Math 20 - Winter 2005 - Final Exam Solutions

1. (15 points) Evaluate the following definite integrals.

(a) $\int_0^2 x^2 e^{x^3}\,dx$

Let $u = x^3$, $du = 3x^2 dx \Rightarrow \frac{1}{3}du = x^2 dx$.

Also, $x=0 \Rightarrow u=0$, $x=2 \Rightarrow u=8$.

So $$\int_0^2 x^2 e^{x^3} dx = \int_0^8 e^u \cdot \tfrac{1}{3} du$$

$$= \frac{1}{3}\int_0^8 e^u du = \frac{1}{3} e^u \Big]_{u=0}^{u=8} = \boxed{\frac{1}{3}(e^8 - 1)}$$

(b) $\int_0^1 x^2 e^{3x}\,dx$

Int.-by-Parts: Let $u = x^2$ $\quad du = 2x\,dx$

$dv = e^{3x}dx \quad v = \frac{1}{3}e^{3x}$

So $$\int_0^1 x^2 e^{3x} dx = x^2 \cdot \tfrac{1}{3}e^{3x}\Big]_0^1 - \int_0^1 \tfrac{2}{3} x e^{3x} dx$$

$$= \tfrac{1}{3}x^2 e^{3x}\Big]_0^1 - \tfrac{2}{3}\underbrace{\int_0^1 x e^{3x} dx}_{\text{need parts!}}$$

New int-by-parts: $f = x \quad df = dx$

$dg = e^{3x}dx \quad g = \frac{1}{3}e^{3x}$

So $$\int_0^1 x^2 e^{3x} dx = \tfrac{1}{3}x^2 e^{3x}\Big]_0^1 - \tfrac{2}{3}\left(\tfrac{x}{3}e^{3x}\Big]_0^1 - \int_0^1 \tfrac{1}{3}e^{3x} dx\right)$$

$$= \left(\tfrac{1}{3}x^2 e^{3x} - \tfrac{2}{9}x e^{3x}\right)\Big]_0^1 + \tfrac{2}{9}\int_0^1 e^{3x} dx$$

$$= \left(\tfrac{1}{3}x^2 e^{3x} - \tfrac{2}{9}x e^{3x} + \tfrac{2}{27}e^{3x}\right)\Big]_0^1 = \boxed{\left(\tfrac{1}{3} - \tfrac{2}{9} + \tfrac{2}{27}\right)e^3 - \tfrac{2}{27}}$$

$\boxed{2.}$ (30 points) Evaluate the following integrals, showing all of your work.

(a) $\displaystyle\int \frac{x^3}{\sqrt{1-x^2}}\,dx$

Let $u = 1-x^2$, so $du = -2x\,dx$.

Then $\displaystyle\int \frac{x^3}{\sqrt{1-x^2}}\,dx = \int \frac{x^2 \cdot x\,dx}{\sqrt{1-x^2}} = \int \frac{(1-u)\cdot \frac{du}{-2}}{\sqrt{u}}$

$$= -\frac{1}{2}\int \frac{1-u}{\sqrt{u}}\,du$$

$$= -\frac{1}{2}\int\left(\frac{1}{\sqrt{u}} - \frac{u}{\sqrt{u}}\right)du$$

$$= -\frac{1}{2}\int\left(u^{-1/2} - u^{1/2}\right)du$$

$$= -u^{1/2} + \frac{1}{3}u^{3/2} + C = \boxed{-\sqrt{1-x^2} + \frac{1}{3}(1-x^2)^{3/2} + C}$$

Note: there are other equivalent answers that arise using other solution methods (like integration by parts).

(b) $\displaystyle\int \sin^3 x \cos^3 x\,dx$

Write $\cos^2 x = 1-\sin^2 x$, so that $\displaystyle\int \sin^3 x\cos^3 x = \int \sin^3 x\cdot(1-\sin^2 x)\cos x\,dx.$

Now let $u = \sin x$,
and so $du = \cos x\,dx$.

