1. (40 points) Evaluate each of the following integrals, showing all of your reasoning.

(6pts) (a) $\int_3^8 x\sqrt{1+x}\,dx$ Let $\left\{\begin{matrix} u=1+x \\ du=dx\end{matrix}\right\}$, so that $x=u-1$ and $\left\{\begin{matrix} x=3 \Rightarrow u=4 \\ x=8 \Rightarrow u=9\end{matrix}\right\}$; thus, we obtain

$$\int_3^8 x\sqrt{1+x}\,dx = \int_4^9 (u-1)\sqrt{u}\,du = \int_4^9 (u\sqrt{u}-\sqrt{u})\,du = \int_4^9 (u^{3/2}-u^{1/2})\,du$$

$$= \left.\frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2}\right]_4^9$$

$$= \boxed{\left(\frac{2}{5}\cdot 9^{5/2} - \frac{2}{3}9^{3/2}\right) - \left(\frac{2}{5}\cdot 4^{5/2} - \frac{2}{3}\cdot 4^{3/2}\right)}$$

(7pts) (b) $\int e^t \cos t\,dt$ Integrate by parts with $\left\{\begin{matrix} u=e^t & du=e^t dt \\ dv=\cos t\,dt & v=\sin t\end{matrix}\right\}$:

$$\Rightarrow \int e^t\cos t\,dt = e^t\sin t - \int e^t \sin t\,dt.$$ Now integrate by parts with $\left\{\begin{matrix} f=e^t & df=e^t dt \\ dg=\sin t\,dt & g=-\cos t\end{matrix}\right\}$:

$$\Rightarrow \int e^t\cos t\,dt = e^t\sin t - \left[-e^t\cos t + \int e^t\cos t\,dt\right]$$

$$= e^t\sin t + e^t\cos t - \int e^t\cos t\,dt$$; adding $\int e^t\cos t\,dt$ to both sides yields

$$2\int e^t\cos t\,dt = e^t\sin t + e^t\cos t$$

$$\Rightarrow \boxed{\int e^t\cos t\,dt = \frac{1}{2}\left(e^t\sin t + e^t\cos t\right) + C}$$

(7pts) (c) $\int x^2\sqrt{9-x^2}\,dx$

After trying "simple" substitutions with no success, we let $x=3\sin\theta$, so that $dx=3\cos\theta\,d\theta$, and also $\theta=\arcsin\left(\frac{x}{3}\right)$; we obtain

$$\begin{aligned}
\int x^2\sqrt{9-x^2}\,dx &= \int (3\sin\theta)^2\sqrt{9-(3\sin\theta)^2}\cdot 3\cos\theta\,d\theta \\
&= \int 9\sin^2\theta\sqrt{9(1-\sin^2\theta)}\cdot 3\cos\theta\,d\theta \\
&= 81\int \sin^2\theta\cos\theta\sqrt{\cos^2\theta}\,d\theta = 81\int \sin^2\theta\cos^2\theta\,d\theta \\
&= 81\int \tfrac{1}{2}(1-\cos 2\theta)\cdot\tfrac{1}{2}(1+\cos 2\theta)\,d\theta \\
&= \frac{81}{4}\int (1-\cos^2(2\theta))\,d\theta = \frac{81}{4}\int \sin^2(2\theta)\,d\theta \\
&= \frac{81}{4}\int \tfrac{1}{2}(1-\cos 4\theta)\,d\theta = \frac{81}{8}\int (1-\cos 4\theta)\,d\theta \\
&= \frac{81}{8}\left(\theta-\frac{\sin 4\theta}{4}\right)+C = \boxed{\frac{81}{8}\left(\arcsin\left(\frac{x}{3}\right)-\frac{1}{4}\sin\left(4\arcsin\left(\frac{x}{3}\right)\right)\right)+C}
\end{aligned}$$

