① ⓐ f is a PDF, so we need $f \geq 0$ [which is true for C positive]

and $\int_{-\infty}^{\infty} f(t)\,dt = 1$.

So, $$1 = \int_{-\infty}^{\infty} f(t)\,dt = \int_{0}^{\infty} f(t)\,dt \quad \text{[because } f(t)=0 \text{ for } t<0]$$

$$= \int_0^1 Ct\,dt + \int_1^{\infty} Ct^{-3}\,dt$$

$$= C\cdot\left[\frac{t^2}{2}\right]_0^1 + C\cdot\lim_{N\to\infty}\left[\frac{t^{-2}}{-2}\right]_1^N$$

$$= C\cdot\left[\frac{1}{2}-0\right] + C\cdot\lim_{N\to\infty}\left[\frac{N^{-2}}{-2}-\left(-\frac{1}{2}\right)\right]$$

$$= C\cdot\frac{1}{2} + C\cdot\left[0+\frac{1}{2}\right]$$

$$= C.$$

So, only solution is $\boxed{C=1}$.

ⓑ $$\text{prob}(0\leq t\leq 3) = \int_0^3 f(t)\,dt$$

$$= \int_0^1 t\,dt + \int_1^3 t^{-3}\,dt$$

$$= \left.\frac{t^2}{2}\right]_0^1 + \left.\frac{-t^{-2}}{2}\right]_1^3$$

$$= \frac{1}{2}-0+\left(\frac{-3^{-2}}{2}-\left(-\frac{1}{2}\right)\right)$$

$$= 1-\frac{1}{18} = \boxed{\frac{17}{18}}.$$

① ⓒ mean $= \int_{-\infty}^{\infty} t\cdot f(t)\,dt = \int_{0}^{\infty} t\cdot f(t)\,dt$

$$= \int_0^1 t^2\,dt + \int_1^\infty t\cdot t^{-3}\,dt$$

$$= \left[\frac{t^3}{3}\right]_0^1 + \lim_{N\to\infty}\left[-\frac{1}{t}\right]_1^N$$

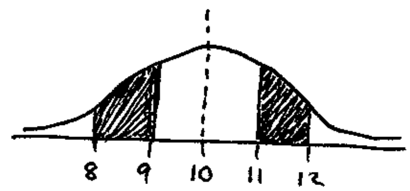
$$= \frac{1}{3} - 0 + \lim_{N\to\infty}\left[-\frac{1}{N} - (-1)\right]$$

$$= \frac{1}{3} + 0 + 1$$

$$= \boxed{\frac{4}{3}\ \text{(weeks)}}.$$

② If X stands for the random variable, then its $\mu = 10\,mg$ and $\sigma = 1\,mg$, and so

$$\text{Prob}(8mg < X < 9mg) = \int_8^9 \frac{1}{\sqrt{2\pi}} e^{-(x-10)^2/2}\,dx$$



$$= \int_{11}^{12} \frac{1}{\sqrt{2\pi}} e^{-(x-10)^2}\,dx \leftarrow$$ (since the PDF is symmetric about the line $x = 10$, can see this by the above sketch of areas)

$$= \int_{-\infty}^{12} \frac{1}{\sqrt{2\pi}} e^{-(x-10)^2}\,dx - \int_{-\infty}^{11} \frac{1}{\sqrt{2\pi}} e^{-(x-10)^2}\,dx$$

$$= 0.98 - 0.84 = \boxed{0.14}.$$

Alternatively, the probability in question can be written using the "standard" bell curve, via $8 = 10 - 2 = \mu - 2\sigma$ and $9 = 10 - 1 = \mu - \sigma$:

$$\text{Prob}(\mu - 2\sigma < X < \mu - \sigma) = \int_{-2}^{-1} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt = \int_1^2 \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\,du,$$

[Via the substitution $u = -t$, or by symmetry]

and this last integral equals $\int_{-\infty}^{2} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\,du - \int_{-\infty}^{1} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\,du = 0.98 - 0.84 = \boxed{0.14}.$

③ⓐ The condition $\int_{-\infty}^{\infty} f(t)\,dt = 1$ implies for this f that

$$1 = \int_0^{24} C\,dt + \int_{24}^{\infty} Ce^{-(t-24)/16}\,dt$$

$$= 24C + C \cdot \int_{24}^{\infty} e^{-(t-24)/16}\,dt$$

$$= 24C + C \cdot \lim_{N\to\infty} \left[-16e^{-(t-24)/16}\right]_{t=24}^{t=N}$$

$$= 24C + 16C \cdot \lim_{N\to\infty} \left(-e^{-(N-24)/16} + e^{0}\right)$$

$$= 24C + 16C \cdot 1 = 40C,$$

so that $\boxed{C = 1/40}$.

