

(1a)

This is an improper integral, since the integrand is undefined at  $x=0$ .

$$\int_0^1 x^{-1/5} dx = \lim_{t \rightarrow 0^+} \int_t^1 x^{-1/5} dx$$

$$= \lim_{t \rightarrow 0^+} \left[ \frac{5}{4} x^{4/5} \right]_t^1 = \frac{5}{4} \lim_{t \rightarrow 0^+} (1 - t^{4/5})$$

$$= \frac{5}{4}$$

(1b)

$$\int_0^2 \frac{1}{\sqrt{x}} dx$$

$$= \lim_{t \rightarrow 0^+} \int_t^2 x^{-1/2} dx$$

$$= \lim_{t \rightarrow 0^+} [2x^{1/2}]_t^2$$

$$= \lim_{t \rightarrow 0^+} [2\sqrt{2} - 2\sqrt{t}]$$

$$= \boxed{2\sqrt{2}}$$

② If  $x \geq 3 > e$ , then  $\ln x > 1$ , so

$\frac{\ln x}{\sqrt{x}} > \frac{1}{\sqrt{x}} > 0$ , so we check if  $\int_3^{\infty} \frac{1}{\sqrt{x}} dx$

diverges.

$$\int_3^{\infty} \frac{1}{\sqrt{x}} dx = \lim_{t \rightarrow \infty} \int_3^t x^{-1/2} dx = \lim_{t \rightarrow \infty} \left[ 2x^{1/2} \right]_3^t$$

$$= 2 \lim_{t \rightarrow \infty} (t^{1/2} - \sqrt{3}) = \infty.$$

Since  $\int_3^{\infty} \frac{1}{\sqrt{x}} dx$  diverges, and  $\frac{\ln x}{\sqrt{x}} > \frac{1}{\sqrt{x}} > 0$

for  $x \geq 3$ , the Comparison Theorem tell us

that  $\int_3^{\infty} \frac{\ln x}{\sqrt{x}} dx$  diverges also.

(3) Since  $|\cos(x)| \leq 1$ ,  $\cos^2(x) \leq 1$  for all  $x$ ,

$$\text{So } \frac{\cos^2 x}{x^3} \leq \frac{1}{x^3}.$$

$$\text{Now, } \int_1^{\infty} \frac{1}{x^3} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^3} dx$$

$$= \lim_{t \rightarrow \infty} \int_1^t x^{-3} dx$$

$$= \lim_{t \rightarrow \infty} \left[ -\frac{1}{2x^2} \right]_1^t$$

$$= \lim_{t \rightarrow \infty} \left[ -\frac{1}{2t^2} + \frac{1}{2} \right]$$

$$= \frac{1}{2}.$$

So  $\int_1^{\infty} \frac{1}{x^3} dx$  is convergent, and by

the comparison test, this means that

$$\int_1^{\infty} \frac{\cos^2(x)}{x^3} dx \text{ converges.}$$

(5a) True. See Example 4 on p. 426.

(5b) True. The assumption means that the radius of convergence of

$$\sum c_n x^n \text{ is } < 2.$$

(5c) True.

$$(6) a) \frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$$

$$1 = A(n+2) + Bn$$

$$n=0: 2A=1$$

$$n=-2: -2B=1$$

$$\frac{1}{n(n+2)} = \frac{1}{2} \left[ \frac{1}{n} - \frac{1}{n+2} \right].$$

$$S_1 = \frac{1}{2} \left[ 1 - \frac{1}{3} \right]$$

$$S_2 = \frac{1}{2} \left[ 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} \right]$$

$$S_3 = \frac{1}{2} \left[ 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} \right]$$

$$S_4 = \frac{1}{2} \left[ 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} \right]$$

$$S_5 = \frac{1}{2} \left[ 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \frac{1}{5} - \frac{1}{7} \right]$$

Now we can start to see the pattern of what's left.

