1. (10 points) Determine whether each of the following improper integrals converges. Explain your reasoning completely.

(a) $\int_{-\infty}^{\infty} \sin(2x)\,dx = \int_{-\infty}^{0} \sin 2x\,dx + \int_{0}^{\infty} \sin 2x\,dx$, and implicitly, the two integrals on the right must converge in order for the integral on the left to converge.

But $\int_0^\infty \sin 2x\,dx = \lim_{N\to\infty} \int_0^N \sin 2x\,dx = \lim_{N\to\infty} \left.\frac{-\cos 2x}{2}\right]_0^N$

$= \lim_{N\to\infty} \frac{1}{2}(1-\cos 2N)$, which does not exist because $\lim_{t\to\infty} \cos t$ does not exist.

Thus, $\int_{-\infty}^{\infty} \sin 2x\,dx$ diverges (because $\int_0^\infty \sin 2x\,dx$ also diverges).

(b) $\int_0^1 \frac{e^x}{\sqrt{x}}\,dx$ This is an improper integral because $\frac{e^x}{\sqrt{x}}$ is discontinuous at 0.

We should be conscious of the fact that the closely related integral

$$\int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}}dx = \lim_{a\to 0^+} \left. 2x^{1/2}\right]_{x=a}^{x=1} = \lim_{a\to 0^+} 2-2\sqrt{a} = 2,$$

and in particular that it converges. We also note that for $0 \le x \le 1$, we have $1 \le e^x \le e$ because e^x is an increasing function, and this implies that $0 < \frac{e^x}{\sqrt{x}} \le \frac{e}{\sqrt{x}}$ for $0 < x \le 1$.

Of course, $\int_0^1 \frac{e}{\sqrt{x}}dx = e \cdot \int_0^1 \frac{1}{\sqrt{x}}dx$ converges because $\int_0^1 \frac{1}{\sqrt{x}}dx$ does, and therefore $\int_0^1 \frac{e^x}{\sqrt{x}}dx$ converges by the Comparison Test for Improper Integrals.

(Note that we didn't have to compute its value!)

2. (10 points) For this problem, use the following information about any normal ("bell-shaped" or "Gaussian") probability density function f:

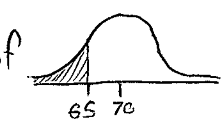
- f has the general form $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$
- $\displaystyle\int_{-\infty}^{\mu+\sigma} f(x)dx \approx .84$
- $\displaystyle\int_{-\infty}^{\mu+2\sigma} f(x)dx \approx .98$

Now suppose that the speeds of vehicles on a highway with speed limit 65 mph are normally distributed, with mean 70 mph and standard deviation 5 mph.

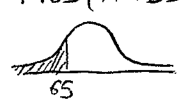
(a) What is the probability that a randomly chosen vehicle is traveling at a legal speed (65 or under)? Justify your answer by writing an integral expression that represents this probability and showing how to evaluate this integral.

Since $\mu = 70$ and $\sigma = 5$, we have $65 = 70 - 5 = \mu - \sigma$. Thus the desired probability can be expressed (and drawn) as:

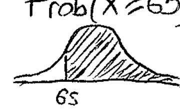
$$\text{Prob}(X \leq 65) = \int_{-\infty}^{65} \frac{1}{5\sqrt{2\pi}} e^{-(x-70)^2/2\cdot 5^2} dx = \int_{-\infty}^{\mu-\sigma} f(x)dx = \text{Area of}$$



To find the value, note that $\text{Prob}(X \leq 65) = 1 - \text{Prob}(X \geq 65)$

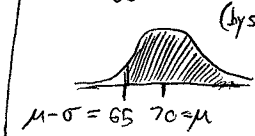


$\underset{\text{(since PDF)}}{=} 1 -$

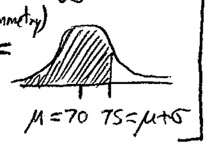


; this latter probability can be found as follows:

$$\text{Prob}(X \geq 65) = \int_{65}^{\infty} f(x)dx = \int_{-\infty}^{75} f(x)dx \approx 0.84;$$



(by symmetry) $=$



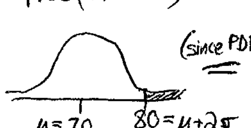
thus,

$$\text{Prob}(X \leq 65) \approx 1 - 0.84 = \boxed{0.16}.$$

(b) If the Highway Patrol are instructed to ticket motorists driving 80 mph or more, what percentage of motorists are targeted? Again use an integral to express your answer, and evaluate it with justification.