ⓑ We can use the calculation from part ⓐ, or else use the fact that f is a PDF to write

$$\int_{24}^{\infty} f(t)\,dt = 1 - \int_{-\infty}^{24} f(t)\,dt$$

$$= 1 - \int_0^{24} \tfrac{1}{40}\,dt = 1 - \tfrac{24}{40} = \boxed{16/40}.$$

This number is the probability that a randomly selected citizen of Bishkadu is over 24 years of age (or, the <u>fraction</u> of the population of Bishkadu that is over 24).

3c) By definition, if the median age is m, then half the population will be older than m and half will be younger. By part b), since $16/40 < 1/2$, the median m is <u>under</u> 24.

So, we're looking for m such that

$$\frac{1}{2} = \int_{-\infty}^{m} f(t)\,dt = \int_{0}^{m} \frac{1}{40}\,dt \qquad \text{(since } f(t) \text{ will be } \tfrac{1}{40} \text{ on the interval } 0 \le t \le m\text{)}$$

$$= \frac{m}{40},$$

so $\boxed{m=20}$ years of age.

4) a) Since a probability density function must be non-negative, we know that $a \geq 0$. The other condition for a probability density function p is that $\int_{-\infty}^{\infty} p(t)\,dt = 1$. Since

$$\int_{-\infty}^{\infty} p(t)\,dt = \int_{0}^{\infty} a e^{-0.122t}\,dt = \frac{a}{-0.122} e^{-0.122t}\Big|_{0}^{\infty}$$

$$= \left(\lim_{b\to\infty} \frac{a}{-0.122} e^{-0.122b}\right) - \left(\frac{a}{-0.122}\right) = (0) + \frac{a}{0.122} = \frac{a}{0.122}$$

we see that we must have <u>$a = 0.122$</u>.

4 b) $P(t) = \int_{-\infty}^{t} p(u)\,du = \int_0^t 0.122\, e^{-0.122u}\,du = -1\, e^{-0.122u}\Big|_0^t$

$= \left(-e^{-0.122t}\right) - (-1) = \underline{1 - e^{-0.122t}}$

c) The median, m, is given by $\int_{-\infty}^{m} p(t)\,dt = \frac{1}{2}$.

$\int_{-\infty}^{m} p(t)\,dt = \int_0^m 0.122\, e^{-0.122t}\,dt = -e^{-0.122t}\Big|_0^m = \left(-e^{-0.122m}\right) - (-1) = 1 - e^{-0.122m}$

We solve the equation $.5 = 1 - e^{-0.122m}$ for m:

$e^{-0.122m} = 1 - .5 = .5 \Rightarrow \ln\left(e^{-0.122m}\right) = \ln(.5) \Rightarrow -0.122m \overbrace{\ln(e)}^{1} = \ln(.5)$

$\Rightarrow \underline{m = \frac{\ln(.5)}{-0.122}}$ (this is approximately 5.6815 seconds).

d) The mean is given by $\int_{-\infty}^{\infty} t\, p(t)\,dt = 0.122 \int_0^{\infty} t e^{-0.122t}\,dt$

We use integration by parts to find $\int t e^{-0.122t}\,dt$. Let $u = t$, $dv = e^{-0.122t}\,dt$. Then $du = dt$ and $v = \frac{1}{-0.122} e^{-0.122t}$, so

$$\int_0^{\infty} t e^{-0.122t}\,dt = -\frac{1}{0.122} t e^{-0.122t}\Big|_0^{\infty} + \frac{1}{0.122}\int_0^{\infty} e^{-0.122t}\,dt$$

$$= \left(\lim_{t\to\infty} -\frac{1}{0.122} t\, e^{-0.122t}\right) - (0) + \left[-\frac{1}{(0.122)^2} e^{-0.122t}\Big|_0^{\infty}\right]$$

$$= -\frac{1}{0.122}\lim_{t\to\infty}\frac{t}{e^{0.122t}} + -\frac{1}{(0.122)^2}\left[\left(\lim_{t\to\infty} e^{-0.122t}\right) - \left(e^0\right)\right]$$

$$= -\frac{1}{0.122}\lim_{t\to\infty}\frac{t}{e^{0.122t}} - \frac{1}{(0.122)^2}\left[0 - 1\right] = -\frac{1}{0.122}\lim_{t\to\infty}\frac{t}{e^{0.122t}} + \frac{1}{(0.122)^2}$$

