Math 42: Calculus
Final Exam — March 19, 2007

Name : ____

Section Leader (Circle one) : Buyukboduk Chang Lee Segerman Zhang

Section Time (Circle one): 11:00 1:15

- This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted.
- In order to receive full credit, please show all of your work and justify your answer. You do not need to simplify your answers unless specifically instructed to do so.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- Please sign the following:

> "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

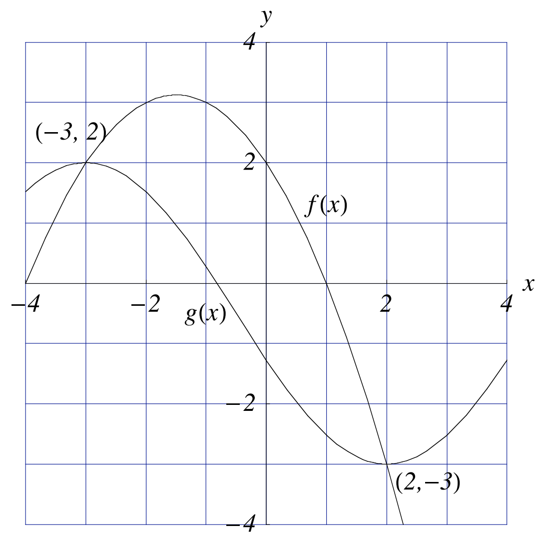
Signature: ____________________________

The following boxes are strictly for grading purposes. Please do not mark.

1	9		**7**	12	
2	10		**8**	16	
3	12		**9**	13	
4	15		**10**	12	
5	21		**11**	6	
6	11		**12**	13	
			Total	150	

GOOD LUCK ON THE REST OF YOUR EXAMS, AND HAVE A GREAT SPRING BREAK!

1. (9 points) Let R be the region bounded by the curves $y = f(x)$ and $y = g(x)$ shown in the graph below.



(a) Set up a definite integral that will give the area of the region R.

(b) Set up a definite integral that will give the volume of the solid generated when the region R is revolved about the line $y = 4$.

(c) If the base of a solid V is the region R, and if all cross-sections of V that are perpendicular to the x-axis are squares, set up a definite integral that will give the volume of V.

2. (10 points) According to Newton's Law of Gravity, an object near the earth at a distance of r meters from the center of the earth feels a gravitational force of $F = k/r^2$ Newtons, where k is a positive constant that depends on the mass of the object.

   (a) In a revision of an earlier doomsday prediction, scientists are now forecasting that in the year 2029, asteroid 2004-MN4 will have moved into an orbiting position about the earth, at a distance of 5×10^7 meters from the center of the earth. In terms of the above constant k, compute the work against gravity which is required to push the asteroid from this position to a more desirable distance of 6×10^7 meters.

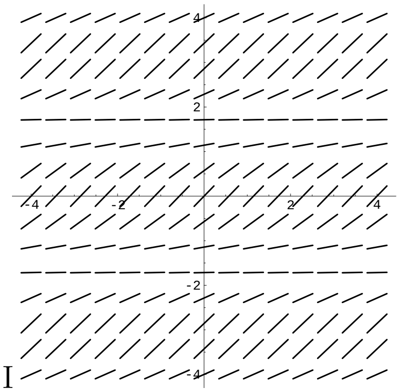
   (b) A Hollywood producer suggests that a further effort be made, to send the asteroid from the position of 6×10^7 meters, completely away from the earth. By using a suitable improper integral, compute (again in terms of k) the amount of work against the pull of gravity which is required to send the asteroid "off to infinity."

3. (12 points) Compute the following, showing all work.

