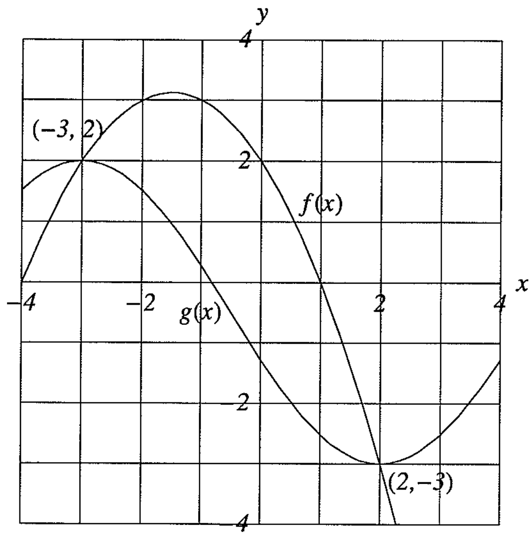
Math 42 – Winter 2007 – Final Exam Solutions

1. (9 points) Let R be the region bounded by the curves $y = f(x)$ and $y = g(x)$ shown in the graph below.



(a) Set up a definite integral that will give the area of the region R.

$$A = \int_{-3}^{2} (f(x) - g(x))\,dx$$

(b) Set up a definite integral that will give the volume of the solid generated when the region R is revolved about the line $y = 4$.

Slice perp to axis of revolution $\Rightarrow$ variable is x; At a coordinate x, the cross-sections are washers, with inner radius $4-f(x)$ and outer radius $4-g(x)$;

thus $$\text{Vol} = \int_{-3}^{2} \text{Area}(x)\,dx = \boxed{\int_{-3}^{2} \pi(4-g(x))^2 - \pi(4-f(x))^2\,dx}.$$

(c) If the base of a solid V is the region R, and if all cross-sections of V that are perpendicular to the x-axis are squares, set up a definite integral that will give the volume of V.

Slices are perp to x-axis $\Rightarrow$ variable is x;

At a coordinate x, cross-sections are squares of length $f(x)-g(x)$;

thus $$\text{Vol} = \int_{-3}^{2} \text{Area}(x)\,dx = \boxed{\int_{-3}^{2} (f(x)-g(x))^2\,dx}.$$

2. (10 points) According to Newton's Law of Gravity, an object near the earth at a distance of r meters from the center of the earth feels a gravitational force of $F = k/r^2$ Newtons, where k is a positive constant that depends on the mass of the object.

   (a) In a revision of an earlier doomsday prediction, scientists are now forecasting that in the year 2029, asteroid 2004-MN4 will have moved into an orbiting position about the earth, at a distance of 5×10^7 meters from the center of the earth. In terms of the above constant k, compute the work against gravity which is required to push the asteroid from this position to a more desirable distance of 6×10^7 meters.

Work in exerting a variable force $F(r)$ along a distance : $W = \int_{r_{start}}^{r_{end}} F(r)\,dr.$

Thus, $W = \int_{5\times10^7}^{6\times10^7} \frac{k}{r^2}\,dr = -\frac{k}{r}\Big]_{r=5\times10^7}^{r=6\times10^7} = -k\left(\frac{1}{6\times10^7} - \frac{1}{5\times10^7}\right)$

$= \frac{k}{10^7}\left(\frac{1}{5} - \frac{1}{6}\right) = \boxed{\frac{k}{10^7} \cdot \frac{1}{30} \text{ Newton-meters.}}$

   (b) A Hollywood producer suggests that a further effort be made, to send the asteroid from the position of 6×10^7 meters, completely away from the earth. By using a suitable improper integral, compute (again in terms of k) the amount of work against the pull of gravity which is required to send the asteroid "off to infinity."

Work $= \int_{6\times10^7}^{\infty} \frac{k}{r^2}\,dr = \lim_{N\to\infty} \int_{6\times10^7}^{N} \frac{k}{r^2}\,dr = \lim_{N\to\infty} -\frac{k}{r}\Big]_{r=6\times10^7}^{r=N}$

$= \lim_{N\to\infty}\left(-\frac{k}{N} + \frac{k}{6\times10^7}\right) = \boxed{\frac{k}{6\times10^7} \text{ Newton-meters.}}$

3. (12 points) Compute the following, showing all work.

(a) $\int_{-1}^{1} \frac{1}{x^{2/3}}\,dx$

This is an improper integral, because integrand is discontinuous at $x=0$.

