SOLUTIONS

1. (42 points) Evaluate each of the following integrals, showing all of your reasoning.

7 points (a) $\int t^2 \cos 2t \, dt$

Integrate by parts: $\begin{cases} u = t^2 & du = 2t\,dt \\ dv = \cos 2t\,dt & v = \frac{1}{2}\sin 2t \end{cases}$ ← Chain Rule!

$$= \frac{t^2}{2}\sin 2t - \int t\sin 2t\,dt$$

Integrate by parts: $\begin{cases} p = t & dp = dt \\ dq = \sin 2t\,dt & q = -\frac{1}{2}\cos 2t \end{cases}$

$$= \frac{t^2}{2}\sin 2t - \left(-\frac{t}{2}\cos 2t + \int \frac{1}{2}\cos 2t\,dt\right)$$

$$= \frac{t^2}{2}\sin 2t + \frac{t}{2}\cos 2t - \frac{1}{2}\int \cos 2t\,dt$$

$$= \boxed{\frac{t^2}{2}\sin 2t + \frac{t}{2}\cos 2t - \frac{1}{4}\sin 2t + C}$$

6 points (b) $\int \frac{dx}{x \ln x \, \ln(\ln x)}$

Let $\begin{cases} u = \ln x \\ du = \frac{1}{x}dx \end{cases}$

$$= \int \frac{du}{u \ln u}$$

Let $\begin{cases} v = \ln u \\ dv = \frac{1}{u}du \end{cases}$

$$= \int \frac{dv}{v}$$

$$= \ln|v| + C = \ln|\ln u| + C = \boxed{\ln|\ln(\ln x)| + C}.$$

$\boxed{7\text{ points}}$ (c) $\int \frac{z^2+z-1}{z^2+1}\,dz$

Perform long division to obtain $\frac{z^2+z-1}{z^2+1} = 1 + \frac{z-2}{z^2+1}$.

$$\begin{array}{r} 1 \\ z^2+1 \,\big)\, \overline{z^2+z-1} \\ \underline{\cancel{z^2}\quad +1} \\ \cancel{z}-2 \end{array}$$

Then $\int \frac{z^2+z-1}{z^2+1}dz = \int\left(1+\frac{z-2}{z^2+1}\right)dz = \int dz + \int \frac{z}{z^2+1}dz + \int \frac{-2}{z^2+1}dz$.

Now $\int \frac{z}{z^2+1}dz = \int \frac{\frac{1}{2}dw}{w}$ for $\left\{\begin{matrix} w = z^2+1 \\ dw = 2z \end{matrix}\right\}$, so equals $\frac{1}{2}\int \frac{dw}{w} = \frac{1}{2}\ln|w| + C$.

Also, $\int \frac{-2}{z^2+1}dz = -2\int \frac{1}{z^2+1}dz = -2\arctan(z) + C$.

Thus, $\int \frac{z^2+z-1}{z^2+1}dz = \boxed{z + \frac{1}{2}\ln|z^2+1| - 2\arctan z + C}$.

$\boxed{7\text{ points}}$ (d) $\int_{\sqrt{10}}^{5} \frac{1}{x^2\sqrt{x^2-9}}\,dx$

After trying some more straightforward substitutions to no avail, we try the trig substitution $\left\{\begin{matrix} x = 3\sec\theta \\ dx = 3\sec\theta\tan\theta\,d\theta \end{matrix}\right\}$. Thus $\theta = \sec^{-1}\left(\frac{x}{3}\right) = \operatorname{arcsec}\left(\frac{x}{3}\right)$,

so that if $x=\sqrt{10}$, then $\theta = \operatorname{arcsec}\left(\frac{\sqrt{10}}{3}\right)$, etc.

