Math 42 - Winter 2007 - Exam I Solutions

1. (40 points) Evaluate each of the following integrals, showing all of your reasoning.

(a) $\displaystyle\int_3^4 \frac{x}{\sqrt{25-x^2}}\,dx$

No need for trig substitution!

Just let $\begin{cases} u = 25-x^2 \\ du = -2x\,dx \end{cases}$. (So if $x=3$, then $u=16$, and if $x=4$, then $u=9$.)

Then $$\int_3^4 \frac{x\,dx}{\sqrt{25-x^2}} = \int_{16}^9 \frac{-\frac{1}{2}\,du}{\sqrt{u}} = -\frac{1}{2}\int_{16}^9 u^{-1/2}\,du$$

$$= -\frac{1}{2}\cdot 2u^{1/2}\Big]_{16}^9 = -\frac{1}{2}(2\cdot 3 - 2\cdot 4) = \boxed{1}.$$

(If you did by trig substitution, you may have obtained $5\cos(\arcsin\frac{3}{5}) - 5\cos(\arcsin\frac{4}{5})$, which is also equal to 1.)

(b) $\displaystyle\int_1^{e^2} \ln z\,dz$

Integrate by parts: $u = \ln z$ $\quad du = \frac{dz}{z}$

$dv = dz \quad v = z$

Then $$\int_1^{e^2} \ln z\,dz = uv\Big]_{z=1}^{z=e^2} - \int_{z=1}^{z=e^2} v\,du$$

$$= z\ln z\Big]_{z=1}^{z=e^2} - \int_{z=1}^{z=e^2} dz$$

$$= (z\ln z - z)\Big]_{z=1}^{z=e^2}$$

$$= e^2\ln e^2 - e^2 - (\ln 1 - 1) = \boxed{e^2+1}.$$

(c) $\int \cos^3 2t\, dt$ First let $u = 2t$, so $du = 2dt$.

Now using the Pythagorean identity, we get

$$\int \cos^3 2t\, dt = \frac{1}{2}\int \cos^3 u\, du = \frac{1}{2}\int \cos u \cdot \cos^2 u\, du$$

$$= \frac{1}{2}\int \cos u \cdot (1-\sin^2 u)\, du$$

$$= \frac{1}{2}\int \cos u\, du - \frac{1}{2}\int \cos u \sin^2 u\, du \quad \text{using } \begin{cases} w = \sin u \\ dw = \cos u\, du \end{cases}$$

$$= \frac{1}{2}\sin u - \frac{1}{6}\sin^3 u + C$$

$$= \boxed{\frac{1}{2}\sin 2t - \frac{1}{6}\sin^3 2t + C}$$

(d) $\int z \arctan z\, dz$

Integrate by parts: $u = \arctan z$ $\quad du = \dfrac{dz}{1+z^2}$

$dv = z\,dz$ $\quad v = \frac{1}{2}z^2$

Then $\int z \arctan z\, dz = \int u\, dv = uv - \int v\, du$

$$= \frac{1}{2}z^2 \arctan z - \int \frac{1}{2}\cdot\frac{z^2}{1+z^2}\, dz.$$

Now $\dfrac{z^2}{1+z^2} = 1 - \dfrac{1}{1+z^2}$ by long division, so

$$\int z\arctan z\, dz = \frac{1}{2}z^2 \arctan z - \frac{1}{2}\int \frac{z^2}{1+z^2}\, dz$$

$$= \frac{1}{2}z^2 \arctan z - \frac{1}{2}\int dz + \frac{1}{2}\int \frac{1}{1+z^2}\, dz$$

$$= \boxed{\frac{1}{2}z^2 \arctan z - \frac{1}{2}z + \frac{1}{2}\arctan z + C}$$

(e) $\int \sin^2 4x\, dx$

Using the double-angle identity $\sin^2 t = \frac{1}{2}(1-\cos 2t)$,

we obtain
$$\int \sin^2 4x\, dx = \int \frac{1}{2}(1-\cos 8x)\, dx$$
$$= \frac{1}{2}\int dx - \frac{1}{2}\int \cos 8x\, dx$$
$$= \boxed{\frac{x}{2} - \frac{\sin 8x}{16} + C}$$