Thus $\int_{-1}^{1} \frac{1}{x^{2/3}}dx = \int_{-1}^{0} \frac{1}{x^{2/3}}dx + \int_{0}^{1} \frac{1}{x^{2/3}}dx = \lim_{b\to 0^-} \int_{-1}^{b} \frac{1}{x^{2/3}}dx + \lim_{a\to 0^+} \int_{a}^{1} \frac{1}{x^{2/3}}dx$

$$= \lim_{b\to 0^-} 3x^{1/3}\Big]_{-1}^{b} + \lim_{a\to 0^+} 3x^{1/3}\Big]_{a}^{1}$$

$$= \lim_{b\to 0^-} (3b^{1/3} - -3) + \lim_{a\to 0^+} (3 - 3a^{1/3}) = 3+3 = \boxed{6}.$$

(b) $\int_{0}^{\sqrt{2}} \frac{x}{\sqrt{9-x^4}}\,dx$

First simplify expression inside radical by writing $u=x^2$.

$u=x^2$
$du=2x\,dx$ $\Rightarrow$ If $x=0$, then $u=0$; if $x=\sqrt{2}$, then $u=2$; $\Rightarrow$

$$\int_{0}^{\sqrt{2}} \frac{x}{\sqrt{9-x^4}}dx = \int_{0}^{2} \frac{\frac{1}{2}du}{\sqrt{9-u^2}} = \frac{1}{2}\int_{0}^{2} \frac{du}{\sqrt{9-u^2}}.$$

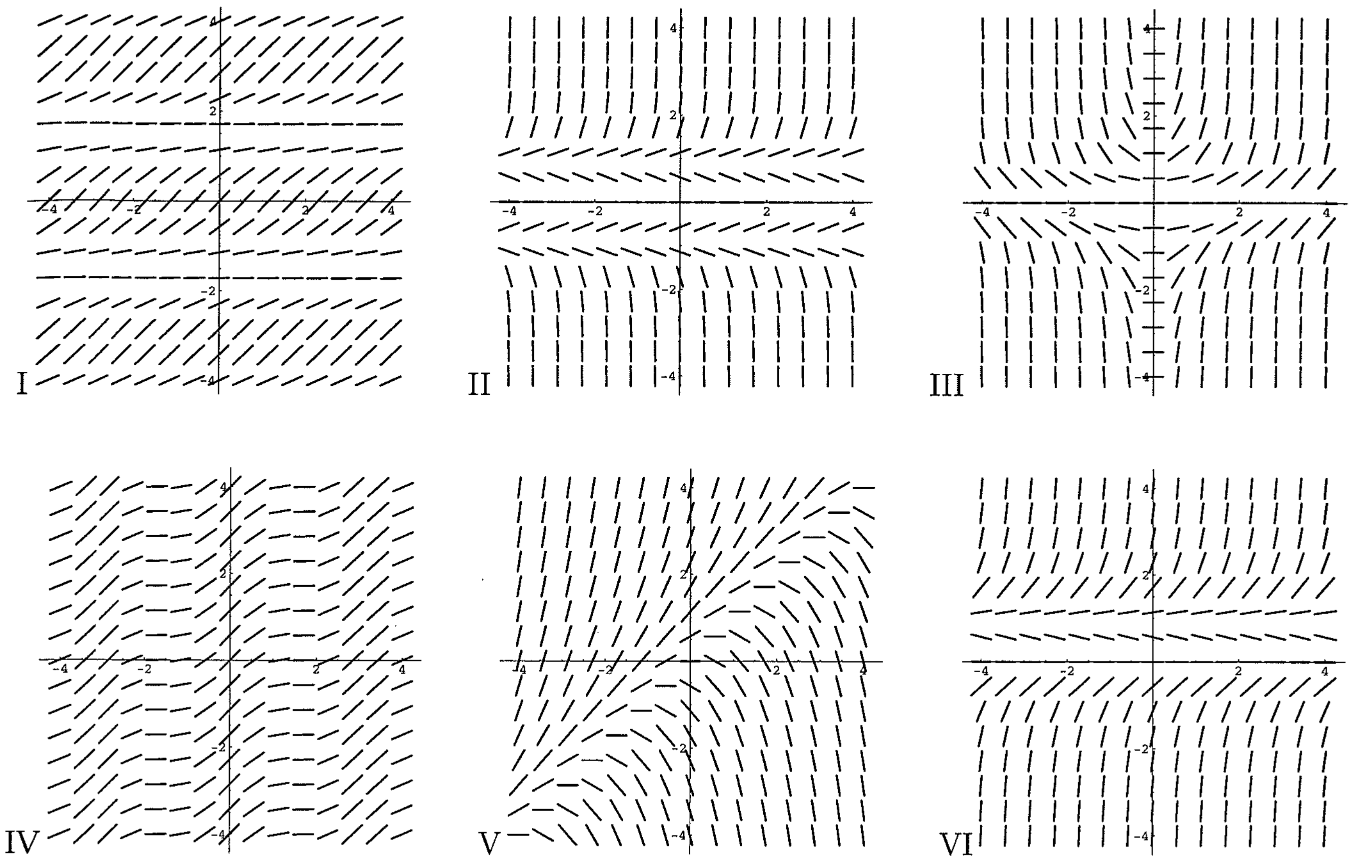
We require a trig substitution of form $\begin{cases} u=3\sin\theta \\ du=3\cos\theta \end{cases}$, so above equals

$$\frac{1}{2}\int_{0}^{\sin^{-1}(\frac{2}{3})} \frac{3\cos\theta}{\sqrt{9-9\sin^2\theta}}d\theta = \frac{1}{2}\int_{0}^{\sin^{-1}(\frac{2}{3})} \frac{3\cos\theta}{\sqrt{9\cos^2\theta}}d\theta$$