$$S_n = \frac{1}{2} \left[ 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right].$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{1}{2} \left[ 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right]$$

$$= \frac{1}{2} \left( 1 + \frac{1}{2} \right) = \frac{3}{4}.$$

$$b) \sum_{k=1}^{\infty} \frac{2^{k-1}}{5^{k+1}} - 3 \sum_{k=1}^{\infty} \frac{1}{5^{k+1}}$$

$$= \left( \frac{1}{25} + \frac{2}{5^3} + \frac{4}{5^4} + \dots \right) - 3 \left( \frac{1}{25} + \frac{1}{5^3} + \frac{1}{5^4} + \dots \right)$$

$a = \frac{1}{25}, r = \frac{2}{5}$   $a = \frac{1}{25}, r = \frac{1}{5}$

$$= \frac{\frac{1}{25}}{1 - \frac{2}{5}} - 3 \frac{\frac{1}{25}}{1 - \frac{1}{5}} = \frac{5}{3} \cdot \frac{1}{25} - 3 \cdot \frac{5}{4} \cdot \frac{1}{25}$$

$$= \frac{1}{15} - \frac{3}{20} = -\frac{5}{60} = -\frac{1}{12}$$

~~$\frac{1}{15} - \frac{3}{20} = \frac{4}{60} - \frac{9}{60} = -\frac{5}{60}$~~

(7) a)  $\lim_{n \rightarrow \infty} e^{-\frac{1}{n}} = e^0 = 1.$

So  $\sum e^{-\frac{1}{n}}$  diverges by the Divergence Test.

b)  $\frac{2n}{(n+3)^{3/2}}$  looks similar to  $\frac{2n}{n^{3/2}} = \frac{2}{\sqrt{n}}.$

$$\lim_{n \rightarrow \infty} \frac{\frac{2n}{(n+3)^{3/2}}}{\frac{2}{\sqrt{n}}} = \lim_{n \rightarrow \infty} \frac{n^{3/2}}{(n+3)^{3/2}} = \left( \lim_{n \rightarrow \infty} \frac{n}{n+3} \right)^{3/2}$$

$$= 1.$$

So by the Limit Comparison Test,

$$\sum_{n=1}^{\infty} \frac{2n}{(n+3)^{3/2}} \text{ does the same thing as } \sum_{n=1}^{\infty} \frac{2}{\sqrt{n}}.$$

Since  $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$  is a divergent p-series,

the series  $\sum_{n=1}^{\infty} \frac{2n}{(n+3)^{3/2}}$  diverges.

c) Let  $f(x) = xe^{-x}$ .

$$f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x} < 0$$

if  $x > 1$ .

So  $f$  is a continuous, decreasing, positive function. By the Integral Test,  $\sum_{n=1}^{\infty} ne^{-n}$  does the same thing as  $\int_1^{\infty} xe^{-x} dx$ .

$$\int xe^{-x} dx \quad \begin{array}{l} u = x \quad du = dx \\ dv = e^{-x} dx \quad v = -e^{-x} \end{array}$$

$$= -xe^{-x} + \int e^{-x} dx = -xe^{-x} - e^{-x} + C.$$

$$\text{So } \int_1^{\infty} xe^{-x} dx = \lim_{t \rightarrow \infty} [-xe^{-x} - e^{-x}]_1^t$$

$$= - \lim_{t \rightarrow \infty} \left( \frac{t}{e^t} + \frac{1}{e^t} - \cancel{2e^{-1}} \right)$$

$$= - \lim_{t \rightarrow \infty} \frac{t}{e^t} + 2e^{-1} \quad \frac{\infty}{\infty} : \text{ use l'Hôpital}$$

$$= - \lim_{t \rightarrow \infty} \frac{1}{e^t} + 2e^{-1} = 2e^{-1}.$$

Since the integral converges, so does the series.

d)  $\frac{3}{k^2+7} < \frac{3}{k^2}$ , and  $3 \sum_{k=1}^{\infty} \frac{1}{k^2}$  is  
a convergent p-series. So the  
series converges by the Comparison Test.

⑧ a)  $f(x) = \frac{1}{x \ln x} > 0$  and is clearly decreasing.