(f) $\int (4-x^2)^{-3/2}\, dx$

After trying some simpler substitutions that don't work well, fall back on a trig-based one: $\begin{cases} x = 2\sin\theta \\ dx = 2\cos\theta\, d\theta \end{cases}$

We get
$$\int (4-x^2)^{-3/2}\, dx = \int (4-4\sin^2\theta)^{-3/2} \cdot 2\cos\theta\, d\theta$$
$$= \int \left[4\cdot(1-\sin^2\theta)\right]^{-3/2} \cdot 2\cos\theta\, d\theta$$
$$= \int (4\cos^2\theta)^{-3/2} \cdot 2\cos\theta\, d\theta$$
$$= \int \frac{2\cos\theta\, d\theta}{8\cos^3\theta} = \frac{1}{4}\int \sec^2\theta\, d\theta = \frac{1}{4}\tan\theta + C$$
$$= \boxed{\frac{1}{4}\tan\left(\arcsin\frac{x}{2}\right) + C}$$

(Note: this answer is equivalent to $\frac{x}{4\sqrt{4-x^2}} + C$.)

2. (10 points) Evaluate the integral $\int \frac{4x^3}{x^3 - x^2 + 3x - 3}\,dx = \int \frac{4x^3}{(x-1)(x^2+3)}\,dx.$

We first perform long division:

$$\begin{array}{r|l} & \quad 4 \\ x^3-x^2+3x-3 & 4x^3 \\ & \underline{4x^3-4x^2+12x-12} \\ & \quad 4x^2-12x+12 \end{array}$$

Thus $$\int \frac{4x^3}{x^3-x^2+3x-3}\,dx = \int 4\,dx + \int \frac{4x^2-12x+12}{x^3-x^2+3x-3}\,dx = 4x + \int \frac{4x^2-12x+12}{x^3-x^2+3x-3}\,dx.$$

We can manage the integrand of the second term by doing partial fraction decomposition:

$$\frac{4x^2-12x+12}{x^3-x^2+3x-3} = \frac{4x^2-12x+12}{(x-1)(x^2+3)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+3}$$

Multiply by $(x-1)(x^2+3)$: $\quad 4x^2-12x+12 = A(x^2+3) + (Bx+C)(x-1).$

If we let $x=1$, then $4-12+12 = A(1+3)$, so $4A=4$, i.e. $\underline{A=1}$.

This simplifies the system to: $4x^2-12x+12 = x^2+3+(Bx+C)(x-1)$

$$\Rightarrow 3x^2-12x+9 = (Bx+C)(x-1)$$
$$= Bx^2+(C-B)x-C.$$

Thus, $\underline{B=3}$ and $\underline{C=-9}$.

We have $$\int \frac{4x^2-12x+12}{x^3-x^2+3x-3}\,dx = \int \frac{1}{x-1}\,dx + \int \frac{3x-9}{x^2+3}\,dx = \ln|x-1| + \int \frac{3x}{x^2+3}\,dx - 9\int \frac{1}{x^2+3}\,dx.$$

We can accomplish the first integral by u-substitution: $\int \frac{3x}{x^2+3}\,dx = \frac{3}{2}\ln(x^2+3)+C.$

