Math 42 – Winter 2007 – Exam 2 Solutions

1. (10 points) At the Stanford Federal Credit Union, the waiting time for teller service is a randomly varying quantity that can be modeled by the following probability density function:

$$f(x) = \begin{cases} \frac{1}{3}e^{-x/3} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases}$$

where x is measured in minutes. It is a fact that the mean waiting time is 3 minutes, according to this PDF. (You do not have to prove this fact!)

(a) Calculate the probability that an experience waiting in line takes longer than average; that is, takes longer than 3 minutes.

$$\text{Prob}(X \geq 3) = \int_3^\infty f(x)\,dx = \int_3^\infty \frac{1}{3}e^{-x/3}\,dx = \lim_{N\to\infty} \int_3^N \frac{1}{3}e^{-x/3}\,dx = \lim_{N\to\infty}\left[-e^{-x/3}\right]_{x=3}^{x=N} =$$

$$= \lim_{N\to\infty}\left(e^{-1} - e^{-N/3}\right) = \boxed{\frac{1}{e}}.$$

Note: can also be computed as

$$\text{Prob}(X \geq 3) = 1 - \text{Prob}(X < 3) = 1 - \int_{-\infty}^3 f(x)\,dx = 1 - \int_0^3 \frac{1}{3}e^{-x/3} =$$

$$= 1 - \left(\left[-e^{-x/3}\right]_{x=0}^{x=3}\right) = 1 - (1 - e^{-1}) = \boxed{\frac{1}{e}}.$$ (This avoids improper integral!)

(b) Determine the median waiting time, showing all reasoning.

Median m is such that $\int_{-\infty}^m f = \frac{1}{2} = \int_m^\infty f$.

Thus, $\frac{1}{2} = \int_{-\infty}^m f(x)\,dx = \int_0^m \frac{1}{3}e^{-x/3}\,dx = -e^{-x/3}\Big]_{x=0}^{x=m} = 1 - e^{-m/3}$,

so $e^{-m/3} = \frac{1}{2}$, meaning $\frac{-m}{3} = \ln\frac{1}{2} = -\ln 2$, so that $\boxed{m = 3\ln 2}$.

(Could also solve $\frac{1}{2} = \int_m^\infty \frac{1}{3}e^{-x/3}\,dx$ for m, but this requires using improper integral....)

2. (10 points)

(a) Find $\int_0^\infty \frac{1}{1+x^2}\,dx$. $= \lim_{N\to\infty} \int_0^N \frac{1}{1+x^2}dx = \lim_{N\to\infty} \Big[\arctan x\Big]_{x=0}^{x=N}$

$$= \lim_{N\to\infty} (\arctan N - \arctan 0)$$

$$= \frac{\pi}{2} - 0 = \boxed{\frac{\pi}{2}}.$$

(b) Determine whether $\int_1^\infty \frac{2}{1+x^5}\,dx$ converges. Explain your reasoning completely.

For all $x \geq 1$, we have

$x^5 \geq x^2 \geq 0$, which implies $1+x^5 \geq 1+x^2 \geq 0$, and so

$$0 \leq \frac{2}{1+x^5} \leq \frac{2}{1+x^2}.$$

Now the integral $\int_1^\infty \frac{2}{1+x^2}dx$ is convergent, since we can compute it as in part (a):

$$\int_1^\infty \frac{2}{1+x^2}dx = 2\cdot \lim_{N\to\infty}\Big[\arctan x\Big]_{x=1}^{x=N} = 2\cdot\left(\frac{\pi}{2} - \arctan 1\right).$$

$$\left(= 2\cdot\left(\frac{\pi}{2}-\frac{\pi}{4}\right).\right)$$

Thus, by the Comparison Theorem for improper integrals,

$\int_1^\infty \frac{1}{1+x^5}dx$ is also convergent. (Note: could also "compare" using $\frac{1}{x^5}$ for $x \geq 1$.)

3. (6 points) A dosage d of a drug is given daily at $t = 0, 1, 2, 3, \ldots$ days. The drug decays exponentially at a rate k in the bloodstream. Thus, the amount in the bloodstream after $n+1$ doses is $d + de^{-k} + de^{-2k} + \cdots + de^{-nk}$.