(6pts) (d) $\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$

$$\left\{\begin{aligned} u&=\sqrt{x} &\Rightarrow&\ \text{If } x=N,\text{ then } u=\sqrt{N} \\ du&=\tfrac{1}{2}x^{-1/2}dx &\Rightarrow&\ 2du=\frac{dx}{\sqrt{x}} \end{aligned}\right\}$$

$$\begin{aligned}
\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx &= \lim_{N\to\infty}\int_1^N \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx \\
&= \lim_{N\to\infty}\int_1^{\sqrt{N}} 2e^{-u}\,du \\
&= \lim_{N\to\infty}\left[-2e^{-u}\right]_1^{\sqrt{N}} \\
&= \lim_{N\to\infty}\left(-2e^{-\sqrt{N}}-(-2e^{-1})\right) \\
&= \lim_{N\to\infty}\left(\frac{2}{e}-\frac{2}{e^{\sqrt{N}}}\right) \\
&= \frac{2}{e}-0=\boxed{\frac{2}{e}}
\end{aligned}$$

(7pts) (e) $\int \frac{1}{(t+2\sqrt{t}+2)\sqrt{t}}\,dt$

Let $u=\sqrt{t} \Leftrightarrow t=u^2$, so that $dt=2u\,du$; we obtain

$$\int \frac{dt}{(t+2\sqrt{t}+2)\sqrt{t}} = \int \frac{2u\,du}{(u^2+2u+2)u} = 2\int \frac{du}{u^2+2u+2}$$

quadratic has no real roots; complete the square!

$$= 2\int \frac{du}{(u+1)^2+1}$$

$$= 2\int \frac{dv}{v^2+1} \qquad \begin{cases} v=u+1 \\ dv=du \end{cases}$$

$$= 2\arctan(v)+C$$

$$= 2\arctan(u+1)+C$$

$$= \boxed{2\arctan(\sqrt{t}+1)+C}$$

(7pts) (f) $\displaystyle\int \frac{2x^2}{x^3 - x^2 - x + 1}\,dx = \int \frac{2x^2}{(x-1)^2(x+1)}\,dx$

Partial fraction decomposition: $\displaystyle\frac{2x^2}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$

multiply by denom $\Rightarrow$ $2x^2 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$

If $x=1$, we obtain $2=2B \Rightarrow \underline{B=1}$.

If $x=-1$, we obtain $2=4C \Rightarrow \underline{C=1/2}$.

Consequently, we find $2x^2 = A(x^2-1) + (x+1) + \frac{1}{2}(x^2-2x+1)$

$= (A+\frac{1}{2})x^2 + (\frac{3}{2}-A)$, so that $\underline{A=\frac{3}{2}}$.

Thus, $\displaystyle\int \frac{2x^2}{(x-1)^2(x+1)}\,dx = \int\left(\frac{3/2}{x-1} + \frac{1}{(x-1)^2} + \frac{1/2}{x+1}\right)dx$

$\displaystyle= \frac{3}{2}\int\frac{dx}{x-1} + \int\frac{1}{(x-1)^2}\,dx + \frac{1}{2}\int\frac{dx}{x+1}$

$\displaystyle= \boxed{\frac{3}{2}\ln|x-1| - \frac{1}{x-1} + \frac{1}{2}\ln|x+1| + C}$

2. (8 points)

(4pts) (a) Set up an integral that represents the length of the curve $y = e^{2x} + 3$ from the point $(0, 4)$ to the point $(1, 3 + e^2)$. Show your steps, but stop before evaluating the integral.

$$L = \int_{x=0}^{x=1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1 + (2e^{2x})^2}\, dx$$

(4pts) (b) Now evaluate the integral you found in part (a); you do not have to simplify the numerical expression you obtain. You may find it useful to know the following integral table entry, which you *do not* have to prove:

$$\int \frac{\sqrt{a^2 + u^2}}{u}\, du = \sqrt{a^2 + u^2} - a \ln \left| \frac{a + \sqrt{a^2 + u^2}}{u} \right| + C$$

Let $u = 2e^{2x}$, so that $du = 4e^{2x} dx$ and thus $\frac{du}{2u} = dx$; furthermore $\left\{ \begin{array}{l} x=0 \Rightarrow u=2 \\ x=1 \Rightarrow u=2e^2 \end{array} \right\}$.