The last integral is done by trig. sub: $\left\{\begin{array}{l} x=\sqrt{3}\tan\theta \\ dx=\sqrt{3}\sec^2\theta\,d\theta \end{array}\right\} \Rightarrow \int \frac{1}{x^2+3}\,dx = \int \frac{\sqrt{3}\sec^2\theta\,d\theta}{3\tan^2\theta+3} =$

$$= \frac{\sqrt{3}}{3}\int \frac{\sec^2\theta}{\tan^2\theta+1}\,d\theta = \frac{\sqrt{3}}{3}\int d\theta = \frac{\sqrt{3}}{3}\arctan\frac{x}{\sqrt{3}},$$

so $$\int \frac{4x^3}{(x-1)(x^2+3)}\,dx = \boxed{4x + \ln|x-1| + \frac{3}{2}\ln(x^2+3) - \frac{9\sqrt{3}}{3}\arctan\frac{x}{\sqrt{3}} + C}.$$

3. (6 points) A rocket has vertical position 0 at time 0. The following chart gives the rocket's upward velocity, in meters per second, at time t seconds.

t	0	2	4	6	8	10	12
v	1	8	25	60	120	200	350

(a) Using all the data in the chart, write a sum representing an estimate of the rocket's height at time $t = 12$, using the *Trapezoidal Rule*. You do not have to simplify or fully evaluate your expression.

The rocket's change in height equals $\int_0^{12} v(t)\,dt$; since initial height is 0 m, this integral also equals the rocket's height at $t = 12$ sec.

With intervals of widths $\Delta t = 2$ sec, there are $n = 6$ such sub-intervals.

Thus, the trapezoidal approximation is T_6:

$$\text{height} = \int_0^{12} v(t)\,dt \approx T_6 = \frac{\Delta t}{2}\Big(v(0) + 2v(2) + 2v(4) + 2v(6) + 2v(8) + 2v(10) + v(12)\Big)$$

$$= \boxed{\frac{2}{2}\Big(1 + 2\cdot 8 + 2\cdot 25 + 2\cdot 60 + 2\cdot 120 + 2\cdot 200 + 350\Big) \text{ meters}}.$$

(b) Do the same using *Simpson's Rule*; again, you do not have to simplify the expression.

Since $n = 6$, which is even, this is a valid value for Simpson's Rule.

The approximation is

$$\text{height} = \int_0^{12} v(t)\,dt \approx S_6 = \frac{\Delta t}{3}\Big(v(0) + 4v(2) + 2v(4) + 4v(6) + 2v(8) + 4v(10) + v(12)\Big)$$

$$= \boxed{\frac{2}{3}\Big(1 + 4\cdot 8 + 2\cdot 25 + 4\cdot 60 + 2\cdot 120 + 4\cdot 200 + 350\Big) \text{ meters}}.$$

4. (8 points) Consider the integral $\int_0^2 \cos(x^2)\,dx$.

(a) Estimate the error made in approximating the value of this integral using the Midpoint Rule using $n = 6$ subintervals. *State your answer in a complete sentence.*

The Midpoint Rule error estimate involves K_2, so we need to estimate the size of $|f''(x)|$ on $[0,2]$, for $f(x) = \cos(x^2)$.

Derivatives: $f'(x) = -2x\sin(x^2)$; $f''(x) = -2\sin(x^2) + (-2x)(2x)(\cos(x^2))$;

so $|f''(x)| = |-(2\sin(x^2) + 4x^2\cos(x^2))| \leq 2\cdot 1 + 4\cdot 2^2\cdot 1 = 18$, due to the fact that $|\sin(x^2)| \leq 1$ and $|\cos(x^2)| \leq 1$, and $|x^2| \leq 4$ on the interval $[0,2]$.

That is, we may take $K_2 = 18$. The Error Bound Formula gives: $|E_M| \leq \frac{K_2(b-a)^3}{24n^2} = \frac{18\cdot 2^3}{24\cdot 6^2} = \frac{1}{6}$; this means that the absolute value of error, or difference between actual value of $\int_0^2 \cos(x^2)dx$ and approximated value, is no more than $\frac{1}{6}$ units.