$$\Rightarrow \int_{\sqrt{10}}^{5} \frac{dx}{x^2\sqrt{x^2-9}} = \int_{\operatorname{arcsec}(\sqrt{10}/3)}^{\operatorname{arcsec}(5/3)} \frac{3\sec\theta\tan\theta\,d\theta}{9\sec^2\theta\sqrt{9\sec^2\theta-9}} = \int_{\operatorname{arcsec}(\sqrt{10}/3)}^{\operatorname{arcsec}(5/3)} \frac{3\sec\theta\tan\theta\,d\theta}{9\sec^2\theta\sqrt{9\tan^2\theta}} = \int_{\operatorname{arcsec}(\sqrt{10}/3)}^{\operatorname{arcsec}(5/3)} \frac{1}{9\sec\theta}\,d\theta$$

$$= \frac{1}{9}\int_{\operatorname{arcsec}(\sqrt{10}/3)}^{\operatorname{arcsec}(5/3)} \cos\theta\,d\theta = \frac{1}{9}\sin\theta\Big]_{\operatorname{arcsec}(\sqrt{10}/3)}^{\operatorname{arcsec}(5/3)} = \boxed{\frac{1}{9}\sin\left(\operatorname{arcsec}\frac{5}{3}\right) - \frac{1}{9}\sin\left(\operatorname{arcsec}\frac{\sqrt{10}}{3}\right)}.$$

$\boxed{7\text{ points}}$ (e) $\displaystyle\int_0^{\pi/2} \sin^5\theta \cos^4\theta \, d\theta$

Let $\left\{\begin{array}{l} u=\cos\theta \\ du=-\sin\theta\, d\theta \end{array}\right\}$, so that
$$\sin^5\theta\cos^4\theta\, d\theta = \cos^4\theta\cdot\sin^2\theta\cdot\sin^2\theta\cdot\sin\theta\, d\theta$$
$$= \cos^4\theta\cdot(1-\cos^2\theta)\cdot(1-\cos^2\theta)\cdot\sin\theta\, d\theta$$
$$= u^4\cdot(1-u^2)(1-u^2)\cdot(-du)$$
$$= -u^4(1-2u^2+u^4)\,du.$$

Furthermore, if $\theta=0$, then $u=\cos 0=1$;
and if $\theta=\pi/2$, then $u=\cos\frac{\pi}{2}=0$.

Thus
$$\int_0^{\pi/2}\sin^5\theta\cos^4\theta\, d\theta = -\int_1^0 u^4(1-2u^2+u^4)\,du = \int_0^1 (u^4-2u^6+u^8)\,du$$
$$= \frac{u^5}{5}-\frac{2u^7}{7}+\frac{u^9}{9}\Bigg]_{u=0}^{u=1} = \boxed{\frac{1}{5}-\frac{2}{7}+\frac{1}{9}}.$$

$\boxed{8\text{ points}}$ (f) $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\, dx$

After trying more "ordinary" substitutions to no avail, we try the trig substitution $\left\{\begin{array}{l} x=\sin\theta \\ dx=\cos\theta\, d\theta \end{array}\right\}$, so that $\theta=\arcsin x$.

$$\Rightarrow \int\frac{x^2\,dx}{\sqrt{1-x^2}} = \int\frac{\sin^2\theta\cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}} = \int\frac{\sin^2\theta\cos\theta}{\sqrt{\cos^2\theta}}\, d\theta$$
$$= \int\sin^2\theta\, d\theta = \int\frac{1}{2}(1-\cos 2\theta)\, d\theta$$
$$= \int\frac{1}{2}\, d\theta - \int\frac{1}{2}\cos 2\theta\, d\theta$$
$$= \frac{\theta}{2}-\frac{\sin 2\theta}{4}+C = \boxed{\frac{\arcsin x}{2}-\frac{\sin(2\arcsin x)}{4}+C}.$$

(This simplifies to $\dfrac{\arcsin x}{2}-\dfrac{x\sqrt{1-x^2}}{2}+C$.)

2. (8 points) Show all steps in evaluating the integral $\int \frac{x^4+1}{x^3-x^2-2x}\,dx = \int \frac{x^4+1}{x(x+1)(x-2)}\,dx$.

We first perform long division to find:

$$\begin{array}{r} x+1 \\ x^3-x^2-2x \overline{)\, x^4 \qquad\quad +1} \\ x^4-x^3-2x^2 \\ \hline x^3+2x^2+1 \\ x^3-x^2-2x \\ \hline 3x^2+2x+1 \end{array}$$

so that $\dfrac{x^4+1}{x^3-x^2-2x} = x+1+\dfrac{3x^2+2x+1}{x^3-x^2-2x}$.