If $u=3\sin\theta$, then $\theta=\sin^{-1}(\frac{u}{3})$.

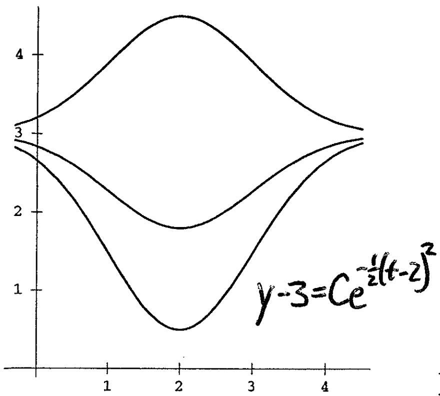
$$= \frac{1}{2}\int_{0}^{\sin^{-1}(\frac{2}{3})} d\theta = \frac{1}{2}\left(\sin^{-1}(\tfrac{2}{3}) - 0\right) = \boxed{\frac{1}{2}\sin^{-1}\left(\frac{2}{3}\right)}.$$

4. (15 points) Match the direction fields below with their differential equations. (The horizontal variable is t; the vertical is y.) Also indicate which two equations do not have matches.

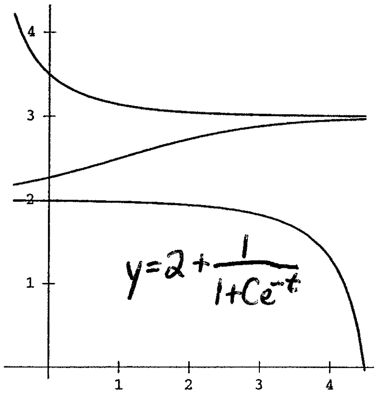


Equation	I, II, III, IV, V, VI, or "none"	Equation	I, II, III, IV, V, VI, or "none"
$dy/dt = ty^2$	III	$dy/dt = \cos^2 y$	I
$dy/dt = y - t$	V	$dy/dt = y(y^2 - 1)$	II
$dy/dt = ty$	none	$dy/dt = \cos^2 t$	IV
$dy/dt = 1 - y$	none	$dy/dt = y(y - 1)$	VI

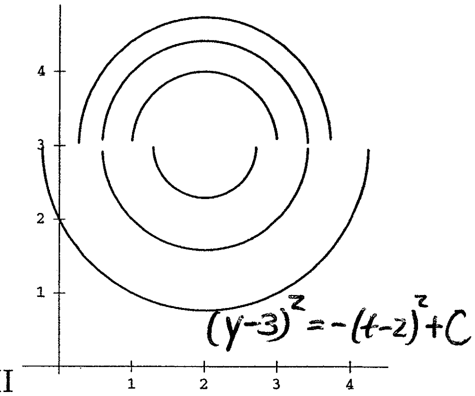
5. (21 points) Each picture below depicts a few possible solution curves to a differential equation chosen from the list at the bottom of the page. (As usual, the t-axis is horizontal, and the y-axis is vertical.) Match each equation to its sketch of solutions; one equation will not have a match.