(b) Again using the Midpoint Rule, how many subintervals n would be necessary to guarantee an error of at most $\frac{1}{1000}$? Give a valid n in simplified form. (As long as you justify your answer, you do not have to worry about finding the best possible value.)

We'll use 18 for K_2, found in part (a).

We wish to find n such that $\frac{K_2\cdot(b-a)^3}{24n^2} \leq \frac{1}{1000}$, because in this case, we'd have $|E_M| \leq \frac{K_2(b-a)^3}{24n^2} \leq \frac{1}{1000}$, as desired.

Thus, we must solve $$\frac{K_2\cdot(b-a)^3}{24n^2} = \frac{18\cdot 2^3}{24\cdot n^2} \leq \frac{1}{1000}$$ for n:

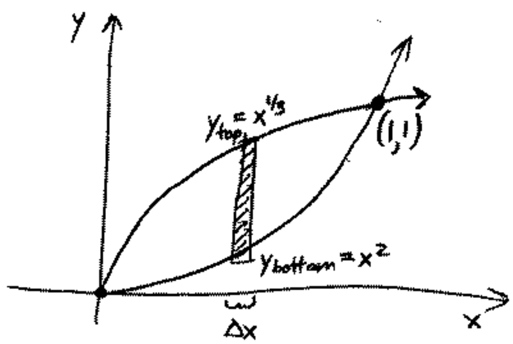
So, $$18\cdot 2^3\cdot 1000 \leq 24n^2,$$

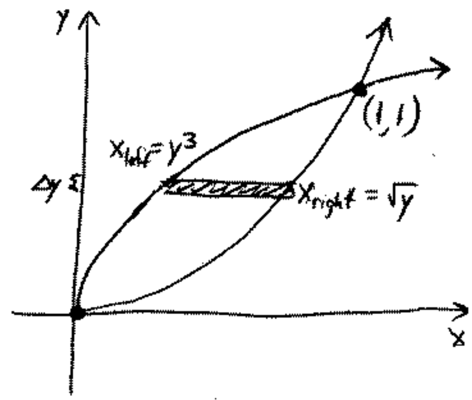
or $$n^2 \geq \frac{18\cdot 8000}{24} = 6000, \quad \text{i.e. } n \geq \sqrt{6000}.$$

Of course, n must be a natural number, so $\boxed{n=80}$ works (because $80^2 = 6400 > 6000$), and in fact any n greater than or equal to 78 would be acceptable with this value of K_2.

5. (10 points)

(a) Set up two distinct integrals, each in terms of a single variable, representing the area of the region in the first quadrant bounded by the curves $y = x^2$ and $y = x^{1/3}$. For each, make sure you justify your answer (draw a picture and mark a sample slice). Don't evaluate either integral.





Vertical slices: at coordinate x,

$$\Delta A = \text{height}(x) \cdot \Delta x$$
$$= (y_{top} - y_{bottom}) \cdot \Delta x$$
$$= (x^{1/3} - x^2)\,\Delta x,$$

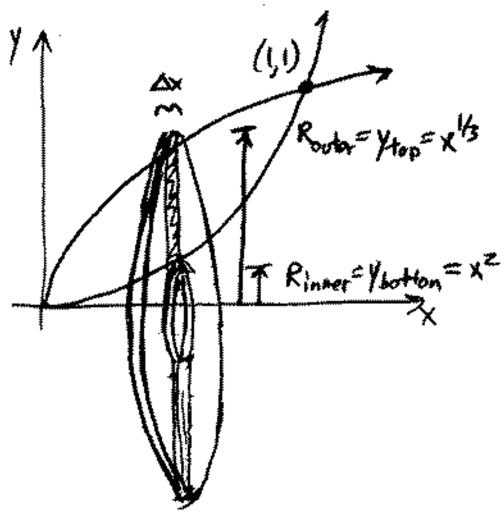
so $A = \int_0^1 (x^{1/3} - x^2)\,dx$.