We obtain

$$L = \int_0^1 \sqrt{1 + (2e^{2x})^2}\, dx = \int_2^{2e^2} \sqrt{1 + u^2} \cdot \frac{du}{2u}$$

$$= \frac{1}{2} \int_2^{2e^2} \frac{\sqrt{1+u^2}}{u}\, du$$

$$= \frac{1}{2}\left( \sqrt{1+u^2} - \ln \left| \frac{1 + \sqrt{1+u^2}}{u} \right| \right) \Bigg|_2^{2e^2}$$

$$= \boxed{\frac{1}{2}\left( \left( \sqrt{1+4e^4} - \ln \left| \frac{1+\sqrt{1+4e^4}}{2e^2} \right| \right) - \left( \sqrt{5} - \ln \left| \frac{1+\sqrt{5}}{2} \right| \right) \right)}$$

3. (10 points) Consider the integral $\int_0^1 f(x)\,dx$, where $f(x) = \sqrt{1+x^3}$.

(5pts) (a) Estimate the error made in approximating the value of this integral using the Trapezoidal Rule with $n = 10$ subintervals. *State your answer in a complete sentence.* You may make use of the fact that $f''(x) = \dfrac{3x(x^3+4)}{4(x^3+1)^{3/2}}$.

The error-bound formula for $E_T$ is $|E_T| \leq \dfrac{K_2(b-a)^3}{12n^2} = \dfrac{K_2 \cdot 1}{1200}$

so we need $K_2$, a number greater than or equal to the maximum value of $|f''|$ on $[0,1]$.

Note that for $x$ in $[0,1]$, we have $\underline{0 \leq x \leq 1}$ and $0 \leq x^3 \leq 1$, so that

$$\underline{4 \leq x^3+4 \leq 5} \quad \text{and} \quad \left\{ \begin{array}{l} 1 \leq (x^3+1)^{3/2} \leq 2^{3/2} \\ \Rightarrow \dfrac{1}{2^{3/2}} \leq \underline{\dfrac{1}{(x^3+1)^{3/2}} \leq 1} \end{array} \right\}, \quad \text{meaning} \quad 0 \leq \frac{3x(x^3+4)}{4(x^3+1)^{3/2}} \leq \frac{3}{4}\cdot 1 \cdot 5 \cdot 1 = \frac{15}{4}.$$

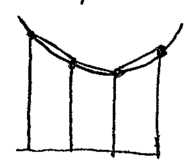
(via multiplication of three inequalities underlined)

Thus, $0 \leq f''(x) \leq \frac{15}{4}$ for $x$ in $[0,1]$, so $|f''(x)| \leq \frac{15}{4}$ for such $x$, and we may take $\underline{K_2 = \frac{15}{4}}$.

It follows that $|E_T| \leq \frac{15}{4800}$, i.e. the Trapezoidal Rule is accurate to within $\frac{15}{4800}$ units of the actual value of the integral.

(2pts) (b) Would the Trapezoidal Rule approximation described in part (a) give an overestimate or underestimate of the actual value, or is it impossible to tell with the information given? Explain briefly.

Since $f''(x) \geq 0$ for $x$ in $[0,1]$, the graph of $f$ is concave upward; the Trapezoidal Rule gives an <u>overestimate</u> in such a situation, since the top edges of trapezoids are secant lines to the graph of $f$, and therefore lie above the curve.



(3pts) (c) Again using the Trapezoidal Rule, how many subintervals $n$ would be necessary to guarantee an error of at most $10^{-6}$? Give a valid $n$ in simplified form. (As long as you justify your answer, you do not have to worry about finding the best possible value.)

We must find $n$ such that $\dfrac{K_2\cdot(b-a)^3}{12n^2} \leq 10^{-6}$, for then $|E_T| \leq \dfrac{K_2\cdot(b-a)^3}{12n^2} \leq 10^{-6}$,

as desired; we'll again take $K_2 = \frac{15}{4}$ and let $a=0, b=1$.

We have $\dfrac{15}{4}\cdot\dfrac{1}{12n^2} \leq 10^{-6}$

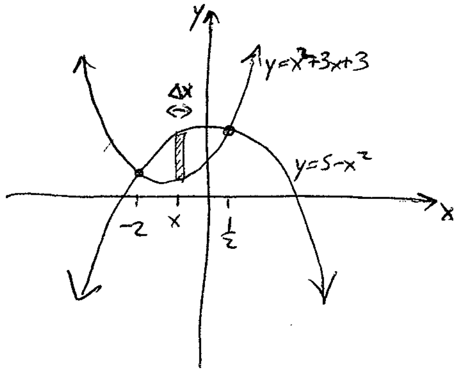
$\Rightarrow \dfrac{15}{48}\cdot 10^6 \leq n^2$

$\Rightarrow n \geq \sqrt{\dfrac{15}{48}\cdot 10^6} = \sqrt{\dfrac{5}{16}}\cdot 10^3 = \dfrac{\sqrt{5}}{4}\cdot 10^3.$

We may take $\underline{n = 750}$, since $750 = \frac{3}{4}\cdot 10^3 = \frac{\sqrt{9}}{4}\cdot 10^3 > \frac{\sqrt{5}}{4}\cdot 10^3$.