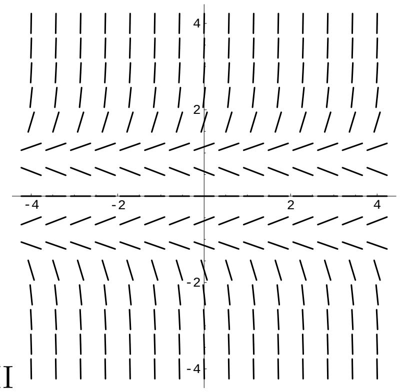
(a) $\displaystyle\int_{-1}^{1} \frac{1}{x^{2/3}}\, dx$

(b) $\displaystyle\int_{0}^{\sqrt{2}} \frac{x}{\sqrt{9 - x^4}}\, dx$

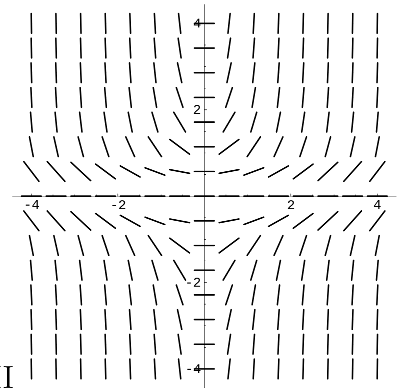
4. (15 points) Match the direction fields below with their differential equations. (The horizontal variable is t; the vertical is y.) Also indicate which two equations do not have matches.



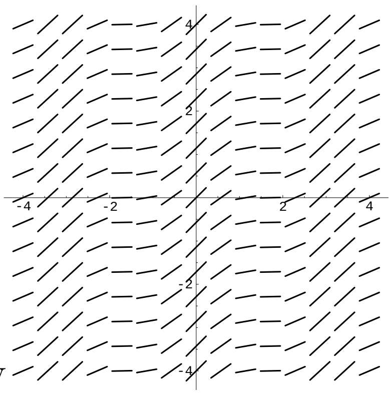
I



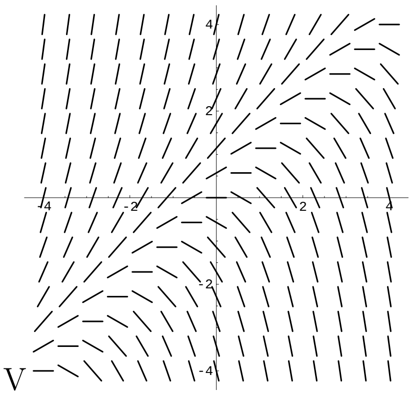
II



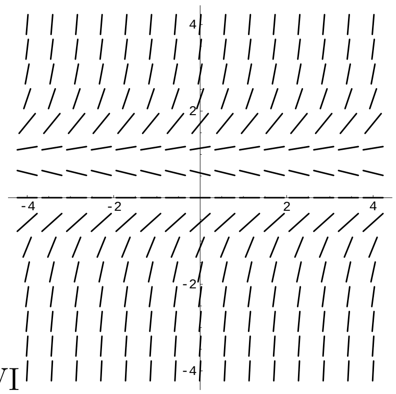
III



IV



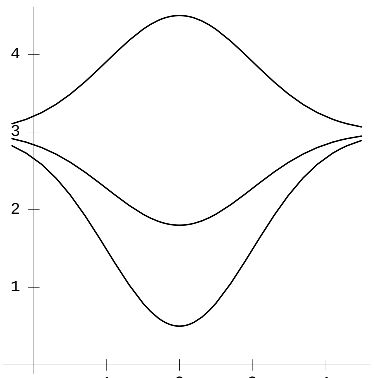
V



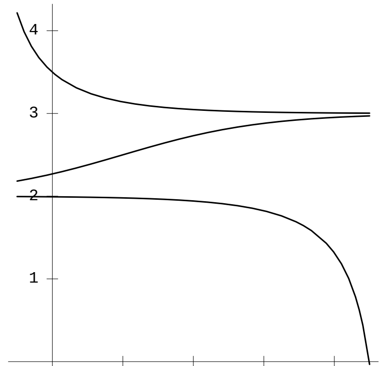
VI

Equation	I, II, III, IV, V VI, or "none"	Equation	I, II, III, IV, V VI, or "none"
$dy/dt = ty^2$		$dy/dt = \cos^2 y$	
$dy/dt = y - t$		$dy/dt = y(y^2 - 1)$	
$dy/dt = ty$		$dy/dt = \cos^2 t$	
$dy/dt = 1 - y$		$dy/dt = y(y - 1)$	

5. (21 points) Each picture below depicts a few possible solution curves to a differential equation chosen from the list at the bottom of the page. (As usual, the t-axis is horizontal, and the y-axis is vertical.) Match each equation to its sketch of solutions; one equation will not have a match.