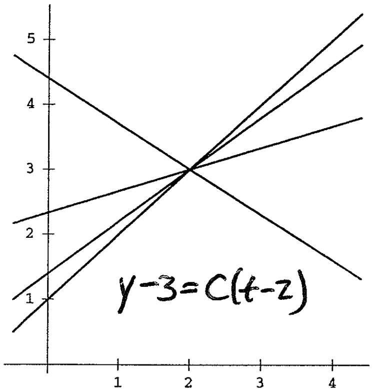
I



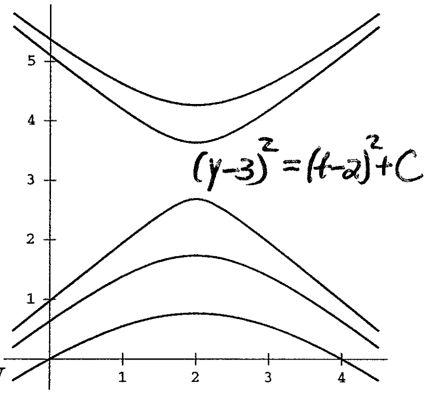
II



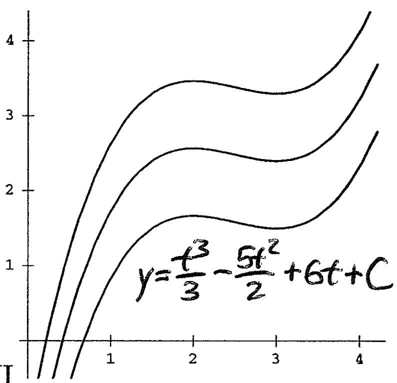
III



IV



V



VI

Equation	I, II, III, IV, V, VI, or "none"	Brief reason { One option is to separate vars and solve, getting eqns above; here are qualitative reasons:
(a) $y' = (t-3)(t-2)$	VI	Only graph VI has slopes that appear to be independent of the value of y; only this eqn. is independent of y.
(b) $y' = -(y-3)(y-2)$	II	In graph II, slope is negative for $y<2$ or $y>3$, regardless value of t; none of the other eqns. are consistent w/this.
(c) $y' = -(y-3)(t-2)$	I	In graph I, slope is 0 for $t=2$, regardless of y; and slope is nearly 0 for y near 3, regardless t. Only eqn (c) has this.
(d) $y' = \dfrac{y-3}{t-2}$	IV	Eqn. predicts positive slope for $t>2$ & $y>3$, or for $t<2$ & $y<3$; that's true of both graphs IV & V, although graph V has an additional property only seen in equation (f). Elimination!
(e) $y' = -\dfrac{y-3}{t-2}$	none	Eqn predicts negative slope for $t>2$ & $y>3$, or for $t<2$ & $y<3$; that's true of graphs I & III, but those graphs also have slope 0 for $t=2$, regardless of y. This isn't true of equation (e).
(f) $y' = \dfrac{t-2}{y-3}$	V	In graph V, slope is 0 for $t=2$, regardless of y; plus, slope is pos for $t>2$ and $y>3$. Only eqn (f) has both properties.
(g) $y' = -\dfrac{t-2}{y-3}$	III	In graph III, slope is 0 for $t=2$, regardless of y; plus, slope is neg for $t>2$ and $y>3$. Although eqn (c) also has these properties, it also requires that slope is 0 near $y=3$, and that's not true of graph III (we need slope nearly undefined near $y=3$).

6. (11 points) A new 15-gallon juice dispenser in Branner Hall's dining room is initially filled with a fruit juice that is 80% orange juice and 20% pineapple juice. Every hour, 10 gallons of juice are consumed. The dispenser is also continuously replenished at this same rate, but due to a supply shortage at Branner, the refilling is being made using an orange-pineapple mixture from neighboring Wilbur Hall that is 40% orange and 60% pineapple. Assume that the dispensed juice is always well-mixed.

(a) Write down a differential equation for $P(t)$, the amount of pineapple juice in Branner's dispenser after t hours. Be sure to state your initial condition, including the units involved.

$P(t)$ = number of gallons of pineapple juice after t hours,

$\frac{dP}{dt}$ = growth rate of pineapple juice in tank, in gal/hr.

Since $\frac{dP}{dt}$ = rate of change (net) = rate in − rate out,

have $$\frac{dP}{dt} = \underbrace{\left(\frac{60}{100}\right)}_{\text{input proportion}}\left(10\ \frac{\text{gal}}{\text{hr}}\right) - \underbrace{\left(\frac{P}{15}\right)}_{\text{current (output) proportion}}\left(10\ \frac{\text{gal}}{\text{hr}}\right),$$ so $\boxed{\frac{dP}{dt} = 6 - \frac{2}{3}P}$,

with $\boxed{P(0) = \frac{20}{100} \cdot 15 = 3 \text{ gal.}}$

(b) By solving the differential equation, find the amount of pineapple juice in the container after 2 hours.

Separate variables: $$\frac{dP}{6-\frac{2}{3}P} = dt \Rightarrow \frac{3\,dP}{18-2P} = dt \Rightarrow \frac{3\,dP}{9-P} = 2dt.$$

Integration yields $$2t + C = \int \frac{3\,dP}{9-P} = -3\ln|9-P|,$$

so $$\ln|9-P| = -\frac{2}{3}t - \frac{C}{3}$$

$$\Rightarrow |9-P| = e^{-2/3t - C/3} = e^{-2/3t} \cdot e^{-C/3}$$

$$\Rightarrow 9-P = \pm e^{-C/3} e^{-(2/3)t} = Ae^{-2t/3} \qquad (A = \pm e^{-C/3})$$

$$\Rightarrow P(t) = 9 - Ae^{-2t/3}.$$

Since $P(0) = 3$, have $3 = 9 - Ae^0 \Rightarrow A = 6$, so $P(t) = 9 - 6e^{-2t/3}$.