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \int_{\ln 2}^{\infty} \frac{1}{u} du \text{ diverges.}$$

$u = \ln x \quad du = \frac{1}{x} dx$

So by the Integral Test,

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} \text{ diverges also.}$$

b) Since  $\lim_{n \rightarrow \infty} \frac{n^3 + 3n + 2}{n^3 + 6} = 1$ , the terms of the series oscillate between being about 1 and about -1. So

$\lim_{n \rightarrow \infty} (-1)^n \frac{n^3 + 3n + 2}{n^3 + 6}$  does not exist,

so by the Divergence Test,  $\sum_{n=0}^{\infty} (-1)^n \frac{n^3 + 3n + 2}{n^3 + 6}$  diverges.

c) Solution I:  $\frac{n}{2^n(n+1)} < \frac{n+1}{2^n(n+1)} = \frac{1}{2^n}$

(or  $\frac{n}{2^n(n+1)} < \frac{n}{2^n n} = \frac{1}{2^n}$ ).

Since  $\sum_{n=1}^{\infty} \frac{1}{2^n}$  is a convergent geometric

series,  $\sum_{n=1}^{\infty} \frac{n}{2^n(n+1)}$  converges by the

Comparison Test.

Solution II:  $\lim_{n \rightarrow \infty} \frac{\left(\frac{n}{2^n(n+1)}\right)}{\left(\frac{1}{2^n}\right)} = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 > 0$ ,

and  $\sum_{n=1}^{\infty} \frac{1}{2^n}$  is a convergent geometric series,

so  $\sum_{n=1}^{\infty} \frac{n}{2^n(n+1)}$  converges by the Limit Comparison Test.



### Solution III:

$$\lim_{n \rightarrow \infty} \frac{\left(\frac{n+1}{2^{n+1}(n+2)}\right)}{\left(\frac{n}{2^n(n+1)}\right)} = \lim_{n \rightarrow \infty} \frac{2^n}{2^{n+1}} \frac{(n+1)^2}{n(n+2)}$$

$$= \frac{1}{2} \lim_{n \rightarrow \infty} \frac{(n+1)^2}{n(n+2)} = \frac{1}{2} < 1,$$

So  $\sum_{n=1}^{\infty} \frac{n}{2^n(n+1)}$  converges by the Ratio Test.

9 a)  $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{4^n (2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{\pi}{2}\right)^{2n} = \cos\left(\frac{\pi}{2}\right) = 0.$

b)  $\sum_{n=1}^{\infty} \frac{1-2^n}{4^n} = \sum_{n=1}^{\infty} \left[ \left(\frac{1}{4}\right)^n - \left(\frac{1}{2}\right)^n \right]$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n - \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$$

$a = \frac{1}{4}, r = \frac{1}{4}$                        $a = \frac{1}{2}, r = \frac{1}{2}$

$$= \frac{\frac{1}{4}}{1 - \frac{1}{4}} - \frac{\frac{1}{2}}{1 - \frac{1}{2}}$$

$$= \frac{1}{3} - 1 = -\frac{2}{3}.$$

⑩ Begin with the Ratio Test:

$$\lim_{n \rightarrow \infty} \left| \frac{\frac{(-2)^{n+1}}{\sqrt{n+1}} (x+3)^{n+1}}{\frac{(-2)^n}{\sqrt{n}} (x+3)^n} \right| = \lim_{n \rightarrow \infty} \left| (-2)(x+3) \frac{\sqrt{n}}{\sqrt{n+1}} \right|$$

$$= 2|x+3| \lim_{n \rightarrow \infty} \sqrt{\frac{n}{n+1}} = 2|x+3|.$$

So we get convergence for  $2|x+3| < 1$ ,

i.e. if  $|x+3| < \frac{1}{2}$ , or  $-\frac{7}{2} < x < -\frac{5}{2}$ .

We still need to check  $x = -\frac{7}{2}, -\frac{5}{2}$ .

$$x = -\frac{7}{2}: \sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n}} \left(-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ diverges}$$

(p-series with  $p = \frac{1}{2}$ ).

$$x = -\frac{5}{2}: \sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n}} \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}.$$

Since  $\frac{1}{\sqrt{n}}$  decreases to 0, this converges by the Alternating Series Test.