(a) Find the level of the drug after "infinitely many" doses. That is, find

$$d + de^{-k} + de^{-2k} + \cdots + de^{-nk} + \cdots$$

This is a geometric series, with initial term d and common ratio $\frac{e^{-(n+1)k}}{e^{-nk}} = \frac{e^{-nk-k}}{e^{-nk}} = e^{-k}$,

so its infinite sum is $\boxed{\frac{d}{1-e^{-k}}}$ (assuming $e^{-k} < 1$, meaning k is positive, which is a likely scenario!).

(b) If $k = 0.1$, what dosage is needed to maintain a drug level of 2?

If we require $\frac{d}{1-e^{-k}} = 2$ for $k = -0.1$, then

$$d = 2(1-e^{-k}) = \boxed{2(1-e^{-0.1})}.$$

4. (24 points) Determine whether each of the series below converges or diverges. Indicate clearly which tests you use and how you apply them.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^4 + n + 1}$

Solution 1: Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ is a p-series with $p = 4 > 1$, it is convergent;

it follows that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^4} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n^4}$ is also convergent.

Next, since $n > 0$, we have that $3n^4 + n + 1 > 3n^4 > 0$, so that

$$0 \leq \frac{1}{3n^4+n+1} < \frac{1}{3n^4} \text{ for all such } n.$$

Thus, by the Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^4+n+1}$ is also convergent.

Solution 2: As before, we note $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges; now we compute

$$\lim_{n\to\infty} \frac{\left(\frac{1}{3n^4+n+1}\right)}{\left(\frac{1}{n^4}\right)} = \lim_{n\to\infty} \frac{n^4}{3n^4+n+1} = \lim_{n\to\infty} \frac{1}{3 + 1/n^3 + 1/n^4} = \frac{1}{3}.$$

Since the limit is nonzero and finite, the conditions of the Limit Comparison Test are satisfied for $a_n = \frac{1}{3n^4+n+1}$ and $b_n = \frac{1}{n^4}$.

Thus, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^4+n+1}$ also converges.

(b) $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{n^3}$ (Intuition: We'd like to guess convergent, since sin is bounded in size and the denominator grows to ∞ "quickly enough"; we need to argue by comparison.)

We first note the important fact that the terms of $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{n^3}$ are all positive: this is because for $n \geq 1$, we have $0 < \frac{1}{n} \leq 1$, and the sine function is positive on the interval $(0,1)$ (since it's positive on $(0,\pi)$ in fact).

This observation allows us to apply the Comparison Test with $a_n = \frac{\sin(1/n)}{n^3}$.

Since $\sin(\frac{1}{n}) \leq 1$ because sin is bounded, we have $0 \leq \frac{\sin(1/n)}{n^3} \leq \frac{1}{n^3}$, so let's take $b_n = \frac{1}{n^3}$.

Note the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent, being a p-series with $p = 3 > 1$.

Thus, the Comparison Test allows us to conclude that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\sin(1/n)}{n^3}$ is also convergent.

(Note: we could have dealt with a trickier series like $\sum_{n=1}^{\infty} \frac{\sin n}{n^3}$ in a similar way, even though these terms aren't all positive. The trick here would be to show first that $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^3}$ converges by the Comparison Test, and then use Absolute Convergence to conclude convergence of the series $\sum_{n=1}^{\infty} \frac{\sin n}{n^3}$.)

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{2^n}{n!\,\ln(n+1)}$

We apply the Ratio Test:

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!\,\ln(n+2)} \cdot \frac{n!\,\ln(n+1)}{2^n}$$

$$= \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \cdot \frac{n!}{(n+1)!} \cdot \frac{\ln(n+1)}{\ln(n+2)} = \lim_{n\to\infty} \frac{2}{n+1} \cdot \frac{\ln(n+1)}{\ln(n+2)}.$$

There's no easy "log rule" to deal with the quotient $\frac{\ln(n+1)}{\ln(n+2)}$; this is an $\frac{\infty}{\infty}$ indeterminate form, and we use L'Hôpital's Rule to deal with it. (But, see also below.)

Note that $\displaystyle\lim_{n\to\infty} \frac{\ln(n+1)}{\ln(n+2)} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n+2}} = \lim_{n\to\infty} \frac{n+2}{n+1} = \lim_{n\to\infty} \frac{1+2/n}{1+1/n} = 1$,

so that $\displaystyle L = \lim_{n\to\infty} \frac{2}{n+1} \cdot \frac{\ln(n+1)}{\ln(n+2)} = 2\cdot 1 \cdot \lim_{n\to\infty} \frac{1}{n+1} = 0.$

Since $L<1$, the series converges.