Then $\displaystyle\int \sin^3 x(1-\sin^2 x)\cos x = \int u^3\cdot(1-u^2)\,du = \int(u^3-u^5)\,du$

$$= \frac{u^4}{4} - \frac{u^6}{6} + C$$

$$= \boxed{\frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + C}$$

(c) $\int x^2(\ln x)^2\,dx$

Int-by-parts: $u=(\ln x)^2 \quad du = 2\ln x \cdot \frac{1}{x}\,dx$

$dv = x^2\,dx \quad v = \frac{x^3}{3}$

So, $\int x^2(\ln x)^2 dx = \frac{x^3}{3}(\ln x)^2 - \int \frac{2}{3}x^2 \ln x\,dx = \frac{x^3}{3}(\ln x)^2 - \frac{2}{3}\int x^2 \ln x\,dx.$

Second int-by-parts: $u_2 = \ln x \quad du_2 = \frac{1}{x}dx$

$dv_2 = x^2 dx \quad v_2 = \frac{x^3}{3}$

So, $\frac{x^3}{3}(\ln x)^2 - \frac{2}{3}\int x^2 \ln x\,dx = \frac{x^3}{3}(\ln x)^2 - \frac{2}{3}\left[\frac{x^3}{3}\ln x - \int \frac{x^2}{3}dx\right]$

$= \frac{x^3}{3}(\ln x)^2 - \frac{2x^3}{9}\ln x + \frac{2}{9}\int x^2 dx = \boxed{\frac{x^3}{3}(\ln x)^2 - \frac{2}{9}x^3 \ln x + \frac{2}{27}x^3 + C}$

(d) $\int \sin^4 x\,dx$

Use $\sin^2 x = \frac{1}{2}(1-\cos 2x)$: $\quad \int \sin^4 x\,dx = \int \left[\frac{1}{2}(1-\cos 2x)\right]^2 dx$

$= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\,dx.$

Now use $\cos^2 2x = \frac{1}{2}(1+\cos 4x)$:

(Here $t = 2x$)

So, $\frac{1}{4}\int(1-2\cos 2x + \cos^2 2x)dx = \frac{1}{4}\int\left(1 - 2\cos 2x + \frac{1}{2}(1+\cos 4x)\right)dx$

$= \frac{1}{4}\int dx - \frac{1}{2}\int \cos 2x\,dx + \frac{1}{8}\int dx + \frac{1}{8}\int \cos 4x\,dx$

$= \frac{x}{4} - \frac{\sin 2x}{4} + \frac{x}{8} + \frac{\sin 4x}{32} + C$

$= \boxed{\frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C}$

$\boxed{3.}$ (32 points) Evaluate the following integrals, showing all of your work.

(a) $\displaystyle\int_4^{12} \frac{x}{\sqrt{1+2x}}\,dx$

Let $u = 1+2x$,

so $du = 2dx$, i.e. $\frac{du}{2} = dx$.

Also, $x = \frac{u-1}{2}$,

and $\begin{cases} \text{if } x=12 \text{ then } u = 1+2\cdot 12 = 25, \\ \text{if } x=4 \text{ then } u = 1+2\cdot 4 = 9. \end{cases}$

$$\int_4^{12} \frac{x}{\sqrt{1+2x}}\,dx = \int_9^{25} \frac{\left(\frac{u-1}{2}\right)}{\sqrt{u}} \cdot \frac{du}{2}$$

$$= \frac{1}{4}\int_9^{25} \frac{u-1}{\sqrt{u}}\,du$$

$$= \frac{1}{4}\int_9^{25} \left(u^{1/2} - u^{-1/2}\right) du$$

$$= \frac{1}{4}\left(\frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2}\right)\Bigg|_9^{25}$$

$$= \frac{1}{4}\left(\frac{2}{3}\cdot 25^{3/2} - 2\cdot 25^{1/2} - \left(\frac{2}{3}\cdot 9^{3/2} - 2\cdot 9^{1/2}\right)\right)$$

$$= \frac{1}{4}\left(\frac{2}{3}\cdot 125 - 2\cdot 5 - \left(\frac{2}{3}\cdot 27 - 2\cdot 3\right)\right) = \boxed{\frac{46}{3}}.$$ (Simplifying unnecessary)

(<u>Note</u>: could also have solved using integration by parts.)