4. (8 points)

(3pts) (a) Set up, but do not evaluate, an integral representing the area of the region bounded by the curves $y = 5 - x^2$ and $y = x^2 + 3x + 3$. As justification, draw a picture with a sample slice labeled.




First find x-coordinates of intersection points:

set $5-x^2 = x^2+3x+3$

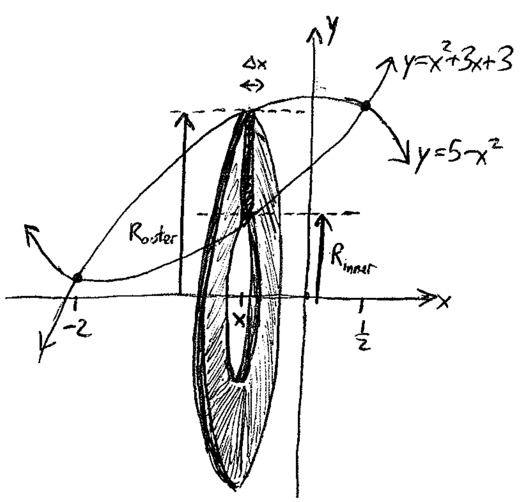
$\Rightarrow 2x^2+3x-2=0$

$\Rightarrow (2x-1)(x+2)=0 \Rightarrow x=\frac{1}{2},\ x=-2.$

For $-2 \le x \le \frac{1}{2}$, a vertical slice at coordinate $x$ of thickness $\Delta x$ has height $y_{Top} - y_{Bottom} = (5-x^2)-(x^2+3x+3)$
$= 2-3x-2x^2$,

so the exact area is $\boxed{\int_{-2}^{1/2} (2-3x-2x^2)\,dx}$.

(5pts) (b) Set up an integral representing the volume obtained by rotating the region from part (a) about the $x$-axis. Make sure you justify your answer (draw and label a diagram). Again, don't evaluate the integral.




Note: We must use washer method (not cyl. shells) if we want a single integral (because $x$ is not a simple function of $y$ for these curves).

For $-2 \le x \le \frac{1}{2}$, a vertical slice at coordinate $x$ of thickness $\Delta x$ rotates around the $x$-axis to produce a washer of

inner radius $R_{inner} = y_{Bottom} = x^2+3x+3$, and

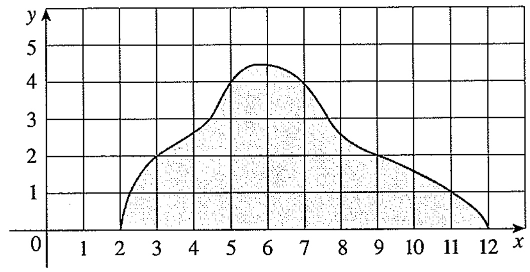
outer radius $R_{outer} = y_{Top} = 5-x^2$,

so the washer's area is $\pi(5-x^2)^2 - \pi(x^2+3x+3)^2$;

thus, the exact volume is

$$\boxed{\int_{-2}^{1/2} \left(\pi(5-x^2)^2 - \pi(x^2+3x+3)^2\right) dx}$$

5. (8 points) Consider the region between the curve and the $x$-axis in the figure below:



(4pts) (a) If the region shown is rotated about the $x$-axis to form a solid, use Simpson's Rule with $n = 10$ to estimate the volume of the solid. (Write an expression involving only numbers, but you do not have to evaluate the expression.)