Note that $80 = 70 + 2\cdot 5 = \mu + 2\sigma$. Thus, since we need $\text{Prob}(X \geq 80) = \int_{80}^{\infty} f(x)dx$, this can be found by the following reasoning:

$$\text{Prob}(X \geq 80) = 1 - \text{Prob}(X \leq 80)$$



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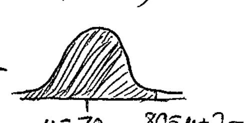

$$= 1 - \int_{-\infty}^{80} f(x)dx \approx 1 - 0.98 = \boxed{0.02}.$$

Thus, approximately $\boxed{\text{2 percent}}$ of motorists will be targeted for tickets.

3. (8 points)

(a) Express $1.53\overline{42} = 1.53424242\ldots$ as a ratio of integers.

We have $1.53\overline{42} = 1.53 + 0.00\overline{42}$

$$= 1.53 + 0.0042 + 0.000042 + 0.00000042 + \cdots$$

$$= \frac{153}{100} + \left(\frac{42}{10^4} + \frac{42}{10^6} + \frac{42}{10^8} + \cdots\right)$$

Geom Series w/initial term $a = 42/10^4$, common ratio $r = 1/100$

$$= \frac{153}{100} + \frac{42/10^4}{1 - 1/100} = \frac{153}{100} + \frac{(42/10^4)}{(99/100)} = \frac{153}{100} + \frac{42}{9900}$$

$$= \boxed{\frac{153 \cdot 99 + 42}{9900}} \left(= \frac{15189}{9900} = \frac{5063}{3300}\right)$$

(b) Determine the values of b for which $1 + \frac{e^{-2b}}{2} + \frac{e^{-4b}}{4} + \frac{e^{-6b}}{8} + \frac{e^{-8b}}{16} + \cdots$ converges, and for those b find the sum. (Your answer will be an expression in terms of b.)

The general term appears to be $\frac{e^{-2nb}}{2^n} = \left(\frac{e^{-2b}}{2}\right)^n$ for $n \geq 0$, so

this is a geometric series with initial term $a = 1$ and common ratio $r = \frac{e^{-2b}}{2}$.

It converges if and only if $|r| < 1$; i.e. $\left|\frac{e^{-2b}}{2}\right| < 1$.

Solving for b, we obtain $e^{-2b} < 2 \iff \boxed{b > -\frac{1}{2}\ln 2}$.

If the series converges, the sum is $S = \frac{a}{1-r} = \boxed{\frac{1}{1 - \left(\frac{e^{-2b}}{2}\right)}}$.

4. (20 points) Determine whether each of the series below converges or diverges. Indicate clearly which tests you use and how you apply them.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{3n^4 - n^2 + 5}$

We'd like to use the Limit Comparison Test with $a_n = \frac{n^3}{3n^4-n^2+5}$ and $b_n = \frac{n^3}{n^4} = \frac{1}{n}$,

but the test requires that a_n and b_n are positive values. Fortunately, this is true in our case, for if $n \geq 1$, then $3n^2 - 1 > 3(n^2-1) \geq 0$,

and so $3n^4 - n^2 + 5 > 3n^4 - n^2 = n^2(3n^2-1) > 0$

since both factors are positive; thus $a_n = \frac{n^3}{3n^4-n^2+5}$ is positive because both its numerator and denominator are positive for $n \geq 1$. (Meanwhile, $b_n = \frac{1}{n} > 0$ for $n \geq 1$ as well.)

Now
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3}{3n^4-n^2+5} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^4}{3n^4-n^2+5}$$
$$= \lim_{n\to\infty} \frac{1}{3 - 1/n^2 + 5/n^4}$$
$$= \frac{1}{3-0+0} = \frac{1}{3} > 0,$$

so the Limit Comparison Test allows us to conclude that either both series $\sum a_n$ and $\sum b_n$ converge, or they both diverge. But $\sum b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series with $p=1$ (a.k.a. the Harmonic Series), so we know it diverges; thus $\sum_{n=1}^{\infty} a_n$ diverges as well.

(b) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{1}{3^n}\right)^2$

We have $a_n = \left(\frac{1}{2^n}+\frac{1}{3^n}\right)^2 = \left(\frac{1}{2^n}\right)^2 + \frac{2}{2^n 3^n} + \left(\frac{1}{3^n}\right)^2 = \frac{1}{4^n} + \frac{2}{6^n} + \frac{1}{9^n}$.