We now find the partial fraction decomposition for the remainder term:

$$\frac{3x^2+2x+1}{x(x+1)(x-2)} = \frac{3x^2+2x+1}{x^3-x^2-2x} = \frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-2}$$

$$\Rightarrow 3x^2+2x+1 = A(x+1)(x-2)+Bx(x-2)+Cx(x+1)$$

Let $x=0 \Rightarrow 1 = A\cdot 1\cdot(-2) \Rightarrow A = -1/2$

Let $x=-1 \Rightarrow 2 = B\cdot(-1)(-3) \Rightarrow B = 2/3$

Let $x=2 \Rightarrow 17 = C\cdot 2\cdot 3 \Rightarrow C = 17/6$

Thus, $\displaystyle\int \frac{x^4+1}{x^3-x^2-2x}\,dx = \int (x+1)\,dx + \int \frac{3x^2+2x+1}{x^3-x^2-2x}\,dx$

$$= \int (x+1)\,dx + \int \frac{-1/2}{x}\,dx + \int \frac{2/3}{x+1}\,dx + \int \frac{17/6}{x-2}\,dx$$

$$= \boxed{\frac{x^2}{2}+x-\frac{1}{2}\ln|x|+\frac{2}{3}\ln|x+1|+\frac{17}{6}\ln|x-2|+C}.$$

3. (6 points) A particle is moved along the x-axis by a force that measures $F(x)$ Newtons at a point x meters from the origin, where values of $F(x)$ are given in the table below.

x	0	2	4	6	8	10	12	14	16
$F(x)$	10.5	9.6	8.8	8.1	7.5	7.0	6.7	6.5	6.4

(a) Using the *Midpoint Rule*, write a sum using *values from the chart* representing an estimate for the work done by the force in moving the object a distance of 16 m. You do not have to simplify your expression.

To be able to use values from the chart for the Midpoint Rule, we can use $n=4$ subintervals ($n=2$ or $n=1$ works, but $n=8$ subintervals wouldn't allow us to utilize values of F that we know from the chart!).

We have $\Delta x = \frac{16-0}{4} = 4$, so

$$M_4 = \Delta x \cdot \left(F\left(\tfrac{0+4}{2}\right) + F\left(\tfrac{4+8}{2}\right) + F\left(\tfrac{8+12}{2}\right) + F\left(\tfrac{12+16}{2}\right)\right)$$
$$= 4\cdot\left(F(2) + F(6) + F(10) + F(14)\right)$$
$$= \boxed{4\cdot\left(9.6 + 8.1 + 7.0 + 6.5\right)} \text{ (Joules).}$$

(b) Do the same using *Simpson's Rule*; again, you do not have to simplify the expression.

For Simpson's Rule, we can use $n=8$: so $\Delta x = \frac{16-0}{8} = 2$, and

$$S_8 = \frac{\Delta x}{3}\cdot\left(F(0) + 4F(2) + 2F(4) + 4F(6) + 2F(8) + 4F(10) + 2F(12) + 4F(14) + F(16)\right)$$
$$= \boxed{\frac{2}{3}\cdot\left(10.5 + 4\times 9.6 + 2\times 8.8 + 4\times 8.1 + 2\times 7.5 + 4\times 7.0 + 2\times 6.7 + 4\times 6.5 + 6.4\right)} \text{ (Joules).}$$

4. (8 points) Consider the integral $\int_0^2 f(x)\,dx$, where $f(x) = e^{-x^2/2}$.

(a) Estimate the error made in approximating the value of this integral using the Trapezoidal Rule with $n = 5$ subintervals. *State your answer in a complete sentence.* You may make use of the fact that $f''(x) = (x^2 - 1)e^{-x^2/2}$.

The error bound on the Trapezoidal Rule approximation depends on K_2, so we must estimate the maximum value of $|f''(x)| = |(x^2-1)e^{-x^2/2}| = |x^2-1| \cdot e^{-x^2}$ on $[0,2]$.

But $0 \leq |x^2-1| \leq 3$ and $0 \leq e^{-x^2} \leq 1$ on $[0,2]$, so we may take $K_2 = 3 \cdot 1 = 3$.