Since $\lim_{t\to\infty}\frac{t}{e^{0.122t}}$ is of the type "$\frac{\infty}{\infty}$", we can apply L'Hospital's Rule.

$$\lim_{t\to\infty}\frac{t}{e^{0.122t}} = \lim_{t\to\infty}\frac{1}{0.122\, e^{0.122t}} = 0.$$

So then

$\int_0^{\infty} t e^{-0.122t}\,dt = \frac{1}{(0.122)^2}$, and the mean is

$$0.122\int_0^{\infty} t e^{-0.122t}\,dt = (0.122)\cdot\frac{1}{(0.122)^2} = \underline{\frac{1}{0.122}} \quad (\approx 8.1967 \text{ seconds})$$

⑤ a) $P(t) = \int_0^t p(x)\,dx$ represents the fraction of the population whose survival time is between 0 and t years. In other words, it is the fraction of people who do not survive beyond t years after treatment.

b) $P(t) = \int_0^t p(x)\,dx = \int_0^t ce^{-cx}\,dx = c\cdot\left(-\frac{1}{c}e^{-cx}\right)\Big|_0^t = -e^{-cx}\Big|_0^t$

$= (-e^{-ct}) - (-e^0) = 1 - e^{-ct}$

c) The probability of surviving at least t years is equal to the fraction of the group whose survival time is $\geq t$. This is the same as one minus the fraction of the group whose survival time is less than t. So $S(t) = 1 - P(t)$, since $P(t)$ = the fraction with survival time between 0 and t. From b), $\underline{S(t)} = 1 - (1 - e^{-ct}) = \underline{e^{-ct}}$.

⑥ ⓐ $$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/2\sigma^2} = \frac{1}{20\sqrt{2\pi}}\, e^{-(x-60)^2/800}$$

So $p(x \in [40, 100])$

$$= \int_{40}^{100} f(x)\,dx = \boxed{\int_{40}^{100} \frac{1}{20\sqrt{2\pi}}\, e^{-(x-60)^2/800}\,dx}$$

ⓑ $$\int_{40}^{100} f(x)\,dx = \int_{40}^{60} f(x)\,dx + \int_{60}^{100} f(x)\,dx$$

$$= \underbrace{\int_{\mu-\sigma}^{\mu} f(x)\,dx}_{\text{(by symmetry)}} + \int_{\mu}^{\mu+2\sigma} f(x)\,dx$$

$$= \int_{\mu}^{\mu+\sigma} f(x)\,dx + \int_{\mu}^{\mu+2\sigma} f(x)\,dx$$

$$\cong .34 + .48 \cong \boxed{.82}$$

⑦ a) $\mu = \int_{-\infty}^{\infty} t\, f(t)\, dt = \int_0^{\infty} t \cdot c e^{-ct}\, dt = c\int_0^{\infty} t e^{-ct}\, dt$

First, we'll do the work to find $\int t e^{-ct}\, dt$, which has to be done using integration by parts. Let $u = t \quad dv = e^{-ct}\,dt$.

Then $du = dt \quad v = -\frac{1}{c}e^{-ct}$.

So

$$\int t e^{-ct}\, dt = t\left(-\frac{1}{c}e^{-ct}\right) - \int \left(-\frac{1}{c}e^{-ct}\right) dt$$

$$= -\frac{t}{c}e^{-ct} + \frac{1}{c}\int e^{-ct}\,dt = -\frac{t}{c}e^{-ct} + \frac{1}{c}\left(-\frac{1}{c}e^{-ct}\right) + \text{const.}$$

Then $\mu = c\int_0^{\infty} t e^{-ct}\,dt = c\left[-\frac{t}{c}e^{-ct} - \frac{1}{c^2}e^{-ct}\right]\Big|_0^{\infty}$

$$= \lim_{b\to\infty} c\left[-\frac{t}{c}e^{-ct} - \frac{1}{c^2}e^{-ct}\right]\Big|_0^{b}$$

$$= \lim_{b\to\infty} c\left[\left(-\frac{b}{c}e^{-c\cdot b} - \frac{1}{c^2}e^{-c\cdot b}\right) - \left(0 - \frac{1}{c^2}\right)\right]$$

$$= \lim_{b\to\infty}\left(-be^{-bc} - \frac{1}{c}e^{-bc} + \frac{1}{c}\right)$$

$$= \left(\lim_{b\to\infty} -be^{-bc}\right) - \left(\lim_{b\to\infty}\frac{1}{c}e^{-bc}\right) + \left(\lim_{b\to\infty}\frac{1}{c}\right)$$

$$= \left(\lim_{b\to\infty}\frac{-b}{e^{bc}}\right) - (0) + \left(\frac{1}{c}\right) \overset{H}{=} \left(\lim_{b\to\infty}\frac{-1}{ce^{bc}}\right) + \frac{1}{c}$$

$$= 0 + \frac{1}{c} = \underline{\frac{1}{c}}.$$

(7) b) We want to find the number s such that

$$\int_0^s ce^{-ct}\,dt = .95$$

$$\int_0^s ce^{-ct}\,dt = -e^{-ct}\Big|_0^s = \left(-e^{-cs}\right) - \left(-e^0\right) = -e^{-cs} + 1$$