Now the three separate series $\sum \frac{1}{4^n}$, $\sum \frac{2}{6^n}$, and $\sum \frac{1}{9^n}$ are each geometric series with common ratio less than 1 ($r = \frac{1}{4}, \frac{1}{6}$, and $\frac{1}{9}$ respectively), so each is a convergent series. It follows that

$$\sum_{n=0}^{\infty} \frac{1}{4^n} + \sum_{n=0}^{\infty} \frac{2}{6^n} + \sum_{n=0}^{\infty} \frac{1}{9^n} = \sum_{n=0}^{\infty} \left(\frac{1}{4^n} + \frac{2}{6^n} + \frac{1}{9^n}\right) = \sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{1}{3^n}\right)^2,$$

and in particular, the latter series is <u>convergent</u>. (In fact, the sum can be computed; it's $\frac{1}{1-1/4} + \frac{2}{1-1/6} + \frac{1}{1-1/9}$.)

[There are a lot of other ways to see this is a convergent series; for instance, one can compute that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{4} < 1$ (details suppressed here!), and so the Ratio Test implies that the series converges.]

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!}$ Let $a_n = 3^n/n!$.

Ratio test: $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{3^{n+1}/(n+1)!}{3^n/n!}$

$$= \lim_{n\to\infty} \frac{3^{n+1}}{3^n} \cdot \frac{n!}{(n+1)!} \qquad \begin{cases} 3^{n+1} = 3 \cdot 3^n \\ (n+1)! = (n+1) \cdot n! \end{cases}$$

$$= \lim_{n\to\infty} 3 \cdot \frac{1}{n+1}$$

$= 0$, which is less than 1,

So the series converges.

(d) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^3}$ (The terms of this series are not all positive, so we can't directly apply a Comparison Test; however, the boundedness of $\sin x$ and the growth of the denominator should already give us a hunch that this will converge.)

What we have to do is instead first consider the series $\sum_{n=1}^{\infty} \frac{|\sin(n)|}{n^3}$ of all positive terms. Since $0 \le |\sin(n)| \le 1$ for all n, we have

$$0 \le \frac{|\sin(n)|}{n^3} \le \frac{1}{n^3};$$

and furthermore, we know $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges since it's a p-series with $p = 3 > 1$. Thus, $\sum_{n=1}^{\infty} \frac{|\sin(n)|}{n^3}$ converges by the Comparison Test.

But this $\sum_{n=1}^{\infty} \frac{|\sin(n)|}{n^3}$ is the series of absolute values of terms of $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^3}$; thus, by the Absolute Convergence Rule, $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^3}$ also converges.

5. (10 points) Suppose that the series $\sum_{n=1}^{\infty} a_n$ converges, for *positive* numbers a_n.

Decide which of the following series must converge, must diverge, or may either converge or diverge (inconclusive). Circle your answer. You do not need to justify your answers.

Key Observation: The above tells us that $\lim_{n\to\infty} a_n = 0$.

(a) $\sum_{n=1}^{\infty} \frac{1}{a_n}$ Converges Diverges Inconclusive

→ Certainly $\lim_{n\to\infty} \frac{1}{a_n} = \infty \neq 0$. Use Test for Divergence.

(b) $\sum_{n=1}^{\infty} (a_n)^3$ Converges Diverges Inconclusive

→ For all sufficiently large n, we'll have $a_n \leq 1$, so that $0 \leq a_n^3 \leq a_n$, and we could apply the Comparison Test to the rest of the series.

(c) $\sum_{n=1}^{\infty} e^{a_n}$ Converges Diverges Inconclusive

→ Certainly $\lim_{n\to\infty} e^{a_n} = e^0 = 1 \neq 0$. Use Test for Divergence.

(d) $\sum_{n=1}^{\infty} (-1)^n a_n$ Converges Diverges Inconclusive

→ Use Absolute Convergence Rule and fact that $\sum a_n$ converges.

(e) $\sum_{n=1}^{\infty} \sqrt{a_n}$ Converges Diverges Inconclusive

→ This could go either way. In case $a_n = \frac{1}{n^2}$, we find $\sum \sqrt{a_n}$ is divergent. But if $a_n = \frac{1}{n^4}$, we find $\sum \sqrt{a_n}$ is convergent.