(In fact, the maximum value of $|x^2-1| \cdot e^{-x^2}$ on $[0,2]$ is 1, but this requires solving a max-min problem & using the Closed Interval Method, so although we could take $K_2 = 1$, it is cumbersome to justify this.) The error-bound formula says $|E_T| \leq \frac{K_2 \cdot (b-a)^3}{12n^2} = \frac{3 \cdot 2^3}{12 \cdot 25} = \frac{2}{25}$;

thus, the error in approximating the integral using T_5 is no greater than $\frac{2}{25}$ in absolute value.

(b) Again using the Trapezoidal Rule, how many subintervals n would be necessary to guarantee an error of at most 10^{-4}? Give a valid n in simplified form. (As long as you justify your answer, you do not have to worry about finding the best possible value.)

We'll again use $K_2 = 3$. We require $\frac{K_2 \cdot (b-a)^3}{12n^2} \leq 10^{-4}$, for then

$$|E_T| \leq \frac{K_2 \cdot (b-a)^3}{12n^2} \leq 10^{-4},$$ as desired.

Thus, we need $\frac{3 \cdot 2^3}{12 \cdot n^2} \leq 10^{-4}$, i.e. $\frac{2}{n^2} \leq 10^{-4}$,

so $n^2 \geq 2 \cdot 10^4$,

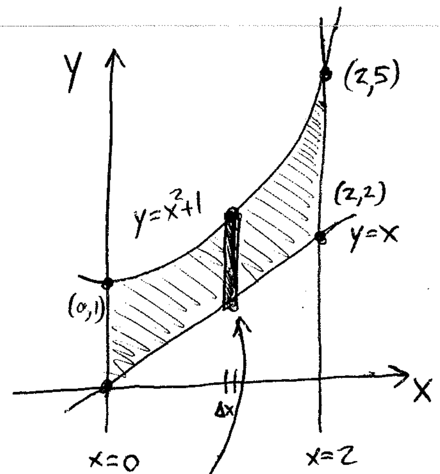
and thus $n \geq \sqrt{2} \cdot 10^2$.

Since $\sqrt{2} < 2$, we may take $\boxed{n = 200}$.

(In fact we may take any whole number n that is at least 142, but this wasn't required. A smaller valid value of K_2, such as $K_2 = 1$, could yield a smaller n.)

5. (10 points) Consider the region R bounded by the curves $y = x$, $y = x^2 + 1$, $x = 0$, and $x = 2$.

(a) Set up, but do not evaluate, an integral that gives the area of the region R. Justify your answer by drawing a picture and marking a sample slice.

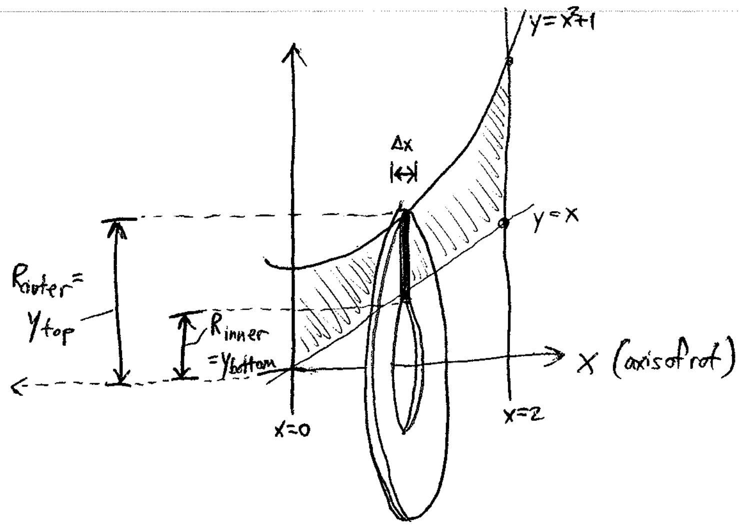


(Note: $y=x$ and $y=x^2+1$ do not intersect, because $x=x^2+1$ has no real solutions.)

$$\left\{\begin{array}{l} \text{height} = y_{top} - y_{bottom} = x^2+1-x \\ \text{width} = \Delta x \end{array}\right\} \Rightarrow \Delta A \approx (x^2+1-x)\Delta x$$

$$\Rightarrow \quad \text{Area} = \int_0^2 (x^2+1-x)\,dx$$

(b) Set up an integral representing the volume of the solid obtained by rotating the region R about the x-axis. Justify your answer by drawing and labeling a picture with a sample slice, and cite the method used, but don't evaluate the integral.