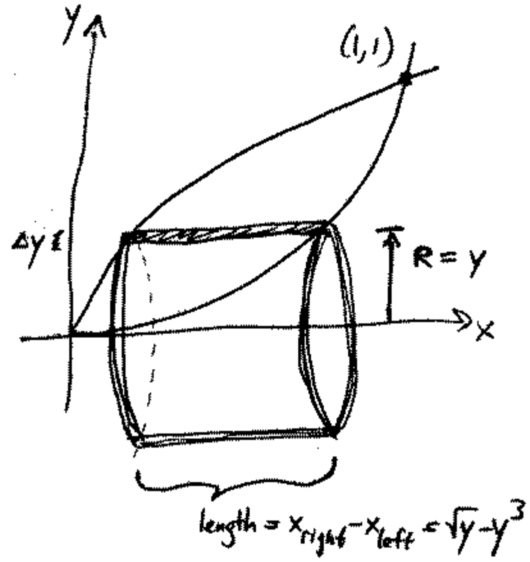
Horizontal slices: at coordinate y,

$$\Delta A = \text{width}(y) \cdot \Delta y$$
$$= (x_{right} - x_{left}) \cdot \Delta y$$
$$= (\sqrt{y} - y^3) \cdot \Delta y,$$

so $A = \int_0^1 (\sqrt{y} - y^3)\,dy$.

(b) Set up two distinct integrals, each in terms of a single variable, representing the volume obtained by rotating the region from part (a) around the x-axis. For each, make sure you justify your answer (draw a picture, label a sample slice, and cite the method used). Don't evaluate either integral.





Vertical slices $\Rightarrow$ perp to rot. axis $\Rightarrow$ Washers.

At coordinate x,

$$\begin{aligned}\Delta V &= A(x)\Delta x\\ &= \left(\pi R_{outer}^2 - \pi R_{inner}^2\right)\Delta x\\ &= \pi\left(x^{2/3} - x^4\right)\Delta x,\end{aligned}$$

so $V = \int_0^1 \pi\left(x^{2/3} - x^4\right)dx$.

Horizontal slices $\Rightarrow$ parallel to rot. axis $\Rightarrow$ cylindrical shells.

At coordinate y,

$$\begin{aligned}\Delta V &= 2\pi(\text{length})(\text{radius})(\text{thickness})\\ &= 2\pi\left(\sqrt{y} - y^3\right)y\,\Delta y,\end{aligned}$$

so $V = \int_0^1 2\pi\left(\sqrt{y} - y^3\right)y\,dy$.

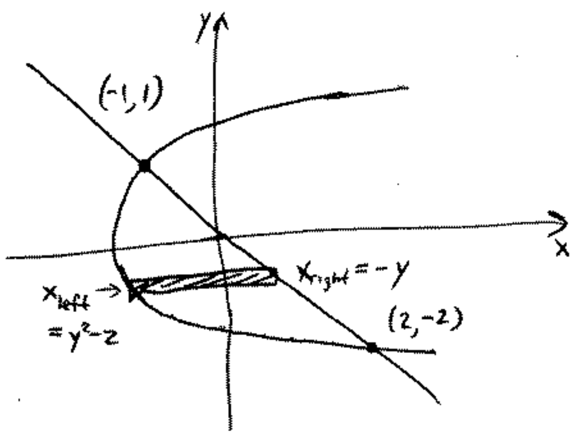
6. (10 points)

(a) Set up, but do not evaluate, an integral representing the area of the region bounded by the curves $x = y^2 - 2$ and $y = -x$. As justification, draw a picture with a sample slice labeled.

Curves intersect when $y^2 - 2 = x = -y$, so $y^2 + y - 2 = 0$

$\Rightarrow (y+2)(y-1) = 0$, i.e. $y = -2, 1$.

When $y=-2$ then $x = 2$; when $y = 1$, then $x = -1$. Graphed below:



Much easier to set integral up in terms of y, i.e., make horizontal slices of thickness Δy.