So the interval of convergence is  $\left(-\frac{7}{2}, -\frac{5}{2}\right]$ .

$$\textcircled{11} \quad \frac{x}{2+x} = x \cdot \frac{1}{2+x} = x \cdot \frac{1}{2(1+\frac{x}{2})}$$

$$= \frac{x}{2} \cdot \frac{1}{1-(-\frac{x}{2})} = \frac{x}{2} \cdot \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n \quad \text{for } |-\frac{x}{2}| < 1$$

or:  $|x| < 2$

$$= \frac{x}{2} \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+1}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n} x^n$$

↑  
this is fine

with  $R=2$ .

$$\textcircled{12} \text{ a) } \sin^{-1} x + C = \int \frac{1}{\sqrt{1-x^2}} dx$$

$$= \int \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2} x^{2n} dx$$

$$= \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2} \frac{x^{2n+1}}{2n+1}$$

Plug in  $x=0$  to find  $C$ :

$$0 + C = 0 \quad \text{so } C = 0.$$

$$\sin^{-1} x = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (2n+1) (n!)^2} x^{2n+1} \quad \text{for } -1 < x < 1.$$

$$b) \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}, \text{ so}$$

$$\frac{\pi}{6} = \sin^{-1}\left(\frac{1}{2}\right) = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (2n+1)(n!)^2} \left(\frac{1}{2}\right)^{2n+1}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(2n)!}{16^n (2n+1)(n!)^2}$$

$$\text{So } \pi = \sum_{n=0}^{\infty} \frac{3 \cdot (2n)!}{16^n (2n+1)(n!)^2}$$

$$\textcircled{13} \lim_{x \rightarrow 0} \frac{e^{x^2} - 1}{\cos x - 1} = \lim_{x \rightarrow 0} \frac{(1 + x^2 + \frac{1}{2}x^4 + \dots) - 1}{(1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots) - 1}$$

$$= \lim_{x \rightarrow 0} \frac{x^2 + \frac{1}{2}x^4 + \dots}{-\frac{x^2}{2} + \frac{x^4}{24} + \dots}$$

$$= \lim_{x \rightarrow 0} \frac{1 + \frac{1}{2}x^2 + \dots}{-\frac{1}{2} + \frac{x^2}{24} + \dots} = \frac{1}{(-\frac{1}{2})} = -2.$$

$$\textcircled{14} \text{ a) } f(x) = x^{4/3} \quad f(27) = 3^4 = 81$$

$$f'(x) = \frac{4}{3} x^{1/3} \quad f'(27) = \frac{4}{3} \cdot 3 = 4$$

$$f''(x) = \frac{4}{9} x^{-2/3} \quad f''(27) = \frac{4}{9} \cdot 3^{-2} = \frac{4}{81}$$

$$f^{(3)}(x) = -\frac{8}{27} x^{-5/3} \quad f^{(3)}(27) = -\frac{8}{27} 3^{-5} = -\frac{8}{3^9}$$

$$f^{(4)}(x) = \frac{40}{81} x^{-8/3}$$

$$T_3(x) = f(27) + f'(27)(x-27) + \frac{f''(27)}{2}(x-27)^2 + \frac{f^{(3)}(27)}{6}(x-27)^3$$

$$= 81 + 4(x-27) + \frac{2}{81}(x-27)^2 - \frac{4}{3^9}(x-27)^3$$

b)  $25 \leq x \leq 29$  means  $|x-27| \leq 2$ .

Since  $f^{(4)}$  is decreasing,

$$|f^{(4)}(x)| = \frac{40}{81} x^{-8/3} \leq \frac{40}{81} \cdot 25^{-8/3} \text{ for } |x-27| \leq 2.$$

So by Taylor's Inequality,

$$|R_3(x)| \leq \frac{1}{24} \cdot \frac{40}{81} 25^{-8/3} \cdot 2^4, \text{ whatever that is.}$$