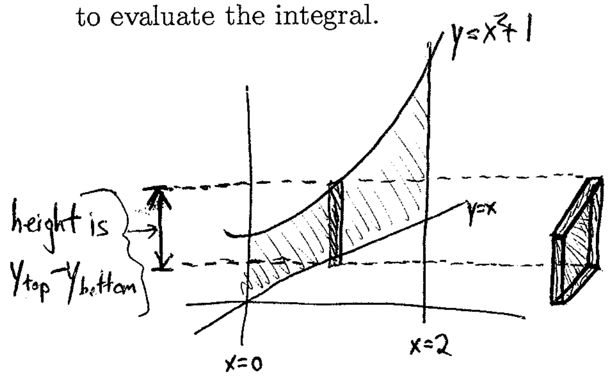


Vertical slices create <u>washers</u> as cross-sections.

$$\Delta V \approx A(x)\,\Delta x = \left(\pi R_{outer}^2 - \pi R_{inner}^2\right)\Delta x = \left(\pi (x^2+1)^2 - \pi x^2\right)\Delta x$$

$$\Rightarrow \text{Volume} = \int_0^2 A(x)\,dx = \int_0^2 \left(\pi(x^2+1)^2 - \pi x^2\right)dx \,.$$

(c) If the base of a solid V is the region R, and if all cross-sections of V that are perpendicular to the x-axis are squares, write a definite integral that will give the volume of V. You don't have to evaluate the integral.



square has side-length equal to <u>height</u> of slice

$$\Delta V \approx A(x)\,\Delta x = (\text{length})^2\,\Delta x = \left(y_{top} - y_{bottom}\right)^2 \Delta x = \left(x^2+1-x\right)^2 \Delta x$$

$$\Rightarrow \text{Volume} = \int_0^2 A(x)\,dx = \int_0^2 (x^2+1-x)^2\,dx \,.$$

6. (10 points)

(a) Set up, but do not evaluate, an integral representing the area of the region bounded by the curves $x = 6 - y^2$ and $y = -x$. As justification, draw a picture with a sample slice labeled.

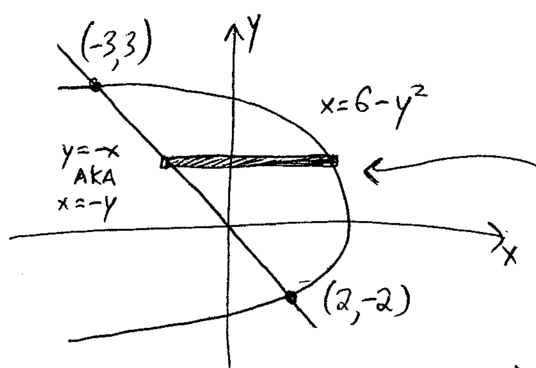
Intersection points of $x=6-y^2$ and $y=-x$:

Set $x=-y=6-y^2 \Rightarrow y^2-y-6=0$

$\Rightarrow (y-3)(y+2)=0$

$\Rightarrow y=3$ or $y=-2$

So the curves intersect at $(x,y)=(-3,3)$ and $(x,y)=(2,-2)$.



$\left\{ \begin{array}{l} \text{length} = x_{right} - x_{left} = 6-y^2-(-y) \\ \text{thickness} = \Delta y \end{array} \right.$

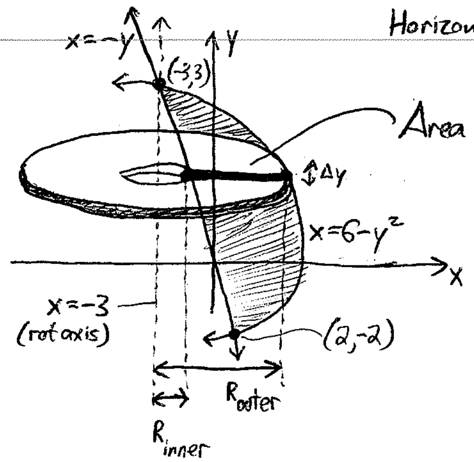
$\Rightarrow$ Area of a horizontal slice at coord y:

$\Delta A \approx (\text{length})\Delta y$

$= (6+y-y^2)\Delta y$.