$-e^{-cs} + 1 = .95 \Rightarrow -e^{-cs} = -.05 \Rightarrow e^{-cs} = .05 \Rightarrow -cs = \ln(.05)$.
So $s = \dfrac{\ln(.05)}{-c}$ will be the time by which 95% of the customers have been served.

(8) a) $\underline{\displaystyle\int_0^{50} \frac{1}{100} e^{-x/100}\,dx}$

b) The average (or mean) lifespan is given by

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} x \cdot \frac{1}{100} e^{-x/100}\,dx.$$

$\displaystyle\int_0^{\infty} x \frac{1}{100} e^{-x/100}\,dx = \frac{1}{100}\int_0^{\infty} x e^{-x/100}\,dx$. We can integrate by parts.

Let $u = x \qquad dv = e^{-x/100}\,dx$
Then $du = dx \qquad v = -100e^{-x/100}$

(Note: $v = \int dv = \int e^{-x/100}\,dx$. So let $w = -x/100$, $dw = -\frac{1}{100}dx$, $-100\,dw = dx$, and get $v = -100\int e^w\,dw = -100e^w = -100e^{-x/100}$.)

(over)

(8) b) (continued)

$$\frac{1}{100}\int_0^\infty x e^{-x/100}\,dx = \frac{1}{100}\left[(x)(-100e^{-x/100})\Big|_0^\infty - \int_0^\infty (-100e^{-x/100})\,dx\right]$$

$$= (-xe^{-x/100})\Big|_0^\infty + \int_0^\infty e^{-x/100}\,dx = (-xe^{-x/100})\Big|_0^\infty - 100e^{-x/100}\Big|_0^\infty$$

$$= \lim_{b\to\infty}(-be^{-b/100}) - 0 - 100\left[\left(\lim_{b\to\infty} e^{-b/100}\right) - e^0\right]$$

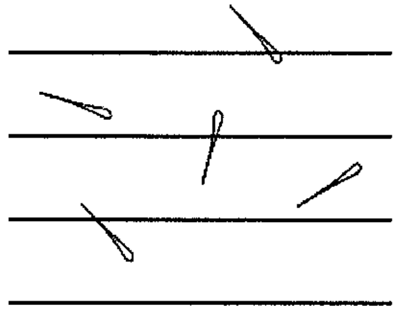
$$= \lim_{b\to\infty}\left(\frac{-b}{e^{b/100}}\right) - 100\left(\lim_{b\to\infty}\frac{1}{e^{b/100}}\right) + 100$$

$$\overset{H}{=} \lim_{b\to\infty}\left(\frac{-1}{\frac{1}{100}e^{b/100}}\right) - 100\cdot(0) + 100$$

$$= \lim_{b\to\infty}\frac{-100}{e^{b/100}} + 100 = 0 + 100 = \underline{100}\quad\text{(days)}$$

Math 20 – Winter 2006 – Final Exam Solutions

9. (14 points) In the original game of Pick Up Sticks, invented by the French naturalist Buffon, a bunch of toothpicks are randomly tossed onto a tabletop that is marked with horizontal parallel lines, and the "successful" tosses are the toothpicks that land touching a line.



Buffon, who was interested in the *angles* at which successful toothpicks are found to lie, discovered that the angles x of successful toothpicks have the following probability distribution:

$$h(x) = \begin{cases} C \sin x & \text{if } 0 \le x \le \pi \\ 0 & \text{otherwise.} \end{cases}$$

where C is a positive constant.

(a) Find C, given that h is a probability density function.

Since h is a PDF, must have $\int_{-\infty}^{\infty} h(x)\,dx = 1$.

So, $$1 = \int_{-\infty}^{\infty} h(x)\,dx$$

$$= \int_0^{\pi} C \sin x\,dx$$

$$= C \int_0^{\pi} \sin x\,dx = C(-\cos x)\Big|_0^{\pi}$$

$$= C(-\cos\pi - (-\cos 0))$$

$$= C(1 + 1) = 2C.$$

Thus, $2C = 1$, so $\boxed{C = \frac{1}{2}}$.