(Avoiding L'Hôpital: $\displaystyle\lim_{n\to\infty} \frac{\ln(n+1)}{\ln(n+2)} = \lim_{n\to\infty} \frac{\ln(n(1+1/n))}{\ln(n(1+2/n))} = \lim_{n\to\infty} \frac{\ln(1+1/n)+\ln n}{\ln(1+2/n)+\ln n}$

$$= \lim_{n\to\infty} \left(\frac{\frac{\ln(1+1/n)}{\ln n}+1}{\frac{\ln(1+2/n)}{\ln n}+1}\right) = \frac{0+1}{0+1} = 1,$$ since $\ln n \to \infty$ and $\begin{cases} 1+\frac{2}{n} \to 1 \\ 1+\frac{1}{n} \to 1. \end{cases}$)

(Avoiding Ratio Test with logs: note $0 \le \frac{2^n}{n!\ln(n+1)} \le \frac{2^n}{n!}$ for $n \ge 2$, since $\ln(n+1) \ge 1$.
Thus, could argue that the series converges by Comparison Test, and using Ratio Test on the much easier series $\sum 2^n/n!$.)

(d) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

Let's check to see if the conditions of the Alternating Series Test are satisfied. The series is $\sum_{n=2}^{\infty} (-1)^n b_n$, where $b_n = \frac{1}{n \ln n}$.

- Clearly $b_n > 0$ for all n, since $\ln n$ is positive for $n > 1$.
- Terms decreasing? (i.e., $b_{n+1} \leq b_n$?) Yes, because $n \ln n$ is an increasing & positive function of $n > 1$, so that $b_n = \frac{1}{n \ln n}$ is decreasing.
- Terms go to 0? Yes, $\lim_{n \to \infty} \frac{1}{n \ln n} = 0$ because $n \ln n \to \infty$.

Thus, the Alternating Series Test tells us that $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converges.

5. (6 points) Suppose we know that $\sum_{n=1}^{\infty} a_n$ converges to 0.8. We are given no other information about this series. For each of the following statements, circle

- **T** if the statement must be true,
- **F** if the statement must be false, and
- **X** if the statement could be either true or false.

You do not need to justify your answers.

T **F** **X** $\lim_{n\to\infty} a_n = 0.8.$ — (If True, we'd be able to use Test for Divergence to say the series diverges.)

T **F** **X** $\lim_{n\to\infty} a_n = 0.$ — (True for <u>any</u> convergent series.)

T **F** **X** $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1.$ — (If True, Ratio Test would say the series diverges.)

T **F** **X** $a_{n+1} < a_n$ for all n. — (Not necessarily true, despite the fact that the terms are approaching 0 in size.)

T **F** **X** $\lim_{n\to\infty} \frac{1}{|a_n|} = \infty.$ — (Must be true when the a_n's are approaching 0.)

T **F** **X** $\lim_{n\to\infty} s_n = 0.8$, where $s_n = a_1 + a_2 + \cdots + a_n$.

(Definition of a convergent series!)

6. (13 points) Find, with complete justification, the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x+5)^n}{3^n\sqrt{n}}$$

Ratio Test: $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x+5)^{n+1}}{3^{n+1}\sqrt{n+1}} \cdot \frac{3^n\sqrt{n}}{(x+5)^n}\right|$

$= \lim_{n\to\infty} \left|\frac{(x+5)^{n+1}}{(x+5)^n}\right| \cdot \frac{3^n}{3^{n+1}} \cdot \frac{\sqrt{n}}{\sqrt{n+1}} = \frac{|x+5|}{3} \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}}.$

Now $\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = \lim_{n\to\infty} \sqrt{\frac{1}{1+1/n}} = \sqrt{\frac{1}{1+0}} = 1$, so

$L = \frac{|x+5|}{3}$. Since $L<1$ gives convergence, we have convergence for

$|x+5|<3$, i.e. for $-3<x+5<3$, i.e. for $\underline{-8<x<-2}$.

We must also check for convergence at endpoints of this interval.

Case $\underline{x=-2}$: $\sum_{n=1}^{\infty} \frac{(-2+5)^n}{3^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{3^n}{3^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which is a p-series

with $p=\frac{1}{2}\leq 1$, so it $\underline{\text{diverges}}$.