(b) $\int \frac{\ln x}{x^{1/3}}\,dx$

Integration by parts:

$u = \ln x \qquad du = \frac{dx}{x}$

$dv = x^{-1/3}dx \qquad v = \frac{3}{2}x^{2/3}$

So,

$$\int \frac{\ln x}{x^{1/3}}dx = \frac{3}{2}x^{2/3}\cdot \ln x - \int \frac{3}{2}x^{2/3}\cdot\frac{1}{x}\,dx$$

$$= \frac{3}{2}x^{2/3}\cdot \ln x - \frac{3}{2}\int x^{-1/3}dx$$

$$= \frac{3}{2}x^{2/3}\ln x - \frac{3}{2}\cdot\frac{3}{2}x^{2/3} + C$$

$$= \boxed{\frac{3}{2}x^{2/3}\ln x - \frac{9}{4}x^{2/3} + C}.$$

(c) $\int \sqrt{t}\, e^{\sqrt{t}}\, dt$

Substitution: let $u = \sqrt{t}$, so $du = \frac{1}{2} t^{-1/2} dt$.

(Also, since $t = u^2$, have $dt = 2u\,du$.)

So, $$\int \sqrt{t} e^{\sqrt{t}}\, dt = \int u e^u \cdot (2u\,du) = 2\int u^2 e^u\, du.$$

Now, integration by parts. Let $f = u^2$, $g' = e^u$, so: $f' = 2u$, $g = e^u$,

and $$\int \sqrt{t} e^{\sqrt{t}} dt = 2\int u^2 e^u du$$
$$= 2\left[u^2 e^u - \int 2u e^u du \right]$$
$$= 2u^2 e^u - 4\int u e^u du.$$

Need another integration by parts: $h = u$, $k' = e^u$, so: $h' = 1$, $k = e^u$,

and $$\int \sqrt{t} e^{\sqrt{t}} dt = 2u^2 e^u - 4\int u e^u du$$
$$= 2u^2 e^u - 4\left[u e^u - \int e^u du \right]$$
$$= 2u^2 e^u - 4u e^u + 4\int e^u du$$
$$= 2u^2 e^u - 4u e^u + 4e^u + C$$
$$= \boxed{2te^{\sqrt{t}} - 4\sqrt{t}\, e^{\sqrt{t}} + 4e^{\sqrt{t}} + C}$$

(d) $\displaystyle\int_{-\pi/2}^{\pi/2} x\cos^2 x\,dx$

Solution #1

$$\int_{-\pi/2}^{\pi/2} x\cos^2 x\,dx = \int_{-\pi/2}^{\pi/2} x\cdot\tfrac{1}{2}(1+\cos 2x)\,dx = \frac{1}{2}\int_{-\pi/2}^{\pi/2} x\,dx + \frac{1}{2}\underbrace{\int_{-\pi/2}^{\pi/2} x\cos 2x\,dx}$$

Integration-by-Parts:

$u = x \qquad du = dx$

$dv = \cos 2x\,dx \qquad v = \frac{1}{2}\sin 2x$

$$= \frac{1}{2}\cdot\frac{x^2}{2}\Big]_{-\pi/2}^{\pi/2} + \frac{1}{2}\cdot\left(\frac{1}{2}x\sin 2x\Big]_{-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2}\frac{1}{2}\sin 2x\,dx\right)$$

$$= \left(\frac{x^2}{4} + \frac{x\sin 2x}{4} + \frac{\cos 2x}{8}\right)\Big]_{-\pi/2}^{\pi/2}$$

$$= \frac{(\pi/2)^2}{4} + \frac{(\pi/2)\sin\pi}{4} + \frac{\cos\pi}{8} - \left(\frac{(-\pi/2)^2}{4} + \frac{(-\pi/2)\sin(-\pi)}{4} + \frac{\cos(-\pi)}{8}\right)$$

$$= \boxed{0} \quad \text{(all corresponding terms cancel)}$$

Solution #2

The function $f(x) = x\cos^2 x$ is <u>odd</u>, because

$$f(-x) = (-x)\cdot\cos^2(-x) = (-x)\cdot(\cos(-x))^2$$

$$= (-x)\cdot(\cos x)^2 = -x\cos^2 x = -f(x),$$

and thus any integral of form $\int_{-a}^{a} f(x)\,dx$ is equal to zero;

we can conclude $\displaystyle\int_{-\pi/2}^{\pi/2} x\cos^2 x\,dx = \boxed{0}$.

4. (30 points) Find each of the following, showing all of your work.

(a) $\int \tan x \, dx \qquad \left(\text{Hint: } \tan x = \dfrac{\sin x}{\cos x}\right)$

Let $u = \cos x$,

so $du = -\sin x\,dx$.

Thus $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int -\frac{du}{u} = -\int \frac{du}{u}$

$$= -\ln|u| + C$$

$$= \boxed{-\ln|\cos x| + C}.$$

(b) $\int x\sqrt{4+x}\,dx$

Note — you can also use integration by parts; the answer looks a bit different than the below (but is still equivalent)

Let $u = 4+x$,

so $du = dx$ and $x = u - 4$.

Thus $\int x\sqrt{4+x}\,dx = \int (u-4)\sqrt{u}\,du$

$$= \int \left(u\sqrt{u} - 4\sqrt{u}\right) du$$

$$= \int \left(u^{3/2} - 4u^{1/2}\right) du$$

$$= \frac{u^{5/2}}{5/2} - \frac{4u^{3/2}}{3/2} + C$$

$$= \frac{(4+x)^{5/2}}{5/2} - \frac{4(4+x)^{3/2}}{3/2} + C$$

$$= \boxed{\frac{2}{5}(4+x)^{5/2} - \frac{8}{3}(4+x)^{3/2} + C}$$

(lots of equivalent answers)

(c) $\int (\sin^2 x - \cos^2 x + \sec^2 x)\, dx$

Use: $\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$\cos^2 x = \frac{1}{2}(1+\cos 2x)$

So, $\sin^2 x - \cos^2 x = \frac{1}{2}(1-\cos 2x-(1+\cos 2x)) = \frac{1}{2}(-2\cos 2x) = -\cos 2x$.

Thus $\int (\sin^2 x-\cos^2 x+\sec^2 x)dx = \int (-\cos 2x)dx + \int \sec^2 x\, dx$

$= \boxed{-\frac{\sin 2x}{2} + \tan x + C}$.

(d) $\int_1^\infty \frac{\ln x}{x^2}\, dx$

Integrate by parts: $\int u\,dv = uv - \int v\,du$

$u = \ln x$, $du = \frac{dx}{x}$

$dv = \frac{dx}{x^2}$, $v = -\frac{1}{x}$

So $\int \frac{\ln x}{x^2}dx = -\frac{\ln x}{x} + \int \frac{dx}{x^2}$

$= -\frac{\ln x}{x} + \left(-\frac{1}{x}\right) + C$.

5. Evaluate the following integrals. Use whatever method you like, but be sure to show all work.

(a) $\displaystyle\int \frac{dx}{3x(1-\frac{x}{20})}$

Clean up the constants, and set up partial fraction decomposition:

$$\frac{1}{3x(1-\frac{x}{20})} = \frac{1}{\frac{3}{20}x(20-x)} = \frac{20}{3}\cdot\frac{1}{x(20-x)} = -\frac{20}{3}\cdot\frac{1}{x(x-20)}.$$

We must solve $\displaystyle\frac{1}{x(x-20)} = \frac{A}{x} + \frac{B}{x-20}$ for A & B.