For easy reference, here is h:

$$h(x) = \begin{cases} C \sin x & \text{if } 0 \le x \le \pi \\ 0 & \text{otherwise,} \end{cases}$$

(b) Find the mean angle of a successful toothpick, with full mathematical justification.

$$\text{Mean} = \mu = \int_{-\infty}^{\infty} x \cdot h(x)\,dx$$

$$= \int_0^{\pi} x \cdot \tfrac{1}{2}\sin x\,dx$$

$$= \frac{1}{2}\int_0^{\pi} x \sin x\,dx$$

Int. by Parts:

$u = x \qquad du = dx$

$dv = \sin x\,dx \qquad v = -\cos x$

$$= \frac{1}{2}\left[x \cdot (-\cos x)\Big|_0^{\pi} - \int_0^{\pi} -\cos x\,dx\right]$$

$$= -\frac{1}{2}x\cos x\Big|_0^{\pi} + \frac{1}{2}\int_0^{\pi}\cos x\,dx$$

$$= -\frac{1}{2}x\cos x + \frac{1}{2}\sin x\Big]_0^{\pi}$$

$$= -\frac{1}{2}\pi\cos\pi + \frac{1}{2}\sin\pi - \left(-0 + \frac{1}{2}\sin 0\right)$$

$$= -\frac{1}{2}\pi \cdot (-1) + 0 - 0$$

$$= \boxed{\frac{\pi}{2}}.$$

10. (10 points) For this problem, use the following information about any normal ("bell-shaped" or "Gaussian") probability density function f:

- f has the general form $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$
- $\displaystyle\int_{-\infty}^{\mu+\sigma} f(x)dx \approx .84$
- $\displaystyle\int_{-\infty}^{\mu+2\sigma} f(x)dx \approx .98$

In 1959, NASA completed an exhaustive search for candidates to fly in the Mercury program, the nation's inaugural manned space missions. Among the many requirements was that the prospective astronaut be an adult male of height between 58 and 76 inches.

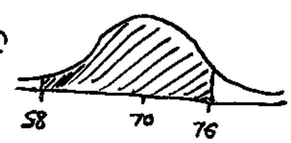
Suppose that, in 1959, the height of an American adult male could be modeled by a random variable with a normal distribution, with mean 70 in and standard deviation 6 in.

Using the above facts and assumptions, find the proportion of adult males in 1959 who would have passed the height requirement for the astronaut program; in other words, find the probability that a randomly selected American adult male had a height between 58 and 76 inches. Justify your answer by writing an integral expression that represents this probability and showing how to evaluate this integral.

Since $\mu = 70$ and $\sigma = 6$, notice that $76 = 70+6 = \mu+\sigma$ and $58 = 70-12 = \mu - 2\sigma$.

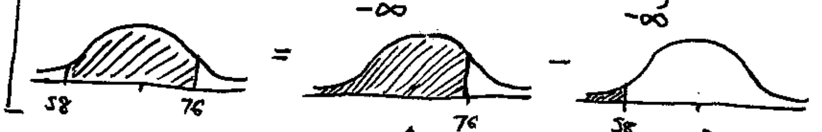
Thus the probability we want can be expressed (and graphically depicted) as:

$$\text{Prob}(58 < X < 76) = \int_{58}^{76} \frac{1}{6\sqrt{2\pi}} e^{-(x-70)^2/2\cdot 6^2}\,dx = \int_{\mu-2\sigma}^{\mu+\sigma} f(x)dx = \text{area of}$$



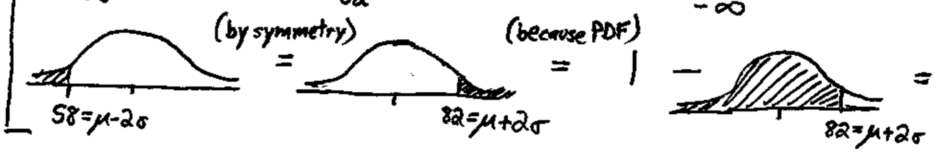
To find the value, first use the following relationship (written both as integrals or as areas):

$$\int_{58}^{76} f(x)\,dx = \int_{-\infty}^{76} f(x)dx - \int_{-\infty}^{58} f(x)\,dx$$



We know the first of the two areas is $\underline{0.84}$, and the second can be found by the below:

$$\int_{-\infty}^{58} f(x)dx = \int_{82}^{\infty} f(x)dx = 1 - \int_{-\infty}^{82} f(x)dx = 1 - 0.98 = 0.02$$



$= 1 - 0.98 = 0.02$

So the area/probability we want is $0.84 - 0.02 = \boxed{0.82}$.