Thus, after 2 hours, there are $\boxed{P(2) = 9 - 6e^{-4/3}}$ gallons present.

7. (12 points) Solve the following initial value problems. You may use any method or result you like, as long as it is fully justified or cited.

(a) $\dfrac{dz}{dt} = z^2 - 2z - 15, \quad z(0) = -1$

Method 1: separation of variables.

$$\frac{dz}{z^2-2z-15} = \frac{dz}{(z-5)(z+3)} = dt, \text{ so } \quad t+C = \int \frac{dz}{(z-5)(z+3)},$$

and we use partial fraction decomposition:

$$\frac{1}{(z-5)(z+3)} = \frac{A}{z-5} + \frac{B}{z+3} \Rightarrow 1 = A(z+3)+B(z-5) = (A+B)z + (3A-5B),$$

so $A=-B$, and $1 = 3A-5B = -8B$, so $B = -\frac{1}{8} = -A$.

Thus $$t+C = \int \frac{dz}{(z-5)(z+3)} = \int \frac{1/8}{z-5}\,dz + \int \frac{-1/8}{z+3}\,dz = \frac{1}{8}\ln|z-5| - \frac{1}{8}\ln|z+3| = \frac{1}{8}\ln\left|\frac{z-5}{z+3}\right|.$$

Must solve for z in terms of t, so $\ln\left|\frac{z-5}{z+3}\right| = 8(t+C) \Rightarrow \frac{z-5}{z+3} = \pm e^{8(t+C)} = De^{8t}$,

Since $z=-1$ when $t=0$, have $D=-3$, so

$$\frac{z-5}{z+3} = -3e^{8t} \Rightarrow z-5 = -3ze^{8t} - 9e^{8t}$$
$$\Rightarrow z(1+3e^{8t}) = 5-9e^{8t} \Rightarrow \boxed{z = \frac{5-9e^{8t}}{1+3e^{8t}}}.$$

Method 2: u-substitution & Logistic eqn:

Let $u = z+3$, so that $\frac{du}{dt} = \frac{dz}{dt}$, and $\frac{du}{dt} = \frac{dz}{dt} = (z-5)(z+3) = (u-8)u$.

The equation $\frac{du}{dt} = u(u-8)$ can be rewritten $\frac{du}{dt} = -8u\left(1-\frac{u}{8}\right)$, which is recognizable as a Logistic equation with $k=-8$ and $K=8$.

We can cite the formula provided as the solution, so that

$$z(t)+3 = u(t) = \frac{8}{1+Ae^{8t}} \Rightarrow z(t) = \frac{8}{1+Ae^{8t}} - 3.$$

Now $z=-1$ when $t=0$, so $-1 = \frac{8}{1+A} - 3 \Rightarrow A=3$. Thus $\boxed{z(t) = \frac{8}{1+3e^{8t}} - 3}$,

which is equivalent to the above.

(b) $\dfrac{dy}{dx} = \dfrac{2y - xy}{x - xy}$, $y(3) = 2$. (Do not attempt to solve for y as a function of x; leave your answer as an implicit curve in x and y.)

Separation of variables will work if we can factor RHS into an x-part and a y-part:

$$\frac{dy}{dx} = \frac{2y - xy}{x - xy} = \frac{y(2-x)}{x(1-y)},$$

so indeed $\frac{1-y}{y} \cdot dy = \frac{2-x}{x} \cdot dx$.

We get $\int \frac{1-y}{y} dy = \int \frac{2-x}{x} dx \Rightarrow \int \left(\frac{1}{y} - 1\right) dy = \int \left(\frac{2}{x} - 1\right) dx$

$$\Rightarrow \ln|y| - y = 2\ln|x| - x + C$$

$$\Rightarrow \ln|y| - 2\ln|x| = y - x + C$$

$$\Rightarrow \ln\left|\frac{y}{x^2}\right| = y - x + C$$

$$\Rightarrow \frac{y}{x^2} = \pm e^{y-x+C} = Ae^{y-x},$$

so since $y=2$ when $x=3$, we have

$$\frac{2}{9} = Ae^{-1} \Rightarrow A = \frac{2e}{9}.$$

Thus $\boxed{y - \frac{2e}{9} \cdot x^2 e^{y-x} = 0}$ is the equation of the curve.

8. (16 points) A certain population of animals is affected by seasonal variations. The rate at which the population grows is proportional to both the current population size P and to $\cos^2(\frac{\pi}{6}t)$; i.e., it is proportional to their product. (Here t is the time measured in months.) Suppose the initial relative growth rate (i.e., $\frac{1}{P}P'$ when $t = 0$) is $\frac{1}{20}$ per month.