Clearing denominators by multiplying by $x(x-20)$, we get

$$1 = A(x-20) + Bx.$$

Thus, $A+B=0$ and

$-20A=1$.

It follows that $A=-\frac{1}{20}$ and $B=\frac{1}{20}$. We have

$$\int \frac{dx}{3x(1-\frac{x}{20})} = -\frac{20}{3}\int \frac{dx}{x(x-20)} = -\frac{20}{3}\int\left(\frac{-1/20}{x} + \frac{1/20}{x-20}\right)dx$$

$$= -\frac{1}{3}\int\left(\frac{-1}{x} + \frac{1}{x-20}\right)dx$$

$$= \frac{1}{3}\int \frac{1}{x}dx - \frac{1}{3}\int\frac{dx}{x-20}$$

$$= \boxed{\frac{1}{3}\ln|x| - \frac{1}{3}\ln|x-20| + C}$$

(b) $\displaystyle\int \frac{t(t^{10}+1)}{t^4+5}\,dt \quad = \int \frac{t^{11}+t}{t^4+5}\,dt$

Since degree(numerator) $\geq$ degree(denominator), we'll first divide:

$$\begin{array}{r l}
 & t^7 - 5t^3 \\
t^4+5 \,\big) & t^{11} \qquad\qquad + t \\
 & \underline{t^{11}+5t^7} \\
 & -5t^7 \\
 & \underline{-5t^7 - 25t^3} \\
 & 25t^3 + t
\end{array}$$

Thus, $\displaystyle\int \frac{t^{11}+t}{t^4+5}\,dt = \int\left(t^7 - 5t^3 + \frac{25t^3+t}{t^4+5}\right)dt.$

[Notice that partial fractions won't go anywhere since t^4+5 does not factor!]

Some initial success comes by splitting into a sum of integrals, so we do that:

$$\int\left(t^7-5t^3+\frac{25t^3+t}{t^4+5}\right)dt = \int(t^7-5t^3)\,dt + \int\frac{25t^3}{t^4+5}\,dt + \int\frac{t}{t^4+5}\,dt.$$

Can do first integral by power rule: $\displaystyle\int(t^7-5t^3)\,dt = \frac{t^8}{8} - \frac{5t^4}{4}.$

The second integral works with $\left.\begin{array}{l} u=t^4+5 \\ du=4t^3\,dt \end{array}\right\} \Rightarrow \displaystyle\int\frac{25t^3}{t^4+5}\,dt = \frac{25}{4}\int\frac{du}{u} = \frac{25}{4}\ln|t^4+5|.$

The third integral simplifies with $\left.\begin{array}{l} v=t^2 \\ dv=2t\,dt \end{array}\right\} \Rightarrow \displaystyle\int\frac{t}{t^4+5}\,dt = \frac{1}{2}\int\frac{dv}{v^2+5},$

and now let $\left.\begin{array}{l} v=\sqrt{5}\tan\theta \\ dv=\sqrt{5}\sec^2\theta\,d\theta \end{array}\right\} \Rightarrow \displaystyle\frac{1}{2}\int\frac{dv}{v^2+5} = \frac{1}{2}\int\frac{\sqrt{5}\sec^2\theta\,d\theta}{5(\tan^2\theta+1)} = \frac{1}{2}\int\frac{\sqrt{5}}{5}\,\frac{\sec^2\theta}{\sec^2\theta}\,d\theta$

$$= \frac{1}{2}\int\frac{\sqrt{5}}{5}\,d\theta = \frac{\sqrt{5}}{10}\theta.$$

Since $\theta = \arctan\left(\frac{v}{\sqrt{5}}\right)$ and $v=t^2$, this last expression equals $\frac{\sqrt{5}}{10}\arctan\left(\frac{t^2}{\sqrt{5}}\right).$