6. (12 points) Find, with complete justification, the interval of convergence of the power series

$$\sum_{n=2}^{\infty} \frac{x^n}{n \ln n}$$

Let $a_n = \frac{x^n}{n\ln n}$. Ratio Test: compute $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)\ln(n+1)} \cdot \frac{n\ln n}{x^n}\right| =$

$= \lim_{n\to\infty} |x| \cdot \frac{n\ln n}{(n+1)\ln(n+1)} = \lim_{n\to\infty} |x| \cdot \frac{n}{n+1} \cdot \frac{\ln n}{\ln(n+1)}$. Now, $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+1/n} = \frac{1}{1+0} = 1$;

the $\frac{\infty}{\infty}$ indeterminate limit $\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)}$ can be evaluated with l'Hôpital's Rule:

$$\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = \lim_{n\to\infty} \frac{(1/n)}{(1/(n+1))} = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} 1 + \frac{1}{n} = 1 + 0 = 1.$$

Thus, $L = \lim_{n\to\infty} |x| \cdot \frac{n}{n+1} \cdot \frac{\ln n}{\ln(n+1)} = |x| \cdot 1 \cdot 1 = |x|$, so that we'll have convergence for $L = |x| < 1$,

i.e. $-1 < x < 1$.

Check left endpoint: $x = -1$ yields the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln n} = \frac{1}{2\ln 2} - \frac{1}{3\ln 3} + \frac{1}{4\ln 4} - \cdots\cdots$. Let $b_n = \frac{1}{n\ln n}$.

Now since $n\ln n$ is a positive, increasing function of $n \geq 2$ that grows to ∞ as $n\to\infty$, it follows that (i) $b_n = \frac{1}{n\ln n} > 0$, (ii) $b_{n+1} \leq b_n$ for all $n \geq 2$, and (iii) $\lim_{n\to\infty} b_n = 0$.

Thus, the conditions of the Alternating Series Test are satisfied for $\sum_{n=2}^{\infty} (-1)^n b_n$, so it converges.

Check right endpoint: $x = 1$ yields the series $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$. We'll examine the integral $\int_2^{\infty} \frac{1}{x\ln x}\,dx$ instead, since $f(x) = \frac{1}{x\ln x}$ is positive and decreasing (its reciprocal is positive and increasing as noted above). We have

$$\int_2^{\infty} \frac{1}{x\ln x}\,dx = \lim_{N\to\infty} \int_2^{N} \frac{1}{x\ln x}\,dx = \lim_{N\to\infty} \int_{\ln 2}^{\ln N} \frac{1}{u}\,du = \lim_{N\to\infty} \Big[\ln|u|\Big]_{u=\ln 2}^{u=\ln N}$$

$$\left\{\begin{array}{l} u = \ln x \\ du = \frac{1}{x}dx \\ x=2 \Rightarrow u = \ln 2 \\ x = N \Rightarrow u = \ln N \end{array}\right\}$$

$$= \lim_{N\to\infty} \Big(\ln(\ln N) - \ln(\ln 2)\Big)$$

$= \infty$, since $\ln x$ grows to ∞ as $x \to \infty$.

Thus $\int_2^{\infty} \frac{1}{x\ln x}\,dx$ diverges, and so the Integral Test tells us that $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ diverges as well.

Final answer: the interval of convergence is $\boxed{[-1, 1)}$.

7. (10 points) Match each function below with its power series, listed among the choices below. You do not need to justify your answers. (Not all of the series listed have a match.)

(a) $1 - 2x + 3x^2 - 4x^3 + \cdots$

(b) $\frac{1}{10} + \frac{x}{100} + \frac{x^2}{1000} + \frac{x^3}{10000} + \cdots$

(c) $1 - x^2 + x^4 - x^6 + \cdots$

(d) $1 - \frac{2^2}{2!}x^2 + \frac{2^4}{4!}x^4 - \frac{2^6}{6!}x^6 + \cdots$

(e) $2x - \frac{2^3}{3!}x^3 + \frac{2^5}{5!}x^5 - \frac{2^7}{7!}x^7 + \cdots$

(f) $\frac{x}{9} + \frac{x^3}{9^2} + \frac{x^5}{9^3} + \frac{x^7}{9^4} + \cdots$

(g) $x^2 - 3x^3 + \frac{3^2}{2!}x^4 - \frac{3^3}{3!}x^5 + \cdots$

(h) $1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots$

(i) $\frac{1}{3}x^3 - \frac{3}{4}x^4 + \frac{3^2}{5 \cdot 2!}x^5 - \frac{3^3}{6 \cdot 3!}x^6 + \cdots$

