Math 42 — Probability Practice Problems

1. [2004 Exam] When a certain farm's crop is sprayed with insecticide, the amount of time that the crop actually stays free of bugs is a randomly varying quantity. The probability density function on this "bug-free" length of time is given by

$$f(t) = \begin{cases} Ct & \text{if } 0 \leq t \leq 1, \\ \dfrac{C}{t^3} & \text{if } t > 1, \\ 0 & \text{otherwise,} \end{cases}$$

   where t is measured in weeks, and C is a positive constant.

   (a) Find C, using the fact that f is a probability density function.

   (b) What is the probability that the crop's bug-free time lasts no more than 3 weeks?

   (c) Find the mean amount of time that the crop is free of bugs.

2. [2005 Exam] For this problem, use the following information about any normal ("bell-shaped" or "Gaussian") probability density function f:

   - f has the general form $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$
   - $\displaystyle\int_{-\infty}^{\mu+\sigma} f(x)dx \approx .84$
   - $\displaystyle\int_{-\infty}^{\mu+2\sigma} f(x)dx \approx .98$

   Suppose that a quality-assurance tester has determined that the amount of sodium in a bottle of Babbling Brook Spring Water is a random variable, having a normal distribution with mean 10 mg and standard deviation 1 mg.

   Using the above facts, find the probability that a randomly selected bottle of Babbling Brook contains between 8 and 9 mg of sodium. Justify your answer by writing an integral expression that represents this probability and showing how to evaluate this integral.

3. [2005 Exam] The age of a citizen in the country of Bishkadu is a random variable that can be closely modeled by the following probability density function:

$$f(t) = \begin{cases} C & \text{if } 0 \le t \le 24, \\ Ce^{-(t-24)/16} & \text{if } t \ge 24, \\ 0 & \text{otherwise,} \end{cases}$$

where t is measured in years, and C is a positive constant.

(a) Find C, using the fact that f is a probability density function.

(b) Find $\int_{24}^{\infty} f(t)dt$ and state the meaning of your result in the context of this application.

(c) What is the median age of a citizen of Bishkadu? Show all reasoning.

4. [2000 Exam] In 1950, Daniel Furlough and Frank Barnes recorded the time gaps between successive cars on the Arroyo Seco Freeway in the Los Angeles, California, region. Their data showed that the probability density function on these time gaps was given approximately by

$$p(t) = \begin{cases} 0 & \text{if } t < 0, \\ ae^{-0.122t} & \text{if } t \ge 0, \end{cases}$$

where t is the time in seconds and a is a constant.

(a) Find a.

(b) Calculate the integral

$$\int_{-\infty}^{t} p(u)du$$

and give the result in terms of t. (This function is sometimes known as p's *cumulative density function.*)

(c) Find the median time gap. Show all of your steps, with justification.

(d) Find the mean time gap. Show all of your steps, with justification.

5. [2000 Exam] Consider a group of people that has received treatment for a disease. Let t be the *survival time*, the number of years a person lives after receiving treatment. The probability density function is given by

$$p(t) = \begin{cases} 0 & \text{if } t < 0, \\ Ce^{-Ct} & \text{if } t \ge 0, \end{cases}$$

where C is a fixed positive constant.

(a) What is the practical meaning of the expression $\int_0^t p(x)dx$?

(b) Compute the above expression, giving your answer in terms of t (and the constant C).

(c) The survival function $S(t)$ is the probability that a randomly selected person survives *at least* t years after treatment. Find an expression for the function $S(t)$ and simplify this expression as much as possible.

6. [2004 Exam] For this problem, use the following information about any normal (Gaussian) probability density function f:

   - f has the general form $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
   - $\int_{\mu}^{\mu+\sigma} f(x)dx \approx .34$
   - $\int_{\mu}^{\mu+2\sigma} f(x)dx \approx .48$

   (a) Suppose that a random variable x has a normal probability density function, with mean equal to 60 and standard deviation equal to 20. Write down an integral representing the probability that the random variable will be between 40 and 100. Your final answer should not include the symbols μ or σ.

   (b) Use above information to find the approximate numerical value of the integral in part (a).

7. [2001 Exam] A 1999 workshop held by Stanford's Alliance for Innovation in Manufacturing (AIM) addressed the problem of how and why lines form, and how to handle lines. Stanford Professor J. Michael Harrison, of the Graduate School of Business, gave the following advice: "The delivery time you quote to your customers shouldn't be your average performance, but rather a delivery time you can hope to achieve 95 percent of the time." (Stanford Report, May 19, 1999.) Keeping this advice in mind, let f (defined below) be the distribution function of waiting times of customers, where c is a positive constant:

$$f(t) = \begin{cases} 0 & \text{if } t < 0, \\ ce^{-ct} & \text{if } t \geq 0 \end{cases}$$

   (a) Calculate the mean waiting time. That is, calculate the mean μ of the distribution function f. Give a complete mathematical justification, showing as much work as possible.

   (b) Find the time s by which the probability that a customer has been served is 95 percent (that is, 0.95). Give a complete mathematical justification, showing as much work as possible.

8. [2001 Exam] The probability density function for the lifespan of a certain species of plant in a given environment is estimated to be

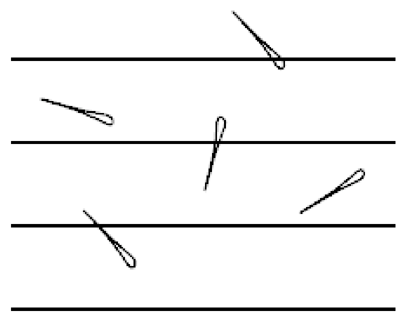
$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ \frac{1}{100}e^{-x/100} & \text{if } x \geq 0, \end{cases}$$

   where x is measured in days.

   (a) Write down an expression that represents the probability that the plant will die within 50 days. You do not need to evaluate this expression.

   (b) Determine the mean lifespan of the plants. Show all of your work, with full mathematical justification.

9. [2006 Exam] In the original game of Pick Up Sticks, invented by the French naturalist Buffon, a bunch of toothpicks are randomly tossed onto a tabletop that is marked with horizontal parallel lines, and the "successful" tosses are the toothpicks that land touching a line.



Buffon, who was interested in the *angles* at which successful toothpicks are found to lie, discovered that the angles x of successful toothpicks have the following probability distribution:

$$h(x) = \begin{cases} C \sin x & \text{if } 0 \le x \le \pi \\ 0 & \text{otherwise.} \end{cases}$$

where C is a positive constant.

(a) Find C, given that h is a probability density function.

(b) Find the mean angle of a successful toothpick, with full mathematical justification.

10. [2006 Exam] For this problem, use the following information about any normal ("bell-shaped" or "Gaussian") probability density function f:

- f has the general form $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$
- $\displaystyle\int_{-\infty}^{\mu+\sigma} f(x)dx \approx .84$
- $\displaystyle\int_{-\infty}^{\mu+2\sigma} f(x)dx \approx .98$

In 1959, NASA completed an exhaustive search for candidates to fly in the Mercury program, the nation's inaugural manned space missions. Among the many requirements was that the prospective astronaut be an adult male of height between 58 and 76 inches.

Suppose that, in 1959, the height of an American adult male could be modeled by a random variable with a normal distribution, with mean 70 in and standard deviation 6 in.

Using the above facts and assumptions, find the proportion of adult males in 1959 who would have passed the height requirement for the astronaut program; in other words, find the probability that a randomly selected American adult male had a height between 58 and 76 inches. Justify your answer by writing an integral expression that represents this probability and showing how to evaluate this integral.