Final answer: $\displaystyle\int\frac{t^{11}+t}{t^4+5}\,dt = \boxed{\frac{t^8}{8} - \frac{5t^4}{4} + \frac{25}{4}\ln(t^4+5) + \frac{\sqrt{5}}{10}\arctan\left(\frac{t^2}{\sqrt{5}}\right) + C}.$

6. Evaluate the following integrals. Use whatever method you like, but be sure to show all work.

(a) $\displaystyle\int \frac{dx}{2x^2 - 6x}$

$$= \int \frac{dx}{2x(x-3)} = \frac{1}{2}\int \frac{dx}{x(x-3)}.$$

Set up partial fraction decomposition for $\frac{1}{x(x-3)}$: $\frac{1}{x(x-3)} = \frac{A}{x} + \frac{B}{x-3}$

(mult both sides by $x(x-3)$) $\Rightarrow$ $1 = A(x-3) + Bx$

$$\Rightarrow \begin{cases} A+B=0 \text{ and} \\ -3A = 1 \end{cases}$$

$\Rightarrow$ $A = -\frac{1}{3}$ and $B = \frac{1}{3}$.

Thus $\displaystyle\frac{1}{2}\int \frac{dx}{x(x-3)} = \frac{1}{2}\int\left(\frac{-1/3}{x} + \frac{1/3}{x-3}\right)dx$

$$= -\frac{1}{6}\int \frac{1}{x}dx + \frac{1}{6}\int \frac{1}{x-3}dx$$

$$= \boxed{-\frac{1}{6}\ln|x| + \frac{1}{6}\ln|x-3| + C}.$$

(b) $\displaystyle\int \frac{dx}{x^3\sqrt{9-x^4}}$

We suspect trig substitution plays a role, but we require radicals to be of form $\sqrt{a^2-u^2}$ and so forth. This suggests we try (after noting other simpler efforts fail) the substitution $\begin{cases} u = x^2 \\ du = 2x\,dx. \end{cases}$

Then $$\int \frac{dx}{x^3\sqrt{9-x^4}} = \int \frac{2x\,dx}{2x^4\sqrt{9-x^4}} = \int \frac{du}{2u^2\sqrt{9-u^2}}.$$

Now we can try $\begin{cases} u = 3\sin\theta \\ du = 3\cos\theta\,d\theta \end{cases}$, so

$$\int \frac{du}{2u^2\sqrt{9-u^2}} = \int \frac{3\cos\theta\,d\theta}{18\sin^2\theta\sqrt{9-9\sin^2\theta}} = \int \frac{3\cos\theta\,d\theta}{18\sin^2\theta\cdot 3\cos\theta} = \frac{1}{18}\int \csc^2\theta\,d\theta$$

$$\left(\text{Note } \frac{1}{\sin^2\theta} = \csc^2\theta\right) \qquad = -\frac{1}{18}\cot\theta + C.$$

Since $\theta = \arcsin\left(\frac{u}{3}\right)$ and $u = x^2$, we have

$$\int \frac{dx}{x^3\sqrt{9-x^4}} = -\frac{1}{18}\cot\theta + C = \boxed{\frac{-1}{18}\cot\left(\arcsin\left(\frac{x^2}{3}\right)\right) + C}.$$

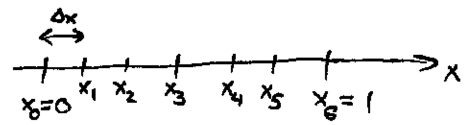
(Optional simplification: the above equals $-\frac{1}{18}\cdot\frac{\sqrt{9-x^4}}{x^2} + C$ by triangle trig.)