Case $\underline{x=-8}$: $\sum_{n=1}^{\infty} \frac{(-8+5)^n}{3^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-3)^n}{3^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. If $b_n = \frac{1}{\sqrt{n}}$, are the

conditions of the Alternating Series Test satisfied? (Clearly $b_n>0$ for $n\geq 1$.)

- Decreasing terms, i.e. $b_{n+1}\leq b_n$? $\underline{\text{Yes}}$, because $\sqrt{n}$ is a positive, increasing function of $n\geq 1$, so $\frac{1}{\sqrt{n}}$ is decreasing.
- Terms go to 0? $\underline{\text{Yes}}$, $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ (since $\sqrt{n}\to\infty$).

Thus we get a $\underline{\text{convergent}}$ series.

Final answer: interval of convergence is $\boxed{[-8,-2)}$, i.e. for $\boxed{-8\leq x<-2}$.

7. (8 points) Consider the following power series:

(I) $x^2 - \dfrac{x^6}{3!} + \dfrac{x^{10}}{5!} - \cdots$

(II) $1 - \dfrac{x^4}{2!} + \dfrac{x^8}{4!} - \cdots$

(III) $\dfrac{x^2}{2!} - \dfrac{x^4}{4!} + \dfrac{x^6}{6!} - \cdots$

(IV) $x - x^3 + x^5 - \cdots$

(V) $x^2 - \dfrac{x^4}{2!} + \dfrac{x^6}{4!} - \cdots$

Fill in the letter of the series that corresponds to the given function. You do not need to justify your answer. (One of the above series does not have a match.)

Function	I, II, III, IV, or V
$1 - \cos x$	III
$x^2 \cos x$	V
$\sin(x^2)$	I
$\dfrac{x}{1+x^2}$	IV

(The unmatched one, II, is the power series for $\cos(x^2)$ at 0.)

8. (9 points) Let f be a function all of whose derivatives (first, second, etc.) are defined everywhere. Suppose the degree-3 Taylor polynomial for f about -2 is given by

$$T_3(x) = 2 - \frac{3}{8}(x+2)^2 - \frac{1}{12}(x+2)^3.$$

(a) Find $f'(-2)$, $f''(-2)$, and $f'''(-2)$.

Since $c_n = \dfrac{f^{(n)}(-2)}{n!}$, we have $f^{(n)}(-2) = n!\,c_n$;

now using $c_0 = 2$, $c_1 = 0$, $c_2 = -\frac{3}{8}$, and $c_3 = -\frac{1}{12}$, we have

$$f'(-2) = \boxed{0}, \quad f''(-2) = 2!\cdot c_2 = \boxed{-\frac{3}{4}}, \quad \text{and} \quad f'''(-2) = 3!\cdot \frac{-1}{12} = \boxed{-\frac{1}{2}}.$$

(b) Use T_3 to find an approximation for $f(0)$.

$$f(0) \approx T_3(0) = 2 - \frac{3}{8}(0+2)^2 - \frac{1}{12}(0+2)^3 = 2 - \frac{3}{2} - \frac{2}{3} = \boxed{-\frac{1}{6}}.$$

(c) The fourth derivative of f satisfies the inequality $|f^{(4)}(x)| \leq \frac{1}{4}$ over the interval $[-4, 0]$. Use this information to make a statement about the accuracy of the approximation for $f(0)$ that you found in part (b).

By Taylor's Inequality with $M = \frac{1}{4}$ (as given) and $a = -2$, $n = 3$, we have

$$\left| f(x) - T_3(x) \right| \leq \frac{M}{(n+1)!}\cdot |x+2|^{n+1} = \frac{1/4}{4!}|x+2|^4 \text{ for } x \text{ in } [-4,0].$$

Thus, at $x=0$, $|\text{error}| = |f(0) - T_3(0)| \leq \frac{1}{4\cdot 4!}\cdot 2^4 = \frac{16}{4\cdot 4\cdot 3\cdot 2} = \frac{1}{6}$.

This means the approximation of part (b) is accurate to within $\frac{1}{6}$ units.

(Thereby making the approximation somewhat lousy: $f(0)$ could be anything between $-\frac{2}{6}$ and 0!)

9. (14 points)

(a) Compute a power series expansion for e^x, centered at 0. (Show all of your steps.)

