Math 42: Calculus
Final Exam — March 16, 2009

Name : ____

Section Leader: (Circle one)	Fai Chandee	Sukhada Fadnavis	Ha Pham	Ian Weiner	Ziyu Zhang
Section Time: (Circle one)	10:00	11:00	1:15	2:15	

- Complete the following problems. In order to receive full credit, please show all of your work and justify your answers.
- You do not need to simplify your answers unless specifically instructed to do so. You may use any result from class that you like, but if you cite a theorem be sure to verify the hypotheses are satisfied.
- **You have 3 hours.** This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted. If you finish early, you must hand your exam paper to a member of teaching staff.
- Please check that your copy of this exam contains 19 pages and is correctly stapled.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- Please sign the following:

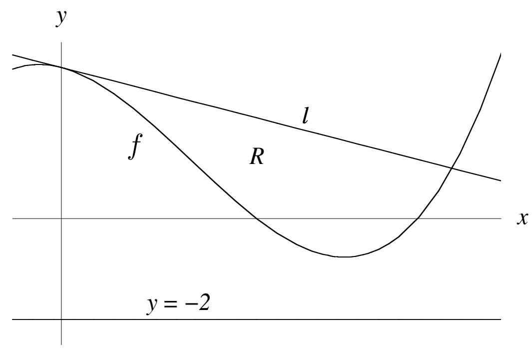
> "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ________________________________

The following boxes are strictly for grading purposes. Please do not mark.

1	10		**8**	8	
2	10		**9**	18	
3	5		**10**	16	
4	12		**11**	10	
5	9		**12**	5	
6	15		**13**	8	
7	11		**14**	13	
			Total	150	

1. (10 points)



Let f be the function given by

$$f(x) = \frac{x^3}{4} - x^2 - \frac{x}{2} + 3.$$

The line l is tangent to the graph of f at $x = 0$.
Let R be the region bounded by f and l.

(a) Find the equation of the tangent line l.

(b) Find the area of R.

(c) Write an integral for the volume of the solid generated when R is rotated about the line $y = -2$. You do not need to evaluate this integral.

2. (10 points) Given a continuous function $f(x)$ on the interval $[a, b]$. Suppose that we wish to approximate the integral

$$\int_a^b f(x)\,dx$$

using one of the basic approximation techniques (Midpoint Rule, Trapezoidal Rule, Simpson's Rule).

Mark each statement below as *true* or *false* by circling **T** or **F**. No justification is necessary.

T **F** The Midpoint Rule always produces a more accurate approximation than the Trapezoidal Rule (for a fixed number of subintervals).

T **F** The smaller the value of K_2 that we choose, the more accurate our Midpoint Rule approximation will be (for a fixed number of subintervals).

T **F** If $f(x)$ is a polynomial function of degree 3, then Simpson's Rule always produces the *actual* value of the integral.

T **F** If $f(x)$ is increasing on the interval $[a, b]$, then the Midpoint Rule always gives an underestimate of the actual value of the integral.

T **F** If $f(x)$ is concave down on the interval $[a, b]$, then the Trapezoidal Rule always gives an overestimate of the actual value of the integral.

3. (5 points) A spring is tested for various properties. It is found to obey Hooke's Law, and it is determined that the work required to stretch it 1 ft beyond its natural length is 12 ft-lb. How much work is needed to stretch it $\frac{3}{4}$ ft beyond its natural length?

   (*Hooke's Law* states that the force required to hold a spring in a given position is proportional to the distance that the spring is stretched from its natural length; that is, if x represents this latter amount, then the force $F = kx$ for some constant k.)

4. (12 points) The time gaps between consecutive bursts of solar particles onto a certain detector is found to be closely modeled by the following probability density function:

$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ \dfrac{C}{(4+x^2)^{3/2}} & \text{if } x \geq 0, \end{cases}$$

where C is a positive constant. Complete the following, giving full justification:

(a) Find C, given that f is a probability density function.

(b) Find the mean time gap.

5. (9 points)

(a) Find (with justification) all values of k such that the function $y = e^{kx}$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = 0.$$

(b) Which of the following families of functions is the solution to the differential equation $\frac{dy}{dt} = 3y + 1$? (Here C stands for any constant.) No justification is necessary; just circle your answer.