(a) Write a differential equation which models the growth of this population.

$P'(t)$ = the rate at which population grows, so

$P'(t) = k \cdot P \cdot \cos^2(\frac{\pi}{6}t)$ for some k (constant of proportionality).

Since $\frac{P'}{P} = \frac{1}{20}$ when $t=0$, we have $k\cos^2 0 = \frac{1}{20}$, so $k = \frac{1}{20}$.

Thus, the differential equation is $\boxed{P' = \frac{1}{20}P\cos^2(\frac{\pi}{6}t)}$.

(b) Suppose the initial population is 400. Use Euler's method with $h = 3$ to estimate the population after 9 months.

Step 0: $(t_0, P_0) = (0, 400)$. We need $\frac{9}{3} = 3$ steps to make the estimate.

Step 1: $t_1 = t_0 + h = 3$, and

$$P_1 = P_0 + h \cdot \frac{1}{20} P_0 \cos^2(\frac{\pi}{6}t_0) = 400 + 3 \cdot \frac{1}{20} \cdot 400 \cdot \cos^2 0$$
$$= 400 + 3 \cdot 20 = \underline{460}.$$

Step 2: $t_2 = t_1 + h = 6$, and

$$P_2 = P_1 + h \cdot \frac{1}{20} \cdot P_1 \cos^2(\frac{\pi}{6}t_1) = 460 + 3 \cdot \frac{1}{20} \cdot 460 \cdot \cos^2\frac{\pi}{2}$$
$$= 460 + 3 \cdot \frac{1}{20} \cdot 460 \cdot 0 = \underline{460}.$$

Step 3: $t_3 = t_2 + h = 9$, and

$$P_3 = P_2 + h \cdot \frac{1}{20} \cdot P_2 \cdot \cos^2(\frac{\pi}{6}t_2) = 460 + 3 \cdot \frac{1}{20} \cdot 460 \cdot \cos^2(\pi)$$
$$= 460 + 3 \cdot 23 \cdot 1 = 460 + 69 = \boxed{529}.$$

(c) Solve the differential equation (again using an initial population of 400) to find an exact expression for the population after t months.

Write $\frac{dP}{dt} = \frac{1}{20} P \cos^2 \frac{\pi}{6} t$; this is a separable equation.

We get $\int \frac{dP}{P} = \int \frac{1}{20} \cos^2\left(\frac{\pi}{6} t\right) dt$, so

$$\ln|P| = \frac{1}{20} \int \cos^2\left(\frac{\pi}{6}t\right) dt = \frac{1}{20} \int \frac{1}{2}\left(1 + \cos \frac{\pi}{3} t\right) dt$$

$$= \frac{1}{40}\left(t + \frac{3}{\pi} \sin \frac{\pi}{3} t \right) + C ,$$

and thus $|P| = e^{\frac{1}{40}\left(t + \frac{3}{\pi}\sin\frac{\pi}{3}t\right) + C} = e^{C} \cdot e^{\frac{1}{40}\left(t + \frac{3}{\pi}\sin\frac{\pi}{3}t\right)}$,

i.e. $P = \pm e^{C} \cdot e^{\frac{1}{40}\left(t + \frac{3}{\pi}\sin\frac{\pi}{3}t\right)} = A e^{\frac{1}{40}\left(t + \frac{3}{\pi}\sin\frac{\pi}{3}t\right)}$. $(A = \pm e^{C})$

With $P(0) = 400$, we obtain $400 = A e^{\frac{1}{40}\left(0 + \frac{3}{\pi}\sin 0\right)} = Ae^{0} = A$,

so the population function is $\boxed{P(t) = 400 e^{\frac{1}{40}\left(t + \frac{3}{\pi}\sin\frac{\pi}{3}t\right)}}$.

(d) Use your answer to part (c) to find the exact value for the population after 9 months.

The exact value is $P(9) = 400 e^{\frac{1}{40}\left(9 + \frac{3}{\pi}\sin 3\pi\right)}$

$$= 400 e^{\frac{1}{40}(9+0)} = \boxed{400 e^{9/40}}.$$

9. (13 points) Two species, A and B, live in a closed ecosystem where they are allowed to interact. Their populations as functions of time, $A(t)$ and $B(t)$ (in numbers of beings; here t is in months), are modeled by the equations

$$\frac{dA}{dt} = -\frac{A}{2} + \frac{AB}{6000}$$
$$\frac{dB}{dt} = 4B - \frac{AB}{50}$$

(a) Describe the nature of the relationship between the two species: is it one of competition, cooperation, or predator and prey, and how can you tell? (If the relationship is predator and prey, don't forget to explain how to tell which species is which.)