Write $y=f(x)$ for the equation of the boundary curve. By the disks method, the exact volume in question is $\int_2^{12} \pi (f(x))^2\,dx$. Thus, with $n=10$ and $\Delta x = \frac{12-2}{10} = 1$, the Simpson's Rule approximation of this integral is

$$S_{10} = \frac{\Delta x}{3}\Big(\pi f(2)^2 + 4\pi f(3)^2 + 2\pi f(4)^2 + 4\pi f(5)^2 + 2\pi f(6)^2 + 4\pi f(7)^2 + 2\pi f(8)^2 + 4\pi f(9)^2 + 2\pi f(10)^2 + 4\pi f(11)^2 + \pi f(12)^2\Big)$$

$$= \boxed{\frac{\pi}{3}\left(0^2 + 4\cdot 2^2 + 2\cdot(2.6)^2 + 4\cdot 4^2 + 2\cdot(4.4)^2 + 4\cdot 4^2 + 2\cdot(2.5)^2 + 4\cdot 2^2 + 2\cdot(1.6)^2 + 4\cdot 1^2 + 0^2\right)}$$

(4pts) (b) If the region is instead rotated about the $y$-axis, use the Midpoint Rule with $n = 5$ to estimate the volume of the resulting solid. (Again, you do not have to evaluate your expression.)

By the cylindrical shells method, the exact volume in question is $\int_2^{12} 2\pi x f(x)\,dx$.

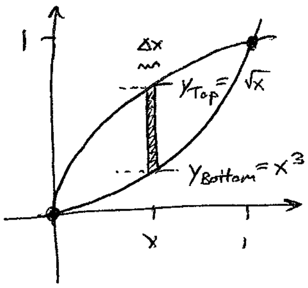
Thus, with $n=5$ and $\Delta x = \frac{12-2}{5} = 2$, the Midpoint Rule approximation of this integral is

$$M_5 = \Delta x\cdot\big(2\pi\cdot 3\cdot f(3) + 2\pi\cdot 5\cdot f(5) + 2\pi\cdot 7\cdot f(7) + 2\pi\cdot 9\cdot f(9) + 2\pi\cdot 11\cdot f(11)\big)$$

$$= \boxed{4\pi\left(3\cdot 2 + 5\cdot 4 + 7\cdot 4 + 9\cdot 2 + 11\cdot 1\right)}$$

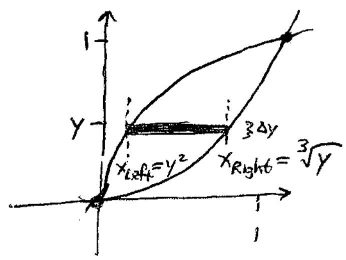
6. (10 points)

(4pts) (a) Set up two distinct integrals, each in terms of a single variable, representing the area of the region in the first quadrant bounded by the curves $y = x^3$ and $y = \sqrt{x}$. For each, justify your answer by drawing a picture and marking a sample slice. Don't evaluate either integral.




$$\text{Area} = \int_{x_{min}}^{x_{max}} (y_{top} - y_{bottom})\,dx$$

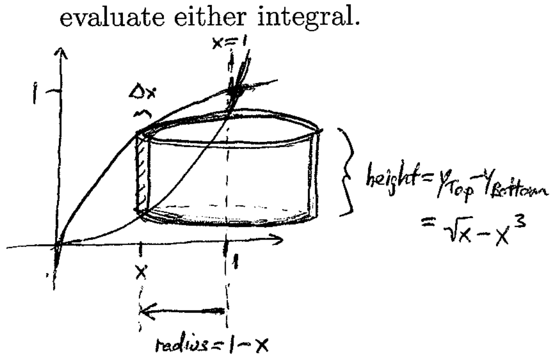
$$= \int_0^1 (\sqrt{x} - x^3)\,dx$$




$$\text{Area} = \int_{y_{min}}^{y_{max}} (x_{Right} - x_{Left})\,dy$$

$$= \int_0^1 (\sqrt[3]{y} - y^2)\,dy$$

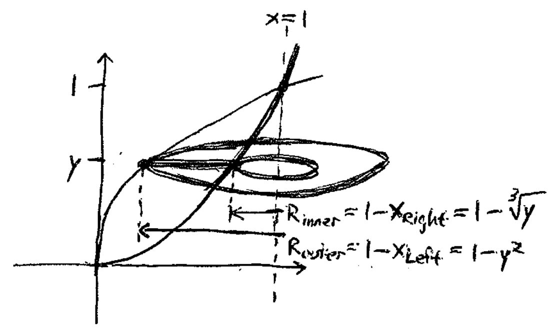
(6pts) (b) Set up two distinct integrals, each in terms of a single variable, representing the volume of the solid obtained by rotating the region from part (a) around the line $x = 1$. For each, make sure you justify your answer (draw a picture, label a sample slice, and cite the method used). Don't evaluate either integral.