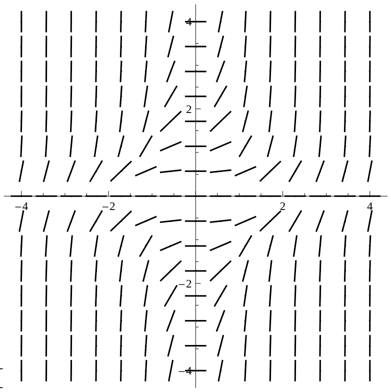
(i) $y = Ce^{3t} + t$

(iii) $y = e^{3t} + t + C$

(ii) $y = e^{3t} - \frac{1}{3} + C$

(iv) $y = Ce^{3t} - \frac{1}{3}$

6. (15 points) Match the direction fields below with their differential equations. (The horizontal variable is t; the vertical is y.) Also indicate which two equations do not have matches.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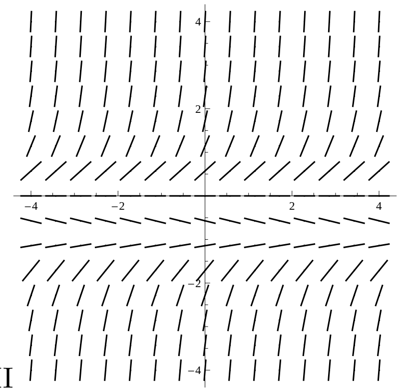
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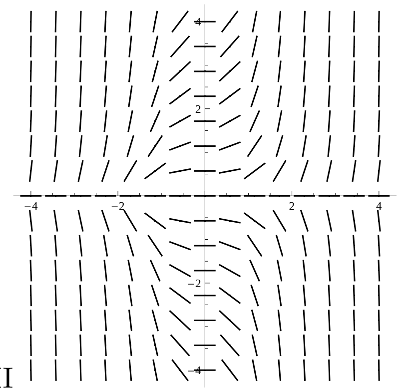
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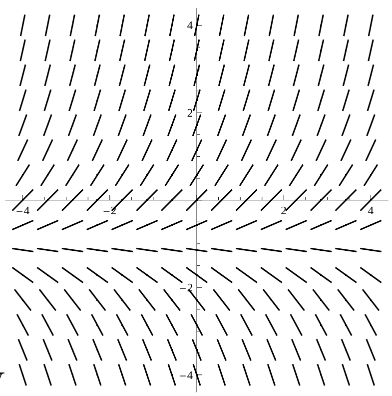
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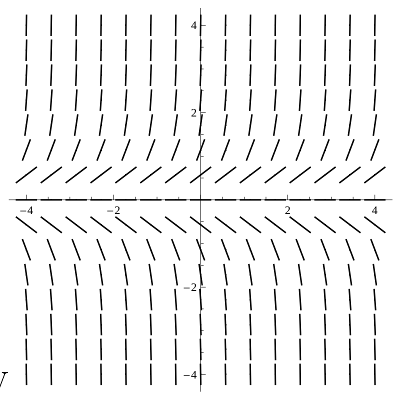
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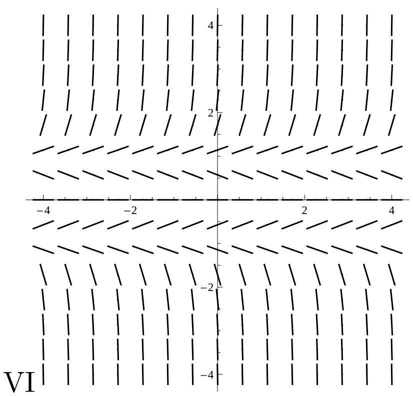

VI

Equation	I, II, III, IV, V VI, or "none"	Equation	I, II, III, IV, V VI, or "none"
$dy/dt = ty^2$		$dy/dt = t^2y^2$	
$dy/dt = t^2y$		$dy/dt = y(y^2-1)$	
$dy/dt = y+t$		$dy/dt = y(y^2+1)$	
$dy/dt = 1+y$		$dy/dt = y(y+1)$	

7. (11 points) A tank initially contains 1000 gallons of water, in which is dissolved 20 pounds of salt. A valve is opened and water containing 0.2 pounds of salt per gallon flows into the tank at a rate of 5 gal/min. The resulting mixture, which is assumed to be always well stirred, drains from the tank at a rate of 5 gal/min.