$\boxed{1.}$ (15 points) Consider the function f, whose formula and derivatives are given below:

$$f(x) = \frac{1}{1+x^2} \qquad f'(x) = \frac{-2x}{(1+x^2)^2} \qquad f''(x) = \frac{-2+6x^2}{(1+x^2)^3} \qquad f'''(x) = \frac{-24x(x^2-1)}{(1+x^2)^4}$$

(a) Let $J = \int_0^1 f(x)\,dx$. Write an expression involving only numbers that estimates the value of J using the Trapezoidal Rule with $n = 6$ subintervals. (You do *not* have to simplify this expression.)

$$n=6 \Rightarrow \Delta x = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}.$$



$$T_6 = \frac{\Delta x}{2}\cdot\left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + 2f(x_5) + f(x_6)\right)$$

$$= \frac{1}{12}\left(f(0) + 2f(\tfrac{1}{6}) + 2f(\tfrac{2}{6}) + 2f(\tfrac{3}{6}) + 2f(\tfrac{4}{6}) + 2f(\tfrac{5}{6}) + f(1)\right)$$

$$= \boxed{\frac{1}{12}\left(\frac{1}{1+0^2} + \frac{2}{1+(1/6)^2} + \frac{2}{1+(2/6)^2} + \frac{2}{1+(3/6)^2} + \frac{2}{1+(4/6)^2} + \frac{2}{1+(5/6)^2} + \frac{1}{1+1^2}\right)}$$

(b) Calculate the value of the "error bound" associated with the approximation above, and explain its significance in a complete sentence. Be as mathematically precise as you can in your reasoning; however, you don't have to simplify all your arithmetic. (You may find the derivatives of f provided above to be helpful.)

- Using the error-bound formula provided comes down to finding K_2 = max. value of $|f''(x)|$ for $0 \le x \le 1$.
- Finding K_2: this is an absolute extremum question, so we can use the Closed Interval Method.

  * Consider $f'''(x) = -\frac{24x(x^2-1)}{(1+x^2)^4}$, the derivative of f''. Notice that f''' can be zero only when $x=0$, $x=\pm 1$, and can never be undefined (denominator always positive), so f''' does <u>not</u> change sign for $0<x<1$, and thus f'' has no <u>local</u> maxima or minima on this interval. (So $|f''(x)|$ has no local max either.)

  * We're left to worry only about the endpoints $x=0$ & $x=1$. Values of $|f''(x)|$ there:

  at $x=0$: $|f''(0)| = \left|\frac{-2}{1^3}\right| = |-2| = 2 \leftarrow \boxed{\text{MAX}}$

  at $x=1$: $|f''(1)| = \left|\frac{-2+6}{(1+1)^3}\right| = \left|\frac{4}{8}\right| = \frac{1}{2}$

  * Thus, K_2 = maxvalue of $|f''(x)|$ on $[0,1]$ = value of $|f''(0)| = 2$.

- Thus, error bound inequality states: $|E_T| \le \frac{K_2\cdot(b-a)^3}{12n^2} = \frac{2}{12\cdot 6^2} = \frac{2}{12\cdot 36} = \frac{1}{216}$.
- In words, this is expressing the fact that the approximation from part (a) is <u>no more than</u> $\frac{1}{216}$ units different from the actual value of $J = \int_0^1 f(x)\,dx$.

$$f(x) = \frac{1}{1+x^2} \qquad f'(x) = \frac{-2x}{(1+x^2)^2} \qquad f''(x) = \frac{-2+6x^2}{(1+x^2)^3} \qquad f'''(x) = \frac{-24x(x^2-1)}{(1+x^2)^4}$$

(c) Now suppose you want to make a Trapezoidal Rule approximation of our $J = \int_0^1 f(x)\,dx$ that is sure to be within 10^{-8} of the true value. How many subintervals would you use? Explain completely; simplify your answer as much as possible.