In the absence of B (i.e. $B=0$), then $\frac{dA}{dt} = -\frac{A}{2}$, so pop A can't survive.
In the absence of A (i.e. $A=0$), then $\frac{dB}{dt} = 4B$, so pop B grows exponentially.
The "$+\frac{AB}{6000}$" term in $\frac{dA}{dt}$ indicates that the growth rate of A is positively affected by interactions with species B; meanwhile, the "$-\frac{AB}{50}$" term in $\frac{dB}{dt}$ indicates that B's growth rate is negatively affected by increased interactions with A.
Thus, this is a relationship of predator and prey, with B as the prey and A as pred.

(b) Explain in words the meaning of the concept "equilibrium solution" for this system, and then compute all these equilibrium solutions.

"Equilibrium solution" means a pair (A,B) of two population sizes that obey the equations but do not change in size over time; that is, sizes of the two populations that can sustain themselves by balancing the natural growth or decay of their sizes against the effects of predation/preying.

To find equilibria, we factor the expressions for growth rate: $\begin{cases} \frac{dA}{dt} = \frac{A}{6000}(B-3000), \\ \frac{dB}{dt} = \frac{B}{50}(200-A). \end{cases}$

If A is constant, then $\frac{dA}{dt}=0$, so $A=0$ or $B=3000$.
But if $A=0$ and B is constant, then $0 = \frac{dB}{dt} = \frac{B}{50}(200-0)$, so $B=0$.
If $B=3000$ and B is constant, then $0 = \frac{dB}{dt} = \frac{3000}{50}(200-A)$, so $A=200$.
Thus the equilibria are $\boxed{(A,B)=(0,0)}$ and $\boxed{(A,B)=(200,3000)}$.

Quick reference:

$$\frac{dA}{dt} = -\frac{A}{2} + \frac{AB}{6000} = \frac{A}{6000}\left(B-3000\right)$$
$$\frac{dB}{dt} = 4B - \frac{AB}{50} = \frac{B}{50}\left(200-A\right)$$

(c) Suppose that at time $t = 0$ months, we have $A(0) = 210$ and $B(0) = 4000$ beings. *Use the differential equations* to predict the two populations in one month's time (i.e., at $t = 1$); be as mathematically precise as possible, and show all reasoning.

At $t=0$, the equations dictate that $\frac{dA}{dt} = \frac{A}{6000}(B-3000) = \frac{210}{6000} \cdot 1000 = \frac{210}{6} = 35$,

and $\frac{dB}{dt} = \frac{B}{50}(200-A) = \frac{4000}{50} \cdot (-10) = -800$,

i.e. species A is <u>growing</u> at a rate of 35 beings/month, and B is <u>shrinking</u> at 800 beings/mo.

We can estimate (by a "linear" approximation, or by an Euler-type method) that in one month, the size of population A will be $210+35 = \boxed{245 \text{ beings}}$, and the size of population B will be $4000-800 = \boxed{3200 \text{ beings}}$.

(Other more accurate guesses are possible, e.g. by making guesses in smaller intervals of time, or by estimating how $\frac{dA}{dt}$ and $\frac{dB}{dt}$ themselves change, but the above answer is sufficient here.)

(d) Will the first month's trend (that you identified in (c)) continue indefinitely? Explain fully how you are able to tell.

<u>No</u>, we will not continue forever seeing B drop and A rise.

To see this, note that as soon as B drops below 3000 in size (which might occur as early as the second month, according to part (c)'s estimate), the sign of $\frac{dA}{dt}$ will be negative, as it depends on the factor $B-3000$.

This will cause species A's population to start dropping in size.

(Other behaviors, such as the eventual re-growth of population B (as soon as A has fallen to below 200), and even potential periodic patterns in the population, could even be predicted with further investigation.)

10. (12 points) Determine whether each of the series below converges or diverges. Indicate clearly which tests you use and how you apply them.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$

Ratio test: $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n!}\right|$

$$= \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{e^{n^2}}{e^{n^2+2n+1}} = \lim_{n\to\infty} \frac{n+1}{e^{2n+1}} \quad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$\left(\text{by L'Hopital}\right) = \lim_{n\to\infty} \frac{1}{2e^{2n+1}} = 0 < 1,$$

so the series converges.

(b) $\sum_{n=1}^{\infty}\left(30-\frac{1}{n^2}\right)$

Note $\lim_{n\to\infty}\left(30-\frac{1}{n^2}\right) = 30-0 = 30 \neq 0$,

so by the <u>Test for Divergence</u>, the series <u>diverges</u>.

11. (6 points) Suppose that the power series

$$\sum_{n=0}^{\infty} a_n(x+2)^n$$

converges if $x = -7$ and diverges if $x = 7$.

Decide which of the following series must converge, must diverge, or may either converge or diverge (inconclusive). Circle your answer. You do not need to justify your answers.