Cylindrical Shells: slice parallel to $x=1$ (vertically)

$$\text{Volume} = \int_{x_{min}}^{x_{max}} (2\pi)(\text{radius})(\text{height})\,dx$$

$$= \int_0^1 2\pi(1-x)(\sqrt{x} - x^3)\,dx$$

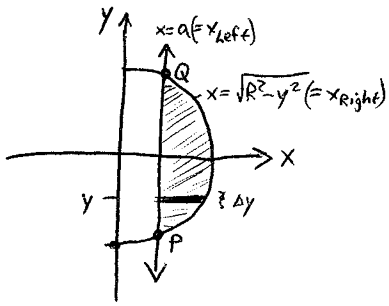


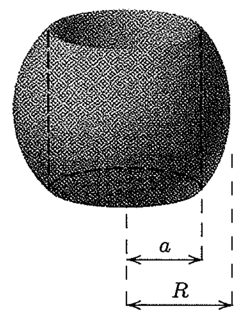

Washers: slice perp. to $x=1$ (i.e. horizontally)

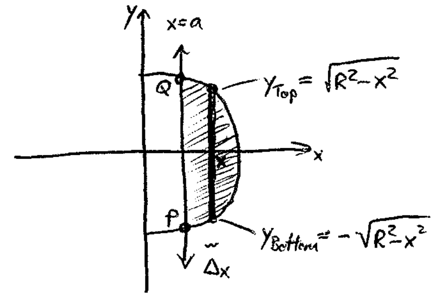
$$\text{Volume} = \int_{y_{min}}^{y_{max}} \pi\left(R_{outer}^2 - R_{inner}^2\right)\,dy$$

$$= \int_0^1 \pi\left((1-y^2)^2 - (1-\sqrt[3]{y})^2\right)\,dy$$

7. (8 points) A napkin ring is made by taking a solid wooden ball (sphere) of radius $R$ and drilling a hole of radius $a$ straight through the center. (The hole is cylindrical in shape with radius $a$, and the resulting solid has flat edges at its top and bottom.) Find the volume of the napkin ring, in terms of $R$ and $a$.




As in either of the sketches above, it is advantageous to view the napkin ring as the solid resulting from rotating about the y-axis the region bounded by the semicircle $x=\sqrt{R^2-y^2}$ and the line $x=a$. The coordinates of the intersection points $P$ and $Q$ are found by setting $x=a$ in the semicircle equation:

$$a=\sqrt{R^2-y^2} \Rightarrow y^2=R^2-a^2 \Rightarrow y=\pm\sqrt{R^2-a^2},$$

so the coordinates of $P$ are $(a,-\sqrt{R^2-a^2})$ and those of $Q$ are $(a,\sqrt{R^2-a^2})$.

<u>Solution #1</u> (washers): Slice perpendicular to the y-axis (i.e., horizontally) as in the diagram above left; for $-\sqrt{R^2-a^2}\le y\le\sqrt{R^2-a^2}$, the slice at coordinate $y$ is rotated about the y-axis to form a washer of outer radius $\underline{R_{outer}=x_{Right}=\sqrt{R^2-y^2}}$ and inner radius $\underline{R_{inner}=x_{Left}=a}$, so that the total volume is

$$\int_{y_{min}}^{y_{max}}\pi\left(R_{outer}^2-R_{inner}^2\right)dy = \int_{-\sqrt{R^2-a^2}}^{\sqrt{R^2-a^2}}\pi\left((R^2-y^2)-a^2\right)dy = \pi\left(R^2y-\frac{y^3}{3}-a^2y\right)\Big|_{y=-\sqrt{R^2-a^2}}^{y=\sqrt{R^2-a^2}}$$

$$=\pi\left((R^2-a^2)y-\frac{y^3}{3}\right)\Big|_{y=-\sqrt{R^2-a^2}}^{y=\sqrt{R^2-a^2}} = \pi\left(\left((R^2-a^2)(\sqrt{R^2-a^2})-\frac{(R^2-a^2)^{3/2}}{3}\right)-\left(-(R^2-a^2)(\sqrt{R^2-a^2})+\frac{(R^2-a^2)^{3/2}}{3}\right)\right)$$