   (a) Write down a differential equation for $S(t)$, the amount of salt in the tank after t minutes. Be sure to state your initial condition, including the units involved.

   (b) By solving the differential equation, find the amount of salt in the tank after 60 minutes.

8. (8 points)

(a) Solve the initial value problem

$$\frac{dy}{dx} = x^3 y^2, \quad y(0) = -1.$$

(b) Is there a function $y(x)$ satisfying the above differential equation, but instead with initial value $y(0) = 0$? Explain.

9. (18 points) At the start of a late-night study session in your dorm, your RA puts out a large bowl of ChexMix®, containing 900 pieces of the delicious snack. Let $y = y(t)$ stand for the amount of ChexMix in the bowl, in *hundreds* of pieces, after t hours.

   (a) The more that's in the bowl, the more people are inclined to take out a snack. Suppose that one-third of the pieces in the bowl are removed each hour. Write a *differential equation* satisfied by y in this case, including the initial value of y.

   (b) Find an expression for $y(t)$ in the above situation.

   (c) For the rest of this problem, suppose that your RA is also continually re-supplying the bowl, adding ChexMix at a rate of $\frac{12}{y}$ hundred pieces per hour. Write a new differential equation satisfied by y in this case.

(d) Find the equilibrium amount of ChexMix in this situation.

(e) Use Euler's method with $h = 2$ to estimate the amount of ChexMix left after 4 hours.

(f) Find an exact expression for $y(t)$ in this situation.

10. (16 points) In a certain closed ecosystem, let functions $x(t)$ and $y(t)$ represent the population sizes (in thousands of beings) of two species, X and Y, respectively; here the time t is measured in months. Suppose further that the population sizes are modeled by the equations

$$\frac{dx}{dt} = x - \frac{x^2}{4} - \frac{xy}{4}$$
$$\frac{dy}{dt} = -\frac{y}{4} + \frac{xy}{4}$$

(a) This system is a predator-prey model. Explain why, and determine which species is predator and which is prey.

(b) Find all equilibrium solutions to this system.

(c) Suppose that at time $t = 0$ months, we have $x(0) = 3$ and $y(0) = 0$. (Thus, there are no beings of species y at any time.) Solve for an explicit formula that gives the population size $x(t)$ in terms of t; what happens to x as t approaches infinity?

For quick reference, here again is the system:

$$\frac{dx}{dt} = x - \frac{x^2}{4} - \frac{xy}{4}$$
$$\frac{dy}{dt} = -\frac{y}{4} + \frac{xy}{4}$$

(d) Suppose instead that at time $t = 0$ months, we have $x(0) = 3$ and $y(0) = 4$. Use the differential equations to predict the sizes of the two populations in one month's time; be as mathematically precise as possible.

(e) For the initial conditions of part (d), consider the *signs* of dx/dt and dy/dt at $t = 0$. Based the prediction you made in part (d), make a further prediction about whether dx/dt or dy/dt will change sign at some point *after* the first month. Explain fully how you are able to tell.

11. (10 points) In each of the problems below, indicate clearly what facts you use and how you apply them.

    (a) Determine whether $\sum_{n=2}^{\infty} \frac{2}{n^2-1}$ converges or diverges; also, *find the sum* if it converges.

(b) Determine whether $\displaystyle\sum_{n=0}^{\infty} \frac{6^n}{5^n + 6^n}$ converges or diverges.

12. (5 points) Suppose we know that the power series

$$\sum_{n=0}^{\infty} c_n (x-4)^n$$

converges if $x = 8$ and diverges if $x = -1$. We are given no other information about this series. For each of the following statements, circle

- **T** if the statement must be true,
- **F** if the statement must be false, and
- **X** if the statement could be either true or false.

You do not need to justify your answers.

T **F** **X** If R is the radius of convergence of the series, then $4 \leq R \leq 5$.