(j) $\frac{1}{10} - \frac{x}{100} + \frac{x^2}{1000} - \frac{x^3}{10000} + \cdots$

Function	Series (choose one of (a) through (j))
$\frac{x}{9-x^2}$	f
$\frac{1}{10-x}$	b
x^2e^{-3x}	g
$\frac{1}{(1+x)^2}$	a
$\int_0^x t^2e^{-3t}\,dt$	i

Key Observations:

- The first two functions may be rewritten as $\frac{(x/9)}{1-(x^2/9)}$ and $\frac{(1/10)}{1-(x/10)}$ respectively; we can now use the geometric series formula $\frac{a}{1-r} = a + ar + ar^2 + \cdots$ $(|r|<1)$.
- The fourth function is $\frac{1}{(1+x)^2} = \frac{d}{dx}\left(\frac{-1}{1+x}\right) = \frac{d}{dx}\left(-1 + x - x^2 + x^3 - \cdots\right)$ by the Geometric Series Rule.
- Note that the fifth function is an antiderivative of the third; there's only one pair of series above that has this same relationship. Alternately, begin by finding $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$, substituting $t = -3x$ and multiplying by x^2 to obtain the series (g); then integrate to get (i).

8. (8 points) Suppose that f is a function with continuous derivatives and $f(5) = 3$, $f'(5) = -2$, $f''(5) = 1$, and $f'''(5) = -3$.

(a) Determine the degree-3 Taylor polynomial T_3 of f about 5.

We need the Taylor coefficients $c_0, \dots, c_3$.

$$c_0 = \frac{f(5)}{0!} = f(5) = 3$$

$$c_1 = \frac{f'(5)}{1!} = f'(5) = -2$$

$$c_2 = \frac{f''(5)}{2!} = \frac{1}{2!} = \frac{1}{2} \; ; \quad c_3 = \frac{f'''(5)}{3!} = \frac{-3}{6} = -\frac{1}{2}.$$

$$\Rightarrow \boxed{T_3(x) = 3 - 2(x-5) + \tfrac{1}{2}(x-5)^2 - \tfrac{1}{2}(x-5)^3}$$

(b) Use the Taylor polynomial that you found in part (a) to approximate $f(4.9)$. Express your answer as a number (or sum of numbers).

$$f(4.9) \approx T_3(4.9) = 3 - 2(4.9-5) + \tfrac{1}{2}(4.9-5)^2 - \tfrac{1}{2}(4.9-5)^3$$

$$= 3 + 2\cdot\frac{1}{10} + \frac{1}{2}\cdot\frac{1}{100} + \frac{1}{2}\cdot\frac{1}{1000}$$

$$= \boxed{3 + \frac{1}{5} + \frac{1}{200} + \frac{1}{2000}}.$$

(c) Suppose $|f^{(4)}(x)| \leq 2$ on the interval $[4.9, 5.1]$. Use this information to make a statement about the accuracy of the approximation that you found in part (b).

By Taylor's Inequality, with $n=3$, $a=5$, and taking $M=2$, we have that

$$|f(x) - T_3(x)| \leq \frac{M}{4!}|x-5|^4 = \frac{2}{24}|x-5|^4 \text{ for all } x \text{ in } [4.9, 5.1].$$

In particular, for $x = 4.9$, we have $|f(4.9) - T_3(4.9)| \leq \frac{2}{24}(0.1)^4 = \frac{1}{12}\cdot\frac{1}{10^4} = \frac{1}{120000}$.

That is, $T_3(4.9)$ is accurate to within $\frac{1}{120000}$ units as an approximation to $f(4.9)$.

9. (12 points)

(a) Compute a power series expansion for $\cos x$, centered at 0. (Show all of your steps.)

We apply the Taylor series recipe for $f(x)=\cos x$ at $a=0$:

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$c_n = f^{(n)}(0)/n!$
0	$\cos x$	1	$1/0! = 1$
1	$-\sin x$	0	0
2	$-\cos x$	-1	$-1/2! = -1/2$
3	$\sin x$	0	0
4	$\cos x$	1	$1/4! = 1/24$
5	$-\sin x$	0	0
6	$-\cos x$	-1	$-1/6!$

(etc.)

Thus, $c_{2n} = \dfrac{(-1)^n}{(2n)!}$ and $c_{2n+1}=0$ for $n\geq 0$. It follows that the power series is

$$\boxed{\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = 1-\frac{x^2}{2}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\cdots}$$

(b) Find a power series expansion for $\displaystyle\int \frac{1-\cos x}{x^2}\,dx$ and determine its radius of convergence.