$$=\boxed{\frac{4}{3}\pi\left(R^2-a^2\right)^{3/2}}.$$

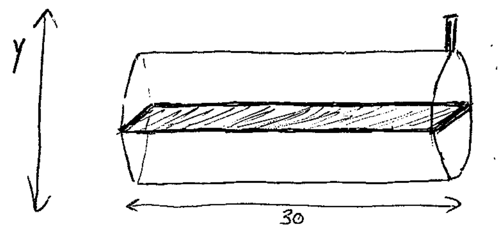
<u>Solution #2</u> (cylindrical shells): Slice parallel to the y-axis (i.e. vertically) as in the diagram above right; for $a\le x\le R$, the slice at coordinate $x$ is rotated about the y-axis to form a cylindrical shell of radius $x$ and height $=y_{Top}-y_{Bottom}=2\sqrt{R^2-x^2}$, so the total volume is

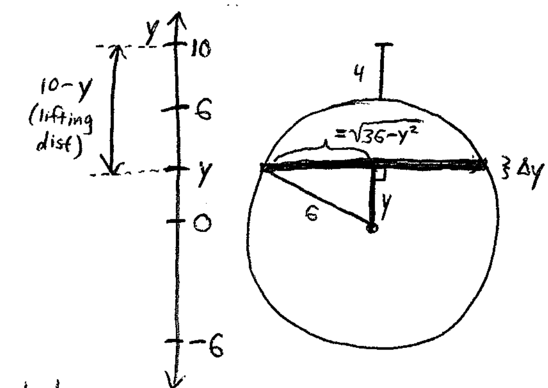
$$\int_{x_{min}}^{x_{max}}2\pi(\text{radius})(\text{height})\,dx = 4\pi\int_a^R x\sqrt{R^2-x^2}\,dx \underset{\uparrow}{=} -\frac{4\pi}{2}\int_{R^2-a^2}^{0}u^{1/2}du=\left(\frac{4\pi}{3}u^{3/2}\right)\Big|_{u=R^2-a^2}^{u=0} = -\frac{4\pi}{3}\cdot\left(-(R^2-a^2)^{3/2}\right)=\boxed{\frac{4\pi}{3}\left(R^2-a^2\right)^{3/2}}.$$

[Let $u=R^2-x^2$; $du=-2x\,dx$]

8. (8 points) A fuel tank buried underground has the shape of a circular cylinder lying lengthwise (so that the axis of symmetry is *horizontal*, i.e., parallel to the ground). The radius of the cylinder is 6 feet, the length is 30 feet, and the top of the tank is 4 feet below ground level.

   Suppose that the tank is completely filled with a liquid that has weight density 40 lb/ft$^3$. Set up an integral in terms of a single variable that represents the amount of work it takes to pump *half* of the liquid out of the tank and up to ground level. Show all your steps in setting up the integral, but do not evaluate it.




Introduce a y-axis along the vertical, with $y=10$ at the ground level and $y=-6$ at the bottom of the tank.

If we slice the tank region into rectangular slices of thickness $\Delta y$ (i.e. slicing perpendicular to the y-axis), then the fuel in each slice is lifted approximately a uniform distance.

According to the diagrams, the slice at coordinate $y$, for $0 \le y \le 6$, is 30 feet long, $\Delta y$ feet thick, and $2\cdot\sqrt{6^2-y^2}$ feet wide (using right triangles in a circle).

Meanwhile, the slice at coordinate $y$ is lifted $10-y$ feet to ground level.

Thus the work to lift the slice at coordinate $y$ is

$$\begin{aligned}\Delta W = (\text{Force})(\text{distance}) &= (\text{Weight})(\text{dist to lift})\\ &= (40\ \text{lb/ft}^3)(\text{Volume})(\text{dist to lift})\\ &= (40)(30)(2\sqrt{36-y^2})(\Delta y)(10-y)\\ &= 2400(10-y)(\sqrt{36-y^2})\Delta y,\end{aligned}$$

so that the total work to lift all fuel between $y=0$ and $y=6$ is

$$\boxed{W = \int_0^6 2400(10-y)(\sqrt{36-y^2})\,dy}.$$