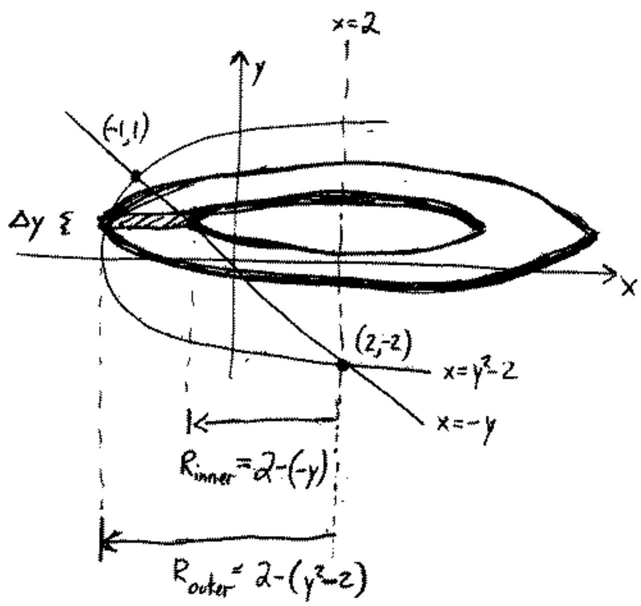
Then $\Delta A = \text{length}(y) \cdot \Delta y$

$= (x_{right} - x_{left}) \Delta y$

$= (-y - (y^2 - 2)) \Delta y$,

and so $A = \displaystyle\int_{-2}^{1} (-y - y^2 + 2)\, dy$.

(b) Set up an integral representing the volume obtained by rotating the region from part (a) around the line $x = 2$. Make sure you justify your answer (draw and label a diagram, and cite the method). Again, don't evaluate the integral.



Horizontal slices ⇒ Slice perp. to rotation axis ⇒ Washers

At coordinate y, $\Delta V = A(y)\,\Delta y$

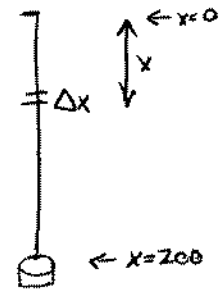
$$= \left(\pi R_{outer}^2 - \pi R_{inner}^2\right)\Delta y$$

$$= \left(\pi(4-y^2)^2 - \pi(2+y)^2\right)\Delta y,$$

so $$V = \int_{-2}^{1} \pi\left((4-y^2)^2 - (2+y)^2\right) dy.$$

7. (8 points) One very rainy day, a bucket is raised from ground level to the top of a building 200 ft high, using a rope having a linear density of 0.1 lb/ft. Initially (at ground level), the bucket weighs 5 pounds; however, with rain continuing to pour at a constant rate, the bucket takes on water and actually weighs 10 pounds by the moment it reaches the top of the building. Assuming the bucket is being raised at a constant rate, how much work is required to pull the bucket (with its rope) to the top?

Solution #1: Break into (A) work to lift rope; (B) work to lift bucket and water.



(A) Slice rope into small pieces of length Δx feet;

any such piece x feet from the top has weight $(\Delta x)(0.1)$ lb and is lifted x ft,

so work to lift this piece is $\Delta W = (\text{weight})(\text{dist}) = 0.1x\Delta x$,

Meaning total work for rope $= W = \int_0^{200} 0.1x\,dx = (0.1)\frac{x^2}{2}\Big]_0^{200} = \underline{2000 \text{ ft}\cdot\text{lb}}$.