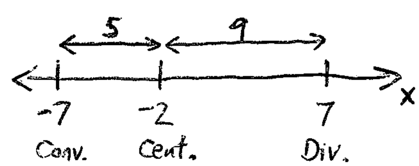
(a) If $x = -8$, the power series	Converges	Diverges	Inconclusive
(b) If $x = 1$, the power series	Converges	Diverges	Inconclusive
(c) If $x = 3$, the power series	Converges	Diverges	Inconclusive
(d) If $x = -11$, the power series	Converges	Diverges	Inconclusive
(e) If $x = 5$, the power series	Converges	Diverges	Inconclusive
(f) If $x = -5$, the power series	Converges	Diverges	Inconclusive

The center $a = -2$. Let R be the (unknown) radius of convergence.

Since $x = -7$ leads to convergence, we know $R \geq |-7-(-2)| = 5$.

Since $x = 7$ leads to divergence, we know $R \leq |7-(-2)| = 9$.

Or, graph the situation on the x-axis:



Thus, any x with $|x-(-2)| > 9$ gives divergence;

any x with $|x-(-2)| < 5$ gives convergence; but since we know nothing more about R, if $|x-(-2)|$ lies between 5 & 9 inclusive, we can't conclude anything.

12. (13 points)

(a) Find, showing all your steps, the degree-4 Taylor polynomial $T_4(x)$ with center 0 for the function

$$f(x) = \sin x + \cos x.$$

n	$f^{(n)}(x)$	$f^{(n)}(0)$	c_n
0	$\sin x + \cos x$	1	$\frac{1}{0!} = 1$
1	$\cos x - \sin x$	1	$\frac{1}{1!} = 1$
2	$-\sin x - \cos x$	-1	$\frac{-1}{2!} = -\frac{1}{2}$
3	$-\cos x + \sin x$	-1	$\frac{-1}{3!} = -\frac{1}{6}$
4	$\sin x + \cos x$	1	$\frac{1}{4!} = \frac{1}{24}$

Thus $T_4(x) = \sum_{n=0}^{4} c_n x^n = \boxed{1 + x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4}$.

(b) Use T_4 to obtain an estimate for $\sin(\frac{1}{10}) + \cos(\frac{1}{10})$. (You do not need to simplify your answer.)

We use $T_4\left(\frac{1}{10}\right)$ to estimate $f\left(\frac{1}{10}\right) = \sin\frac{1}{10} + \cos\frac{1}{10}$; thus,

$$f\left(\tfrac{1}{10}\right) \approx T_4\left(\tfrac{1}{10}\right) = \boxed{1 + \frac{1}{10} - \frac{1}{200} - \frac{1}{6000} + \frac{1}{240000}}.$$

(c) Compute the 5th derivative $f^{(5)}(x)$ of $f(x)$ and explain why

$$|f^{(5)}| \leq 2 \quad \text{for all } x.$$

From part (a), $f^{(4)}(x) = \sin x + \cos x \quad (= f(x)\ !)$,

and so $f^{(5)}(x) = \cos x - \sin x$.

We note that $|f^{(5)}(x)| = |\cos x - \sin x| \leq |\cos x| + |-\sin x| = |\cos x| + |\sin x|$,

and since $|\cos x|$ and $|\sin x|$ are both always at most 1, we have

$$|f^{(5)}| \leq |\cos x| + |\sin x| \leq 1 + 1 = 2.$$

(d) Use the fact from part (c) (even if you were unable to verify it) to draw a conclusion (in sentence form) about the accuracy of your estimate from part (b); be as mathematically precise as you can, and cite all of your reasoning.

By Taylor's inequality, for any x in the interval $[-\frac{1}{10}, \frac{1}{10}]$, we have that

$$|R_4(x)| = |f(x) - T_4(x)| \leq \frac{M}{5!}|x|^5,$$ where M is such that $|f^{(5)}| \leq M$ on $[-\frac{1}{10}, \frac{1}{10}]$.

By (c), since in fact $|f^{(5)}| \leq 2$ for all x, we may take $M = 2$, so

that $$|\text{error}| = |f(x) - T_4(x)| \leq \frac{2}{5!}|x|^5 \quad \text{for } x \text{ in } [-\tfrac{1}{10}, \tfrac{1}{10}].$$

In particular, for $x = \frac{1}{10}$, we have that

$$|\text{error}| = \left|f\left(\tfrac{1}{10}\right) - T_4\left(\tfrac{1}{10}\right)\right| \leq \frac{2}{5!} \cdot \frac{1}{10^5};$$

this means that the estimate using $T_4(\frac{1}{10})$ differs from the true value of $\sin(\frac{1}{10}) + \cos(\frac{1}{10})$

by no more than $\frac{2}{10^5 \cdot 5!}$ units.

(And $\frac{2}{10^5 \cdot 5!} < \frac{2}{10^7}$, so this is excellent.)