I



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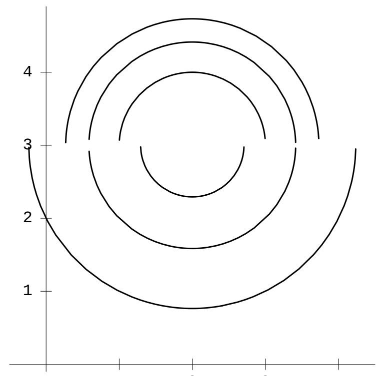
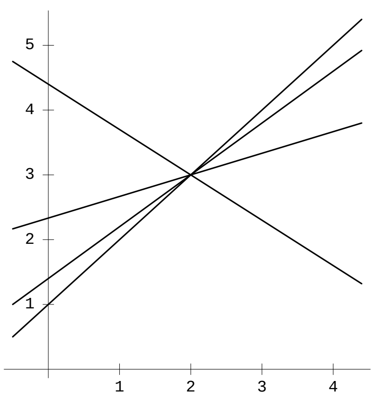
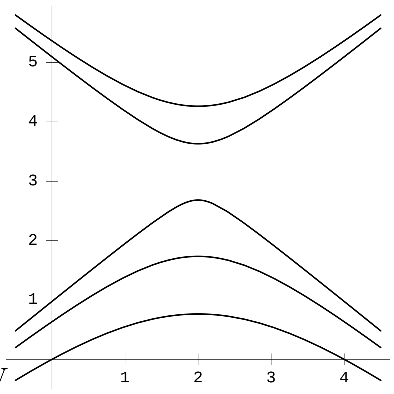


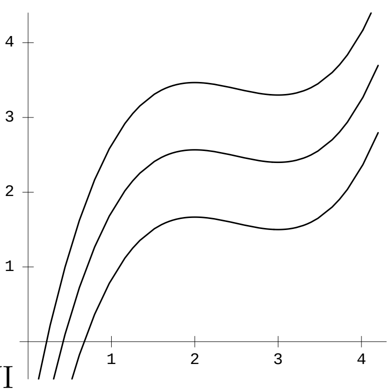
III



IV



V



VI

Equation	I, II, III, IV, V, VI, or "none"	Brief reason
$y' = (t-3)(t-2)$		
$y' = -(y-3)(y-2)$		
$y' = -(y-3)(t-2)$		
$y' = \dfrac{y-3}{t-2}$		
$y' = -\dfrac{y-3}{t-2}$		
$y' = \dfrac{t-2}{y-3}$		
$y' = -\dfrac{t-2}{y-3}$		

6. (11 points) A new 15-gallon juice dispenser in Branner Hall's dining room is initially filled with a fruit juice that is 80% orange juice and 20% pineapple juice. Every hour, 10 gallons of juice are consumed. The dispenser is also continuously replenished at this same rate, but due to a supply shortage at Branner, the refilling is being made using an orange-pineapple mixture from neighboring Wilbur Hall that is 40% orange and 60% pineapple. Assume that the dispensed juice is always well-mixed.

   (a) Write down a differential equation for $P(t)$, the amount of pineapple juice in Branner's dispenser after t hours. Be sure to state your initial condition, including the units involved.

   (b) By solving the differential equation, find the amount of pineapple juice in the container after 2 hours.

7. (12 points) Solve the following initial value problems. You may use any method or result you like, as long as it is fully justified or cited.