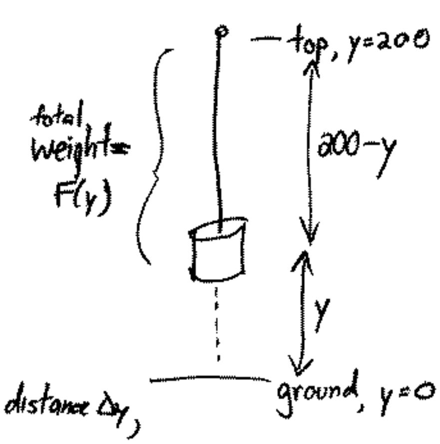
(B) After water has been lifted y feet, weight of bucket is $5+\frac{5}{200}y$ lbs.

(This is because the constant rates imply weight is a linear function of bucket height, and we know weight is 5 when $y=0$ and weight is 10 when $y=200$.)

Thus, process of lifting bucket is the exertion of a variable force $F(y) = 5+\frac{5y}{200}$ over the interval $0 \le y \le 200$: $W = \int_0^{200} F(y)\,dy = \int_0^{200}\left(5+\frac{5y}{200}\right)dy = 5y+\frac{5y^2}{400}\Big]_0^{200} = \underline{1500 \text{ ft}\cdot\text{lb}}$.

Thus, total work $=$ (A) $+$ (B) $= 2000+1500 = \boxed{3500 \text{ ft}\cdot\text{lb}}$.

Solution #2: The whole process of lifting both bucket & rope can be thought of as exerting a variable force over a distance.



After the bucket has been hoisted a distance y, the weight still hanging is (see above discussion):

$$F(y) = \underbrace{5+\frac{5y}{200}}_{\text{bucket}} + \underbrace{(200-y)(0.1)}_{\text{rope, length times density}}$$

Thus, $\Delta W = F(y)\Delta y$ is the work to lift this weight a small distance Δy,

and so $W = \int_0^{200} F(y)\,dy = \int_0^{200}\left(5+\frac{5y}{200}+(200-y)(0.1)\right)dy = (\text{etc.}) = \boxed{3500 \text{ ft}\cdot\text{lb}}$, as before.

8. (8 points) A giant clay ant hill in northeastern Argentina has the shape of a perfect cone, with its circular base on the ground. The base has a 1-foot radius, and the ant hill is 3 feet in height. The density of the material is a uniform 30 lb/ft^3. How much work have the ants done to assemble this ant hill, lifting all material vertically from ground-level?

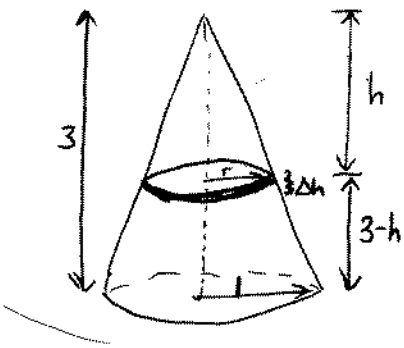


Photo source: Wikipedia

If we slice the cone into horizontal cross-sections of thickness Δh (small), we can use the formula "work = force · distance" to compute the work to lift each slice, since the force and distance are constant on each slice.

Note that the slice located h feet from the top of the cone (and therefore $3-h$ feet from the ground) has radius r, where $\frac{r}{h} = \frac{1}{3}$, i.e. $r = \frac{h}{3}$.

Thus, ΔW = work to lift each slice = force · distance

= (weight of slice)(distance to lift slice)

= (density)(volume of slice)(distance to lift)

= (density)(area)(thickness)(dist. to lift)

$= (30)\left(\pi\left(\frac{h}{3}\right)^2\right)\Delta h \cdot (3-h)$,

so total work = $W = \int_{h=0}^{h=3} 30\pi \cdot \frac{h^2}{9} \cdot (3-h)\,dh = \frac{30\pi}{9}\int_0^3 (3h^2 - h^3)\,dh = \frac{30\pi}{9}\left(h^3 - \frac{h^4}{4}\right)\Big|_0^3$

$= \boxed{\frac{45\pi}{2} \text{ ft} \cdot \text{lb}}$.

(There are other ways to set up the integral, but the outcome is the same.)