Using Taylor's recipe with $f(x)=e^x$, $a=0$:

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$c_n = \frac{f^{(n)}(0)}{n!}$
0	e^x	$e^0=1$	$c_0 = \frac{1}{0!} = 1$
1	e^x	$e^0=1$	$c_1 = \frac{1}{1!} = 1$
2	e^x	$e^0=1$	$c_2 = \frac{1}{2!} = \frac{1}{2!}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
[any n]	e^x	$e^0=1$	$\boxed{c_n = \frac{1}{n!}}$

Thus, the power series is $\sum_{n=0}^{\infty} c_n x^n = \boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!}}$.

(b) Find a power series expansion for $\int e^{-x^3}\,dx$ and determine its radius of convergence.

We have $$\int e^{-x^3}dx = \int\left[\sum_{n=0}^{\infty}\frac{(-x^3)^n}{n!}\right]dx = \int\left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n}}{n!}\right]dx$$

$$= \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n+1}}{(3n+1)\cdot n!} + C}.$$

(Term-by-term, this can be checked: since $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$, we have $e^{-x^3} = 1-x^3+\frac{x^6}{2!}-\frac{x^9}{3!}+\cdots$, and so $\int e^{-x^3}dx = C+x-\frac{x^4}{4}+\frac{x^7}{7\cdot 2!}-\frac{x^{10}}{10\cdot 3!}+\cdots$)

We use the ratio test on this series:

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{3n+4}}{x^{3n+1}}\right|\cdot\frac{3n+1}{3n+4}\cdot\frac{n!}{(n+1)!} = |x^3|\cdot\lim_{n\to\infty}\frac{3n+1}{(n+1)(3n+4)}$$

$$= |x^3|\cdot\lim_{n\to\infty}\frac{3/n+1/n^2}{3+7/n+4/n^2} = |x^3|\cdot\frac{0}{3+0} = 0.$$ Since $L<1$ regardless of x, the series converges for all x, and so the radius of convergence is $\boxed{\infty}$.

(c) Use your answer in part (b) to find a series for $\int_0^{1/2} e^{-x^3}\,dx$.

We have $\displaystyle\int_0^{1/2} e^{-x^3}dx = \left[C + x - \frac{x^4}{4} + \frac{x^7}{7\cdot 2!} - \frac{x^{10}}{10\cdot 3!} + \frac{x^{13}}{13\cdot 4!} - \cdots\right]_{x=0}^{x=1/2} = \left[C + \sum_{n=0}^{\infty}\frac{(-1)^n x^{3n+1}}{(3n+1)\,n!}\right]_{x=0}^{x=1/2}$

$$= \left(C + \frac{1}{2} - \frac{(1/2)^4}{4} + \frac{(1/2)^7}{7\cdot 2!} - \frac{(1/2)^{10}}{10\cdot 3!} + \frac{(1/2)^{13}}{13\cdot 4!} - \cdots\right) - \left(C + 0 + 0 + \cdots\right)$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n (1/2)^{3n+1}}{(3n+1)\,n!} = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{3n+1}\cdot(3n+1)\cdot n!}}\,.$$

(d) If you approximate the definite integral $\int_0^{1/2} e^{-x^3}\,dx$ by taking the partial sum consisting of the first four nonzero terms of the series that you obtained in part (c), what can you say about the accuracy of your approximation? Be as precise as you can, and state your reasoning clearly.

(The partial sum in question is $\frac{1}{2} - \frac{(1/2)^4}{4} + \frac{(1/2)^7}{7\cdot 2!} - \frac{(1/2)^{10}}{10\cdot 3!}$, by the way.)

The series above is of the form $\sum_{n=0}^{\infty}(-1)^n\cdot b_n$, where $b_n = \frac{1}{2^{3n+1}\cdot(3n+1)\cdot n!}$,

and we'll check that it satisfies the conditions of the Alternating Series Test.

- Clearly $b_n > 0$ for $n \geq 0$, as each term in the denominator is positive.
- Terms decreasing, i.e. $b_{n+1} \leq b_n$? Yes, since all of the factors in the denominator are positive, increasing functions of n (2^{3n+1}, $3n+1$, and $n!$).
- Terms go to 0? Yes, $\lim_{n\to\infty} b_n = 0$, because all factors in the denominator go to infinity as n increases.

Thus, by the Remainder Estimate of the Alt. Series Test, the absolute value of error in using the above partial sum is no greater than the fifth (nonzero) term, i.e. the approximation is accurate to within $\frac{(1/2)^{13}}{13\cdot 4!} = \frac{1}{2^{13}\cdot 13\cdot 4!}$, which is tiny!