(a) $\dfrac{dz}{dt} = z^2 - 2z - 15, \quad z(0) = -1$

(b) $\dfrac{dy}{dx} = \dfrac{2y - xy}{x - xy}$, $\quad y(3) = 2$. (Do not attempt to solve for y as a function of x; leave your answer as an implicit curve in x and y.)

8. (16 points) A certain population of animals is affected by seasonal variations. The rate at which the population grows is proportional to both the current population size P and to $\cos^2(\frac{\pi}{6}t)$; i.e., it is proportional to their product. (Here t is the time measured in months.)

Suppose the initial relative growth rate (i.e., $\frac{1}{P}P'$ when $t = 0$) is $\frac{1}{20}$ per month.

(a) Write a differential equation which models the growth of this population.

(b) Suppose the initial population is 400. Use Euler's method with $h = 3$ to estimate the population after 9 months.

(c) Solve the differential equation (again using an initial population of 400) to find an exact expression for the population after t months.

(d) Use your answer to part (c) to find the exact value for the population after 9 months.

9. (13 points) Two species, A and B, live in a closed ecosystem where they are allowed to interact. Their populations as functions of time, $A(t)$ and $B(t)$ (in numbers of beings; here t is in months), are modeled by the equations

$$\frac{dA}{dt} = -\frac{A}{2} + \frac{AB}{6000}$$
$$\frac{dB}{dt} = 4B - \frac{AB}{50}$$

(a) Describe the nature of the relationship between the two species: is it one of competition, cooperation, or predator and prey, and how can you tell? (If the relationship is predator and prey, don't forget to explain how to tell which species is which.)

(b) Explain in words the meaning of the concept "equilibrium solution" for this system, and then compute all these equilibrium solutions.

Quick reference:

$$\frac{dA}{dt} = -\frac{A}{2} + \frac{AB}{6000}$$
$$\frac{dB}{dt} = 4B - \frac{AB}{50}$$

(c) Suppose that at time $t = 0$ months, we have $A(0) = 210$ and $B(0) = 4000$ beings. *Use the differential equations* to predict the two populations in one month's time (i.e., at $t = 1$); be as mathematically precise as possible, and show all reasoning.

(d) Will the first month's trend (that you identified in (c)) continue indefinitely? Explain fully how you are able to tell.

10. (12 points) Determine whether each of the series below converges or diverges. Indicate clearly which tests you use and how you apply them.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \left(30 - \frac{1}{n^2}\right)$

11. (6 points) Suppose that the power series

$$\sum_{n=0}^{\infty} a_n(x+2)^n$$

converges if $x = -7$ and diverges if $x = 7$.

Decide which of the following series must converge, must diverge, or may either converge or diverge (inconclusive). Circle your answer. You do not need to justify your answers.

(a) If $x = -8$, the power series Converges Diverges Inconclusive

(b) If $x = 1$, the power series Converges Diverges Inconclusive

(c) If $x = 3$, the power series Converges Diverges Inconclusive

(d) If $x = -11$, the power series Converges Diverges Inconclusive

(e) If $x = 5$, the power series Converges Diverges Inconclusive

(f) If $x = -5$, the power series Converges Diverges Inconclusive

12. (13 points)

(a) Find, showing all your steps, the degree-4 Taylor polynomial $T_4(x)$ with center 0 for the function

$$f(x) = \sin x + \cos x.$$

(b) Use T_4 to obtain an estimate for $\sin(\frac{1}{10}) + \cos(\frac{1}{10})$. (You do not need to simplify your answer.)

(c) Compute the 5th derivative $f^{(5)}(x)$ of $f(x)$ and explain why

$$|f^{(5)}| \leq 2 \quad \text{for all } x.$$

(d) Use the fact from part (c) (even if you were unable to verify it) to draw a conclusion (in sentence form) about the accuracy of your estimate from part (b); be as mathematically precise as you can, and cite all of your reasoning.