$\Rightarrow \boxed{\text{Area} = \int_{y=-2}^{y=3} (6+y-y^2)\,dy}$.

(b) Set up an integral representing the volume of the solid obtained by rotating the region from part (a) around the line $x = -3$. Make sure you justify your answer (draw and label a diagram, and cite the method). Again don't evaluate the integral.



Horizontal slices create <u>washers</u> in cross-section.

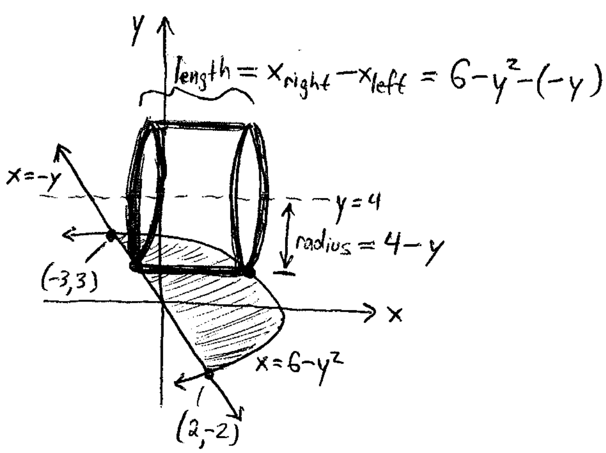
$\text{Area} = \pi R_{outer}^2 - \pi R_{inner}^2$; thickness $= \Delta y$

$R_{inner} = x_{left} - x_{rot\,axis}$
$= -y - (-3) = 3 - y$

$R_{outer} = x_{right} - x_{rot\,axis}$
$= (6-y^2) - (-3) = 9 - y^2$

$$\Rightarrow \text{Volume} = \int_{y=-2}^{y=3} A(y)\,dy = \boxed{\int_{-2}^{3} \left(\pi(9-y^2)^2 - \pi(3-y)^2\right) dy}.$$

(c) Now do the same for the volume of the solid obtained by rotating the region from part (a) around the line $y = 4$.



Horizontal slices create <u>cylindrical shells</u>.

Volume ΔV of shell created by slice at coordinate y: $(-2 \le y \le 3)$

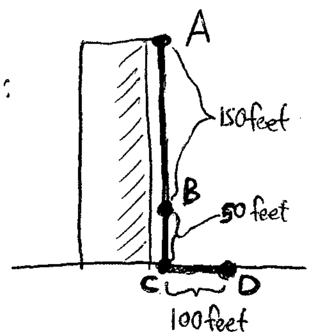
$\Delta V \approx 2\pi\,(\text{radius})(\text{length})(\text{thickness})$
$= 2\pi(4-y)(6-y^2+y)\,\Delta y$

$$\Rightarrow \boxed{\text{Volume} = \int_{-2}^{3} 2\pi(4-y)(6-y^2+y)\,dy}.$$

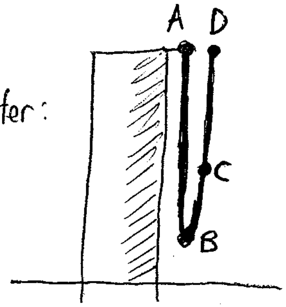
7. (8 points) A 300-foot cable with a linear density of 1.3 lb/ft is suspended by one end from the top of a 200-foot building — so that the other end lies on the ground, which is completely level.

Suppose that we wish to lift the cable's lower end up to the top of the building, while keeping the upper end fixed there, so that the two ends of the cable are level with each other (and the remainder of the cable hangs below, now doubled up). How much work is required to accomplish this? Show all the steps in your computation.

Before:



After:



According to the diagram, we'll break the cable into three portions:

- Segment AB (150 feet long);
- Segment BC (50 feet long);
- Segment CD (100 feet long; this portion begins lying on the ground).

