sum $\boxed{\frac{1}{100} - \frac{1}{4 \cdot 100^2}}$ (two terms) estimates the integral to within (better than) the desired amount.