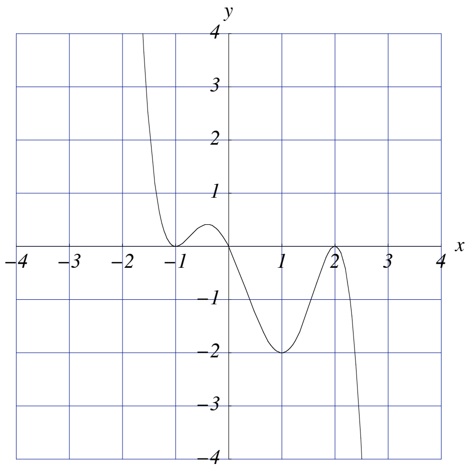
T **F** **X** The series converges for $x = 0$.

T **F** **X** The series diverges for $x = 9$.

T **F** **X** The series diverges for $x > 9$.

T **F** **X** $\lim_{n\to\infty} c_n = 0$.

13. (8 points) The polynomials in the chart below are second-degree Taylor polynomials for functions whose graphs are given below. Match each Taylor polynomial with the appropriate graph.



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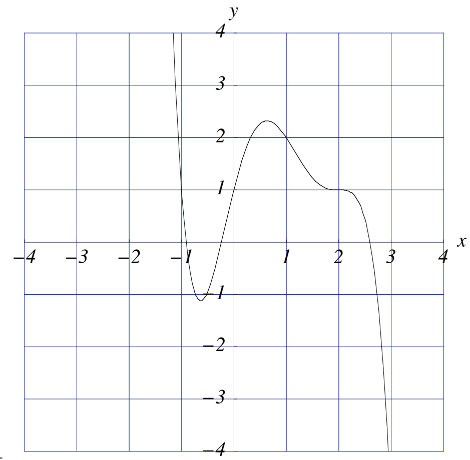

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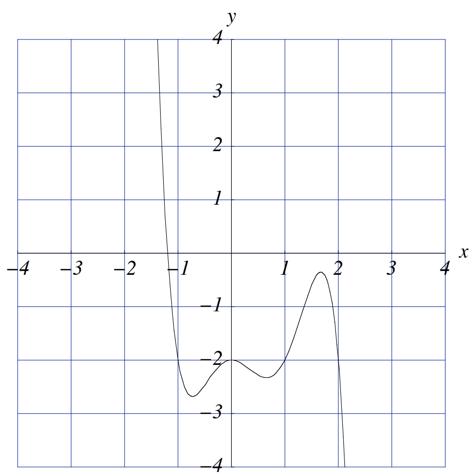

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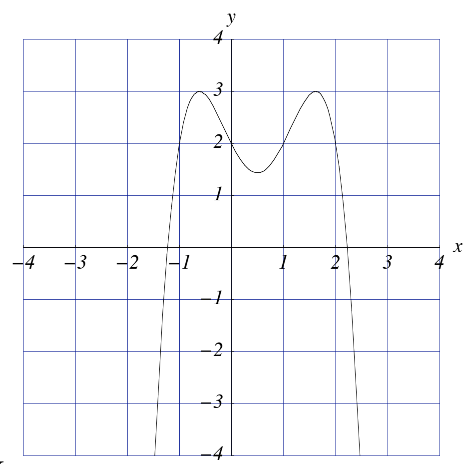

IV

Taylor polynomial	Function (I, II, III or IV)
$T_2(x) = 2 + 2(x-1) + (x-1)^2$	
$T_2(x) = -2 + 2(x-1) + 3(x-1)^2$	
$T_2(x) = -2 + \frac{7}{2}(x-1)^2$	
$T_2(x) = 2 - \frac{3}{2}(x-1) - (x-1)^2$	

14. (13 points)

(a) Find, showing all your steps, the degree-6 Taylor polynomial $T_6(x)$ with center 0 for the function

$$f(x) = e^x + e^{-x}.$$

(b) Use T_6 to obtain an estimate for $\sqrt{e} + \frac{1}{\sqrt{e}}$. (You do not need to simplify your answer.)

(c) Compute the 7th derivative $f^{(7)}(x)$ of $f(x)$ and explain why

$$|f^{(7)}(x)| \leq 4 \quad \text{for all } x \text{ in } \left[-\tfrac{1}{2}, \tfrac{1}{2}\right].$$

(d) Use the fact from part (c) (even if you were unable to verify it) to draw a conclusion, in sentence form, about the accuracy of your estimate from part (b); be as mathematically precise as you can, and cite all of your reasoning.