First, $\dfrac{1-\cos x}{x^2}$ has power series

$$\frac{1}{x^2}\left(1-\left(1-\frac{x^2}{2}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\cdots\right)\right) = \frac{1}{x^2}\left(\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\frac{x^8}{8!}+\frac{x^{10}}{10!}-\cdots\right)$$

$$= \frac{1}{2!}-\frac{x^2}{4!}+\frac{x^4}{6!}-\frac{x^6}{8!}+\frac{x^8}{10!}-\cdots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+2)!}.$$

Next, we find $\displaystyle\int\frac{1-\cos x}{x^2}dx$ has power series

$$C+\frac{x}{2!}-\frac{x^3}{3\cdot 4!}+\frac{x^5}{5\cdot 6!}-\frac{x^7}{7\cdot 8!}+\frac{x^9}{9\cdot 10!}-\cdots = \boxed{C+\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)\cdot(2n+2)!}}.$$

We find the radius of convergence via the Ratio Test. If $a_n = \dfrac{(-1)^n x^{2n+1}}{(2n+1)\cdot(2n+2)!}$, then

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+3}}{(2n+3)\cdot(2n+4)!}\cdot\frac{(2n+1)\cdot(2n+2)!}{(-1)^n x^{2n+1}}\right| = \lim_{n\to\infty}|x^2|\cdot\frac{2n+1}{2n+3}\cdot\frac{(2n+2)!}{(2n+4)!} \leftarrow \left((2n+4)! = (2n+4)(2n+3)(2n+2)!\right)$$

$$=\lim_{n\to\infty}|x^2|\cdot\frac{2n+1}{2n+3}\cdot\frac{1}{(2n+4)(2n+3)} = 0,\ \text{because}\ \lim_{n\to\infty}\frac{2n+1}{(2n+3)^2(2n+4)} = \lim_{n\to\infty}\frac{2/n^2+1/n^3}{(2+3/n)^2(2+4/n)} = 0.$$

Since $L<1$ for all x, the Ratio Test implies that the power series converges for all x; thus, the radius of convergence is $\boxed{\infty}$.

(c) Use your answer in part (b) to find a series for $\int_0^1 \frac{1-\cos x}{x^2}\,dx$.

Under the assumption that $\int_0^1 \frac{1-\cos x}{x^2}\,dx = \int_0^1 \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+2)!}\right) dx = \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\cdot(2n+2)!}\right]_{x=0}^{x=1}$,

we obtain the series $\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\cdot(2n+2)!}} = \frac{1}{2!} - \frac{1}{3\cdot 4!} + \frac{1}{5\cdot 6!} - \frac{1}{7\cdot 8!} + \frac{1}{9\cdot 10!} - \cdots$

(d) If you approximate the definite integral $\int_0^1 \frac{1-\cos x}{x^2}\,dx$ by taking the partial sum consisting of the first four nonzero terms of the series that you obtained in part (c), what can you say about the accuracy of your approximation? Be as precise as you can, and state your reasoning clearly.

The above series is $\sum_{n=0}^{\infty} (-1)^n b_n$, where we write $b_n = \frac{1}{(2n+1)\cdot(2n+2)!}$ for $n \geq 0$.

We note that $(2n+1)\cdot(2n+2)!$ is a <u>positive</u>, <u>increasing</u> function of $n \geq 0$ that grows arbitrarily large as $n \to \infty$, so $b_n = \frac{1}{(2n+1)\cdot(2n+2)!}$ satisfies

(i) $b_n > 0$,

(ii) $b_{n+1} \leq b_n$ for $n \geq 0$, and

(iii) $\lim_{n\to\infty} b_n = 0$.

Thus, we may apply the Alternating Series Test and Remainder Estimate to conclude not only that $\sum_{n=0}^{\infty} (-1)^n b_n$ converges to a value S (a fact we already knew from (b)), but also that the error (remainder) in approximating S using the four terms from $n=0$ to $n=3$ satisfies $|R_3| = |S - S_3| \leq b_4 = \frac{1}{9\cdot 10!}$. In other words, although the described partial sum (consisting of the first four nonzero terms of the series from part (c)) is less than the true value of $\int_0^1 \frac{1-\cos x}{x^2}\,dx$, it is <u>no more than</u> $\frac{1}{9\times 10!}$ different from the true value.