- Work to move AB: No portion of this segment moves at all, so no work is done: $W_{AB} = 0$.
- Work to move CD: Consider a small piece of this portion of cable, of length Δx, located x feet from point D, where $0 \leq x \leq 100$. This piece is lifted an (approximately) uniform amount of $(200-x)$ feet; thus the work to lift this piece is

  $\Delta W \approx (\text{weight})(\text{distance to lift}) = ((1.3 \text{ lb/ft})(\Delta x)) \cdot (200-x)$, and so the total work

  to lift segment CD is $W_{CD} = \int_0^{100} 1.3(200-x)\,dx = 1.3\left(200x - \frac{x^2}{2}\right)\Big]_0^{100} = (1.3)(15000)$ ft·lb.

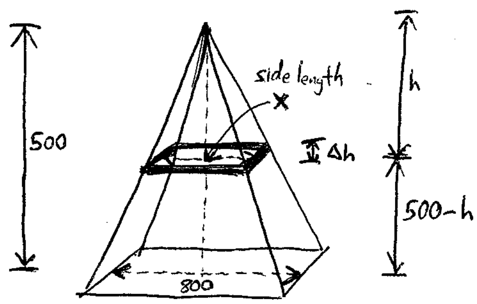
- Work to move BC: This 50-foot portion of cable is "inverted" upside-down around point B; consider a small piece of this segment, located x feet from point B and having length Δx. (Note $0 \leq x \leq 50$ here; the piece at $x = 50$ is at point C.) This piece is lifted $2x$ feet upwards; thus, the work to lift this piece is

  $\Delta W \approx (\text{weight})(\text{distance to lift}) = ((1.3 \text{ lb/ft})(\Delta x)) \cdot 2x$, and so the total work

  to lift this portion is $W_{BC} = \int_0^{50} 1.3 \cdot 2x\,dx = 1.3x^2\Big]_0^{50} = (1.3)(2500)$ ft·lb.

Thus, the total work done is $W_{AB} + W_{BC} + W_{CD} = \boxed{(1.3)(17500) \text{ ft·lb}}$.

8. (8 points) The Great Pyramid of Giza is roughly a solid square pyramid made of limestone; when first constructed it stood 500 feet tall, with a square base of side length 800 feet. (Like any square pyramid, each horizontal cross-section of the pyramid is a square.) The weight density of limestone is 150 lb/ft^3. How much work was done to lift upward, from ground-level, all the limestone that forms this pyramid? [Note: The dimensions were rounded to the nearest 100 feet in this problem!]



If we slice the pyramid into horizontal cross-sections of thickness Δh (small), we can use the formula "work = force·distance" to compute the work to lift each slice, since the force (weight) and distance are constant/uniform on each slice.

Note that the square slice located h feet from the top of the pyramid $(0 \leq h \leq 500)$, i.e. $(500-h)$ feet above the ground, has side length x, where

$$\frac{x}{h} = \frac{800}{500}\ ; \quad \text{i.e.} \quad x = \frac{8}{5}h. \qquad \text{Thus,}$$

$$\begin{aligned}
\Delta W &= \text{work to lift each slice} \\
&\approx \text{force} \cdot \text{distance} = (\text{weight of slice})(\text{distance to lift slice}) \\
&= (\text{density})(\text{volume of slice})(\text{distance to lift slice}) \\
&= (\text{density})(\text{area})(\text{thickness})(\text{dist. to lift}) \\
&= 150 \cdot \left(\tfrac{8}{5}h\right)^2 \cdot \Delta h \cdot (500-h),
\end{aligned}$$

$$\text{so total work} = W = \int_0^{500} 150 \cdot \frac{64}{25} h^2 (500-h)\,dh = \frac{150\cdot 64}{25} \int_0^{500} (500h^2 - h^3)\,dh$$

$$= \frac{150\cdot 64}{25}\left(\frac{500h^3}{3} - \frac{h^4}{4}\right)\Bigg|_0^{500} = \frac{150\cdot 64}{25}\cdot 500^4 \cdot \left(\frac{1}{3}-\frac{1}{4}\right) = \boxed{2\times 10^{12} \text{ ft-lb}}.$$

Fun fact: Via the conversion 10 person hours $\approx$ 1 horsepower hour $\approx 2\times 10^6$ ft-lb, this effort required roughly 10^7 person hours! (And that's just for the upward lifting.)