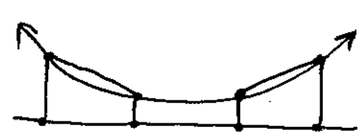
- Value of K_2 from part (b): $K_2 = 2$.
- We want n such that $\frac{K_2 \cdot (b-a)^3}{12n^2} \leq 10^{-8}$, for then we know we'd have $|E_T| \leq \frac{K_2(b-a)^3}{12n^2} \leq 10^{-8}$, which is what we want.
- Inequality we must solve: $\frac{2 \cdot 1^3}{12n^2} \leq 10^{-8}$

$$\Rightarrow \quad n^2 \geq \frac{10^8}{6} \quad \Rightarrow \quad n \geq \sqrt{\frac{10^8}{6}} = \frac{10^4}{\sqrt{6}} = \frac{10000}{\sqrt{6}}.$$

- Note that $\frac{10000}{\sqrt{6}}$ is not itself a whole number: so we need any whole number n larger than this; for instance we could take $\boxed{n = \frac{10000}{2} = 5000}$ subintervals. (In fact $n = 4100$ is okay as well.)

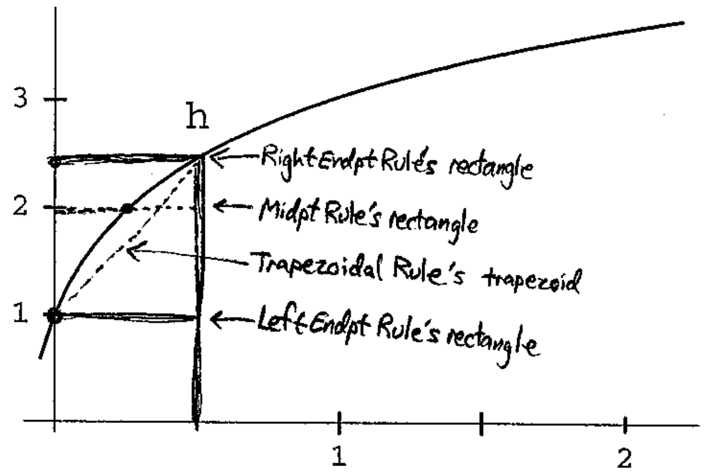
(d) Now consider an arbitrary function $g(x)$. How does the graph of g affect whether an approximation by the Trapezoidal Rule is an overestimate or an underestimate? Explain why this is so. (It might help to draw a picture, but this alone is not sufficient justification.)

The concavity of g affects the Trapezoidal Rule approximation: note from the first sketch that if g is concave down, then the non-perpendicular edges of the trapezoids lie below the curve of g, so approximation is an underestimate of actual area. (The approximating trapezoids have area that's either "less positive" or "more negative," which numerically speaking always means "less", than the area they're approximating.)



As depicted in the second sketch, the reverse is true if g is concave up: the approximation is an overestimate.

2. (15 points) Let h be the function graphed below.



(a) Four students (I, II, III, and IV) approximated the area under the graph of h from $x = 0$ to $x = 2$. They all used the same number of subintervals, but they each used a different method among the ones listed below. Here are their results:

I: 5.4386   II: 5.70486   III: 5.73442   IV: 5.97112

Which result corresponds to which method? Explain.

Method	I, II, III, or IV?	Brief reason
Left Endpoint Rule	I	For uniformly increasing functions, L. Endpt Rule gives "lowest possible" underestimate among these rules.
Right Endpoint Rule	IV	For uniformly increasing functions, R. Endpt Rule gives "highest possible" overestimate among these rules.
Midpoint Rule	III	For conc. dn. funcs, rule gives an overestimate of actual area (and will be greater than trap. rule, which is underest)
Trapezoidal Rule	II	Is equal to the average of L&R Endpt Rules (Also, for conc. dn. funcs, will be less than midpt rule)

(b) Write an expression, involving h evaluated at specific numbers, that represents the Simpson's Rule approximation to the area $\int_0^2 h(x)\,dx$ using $n = 8$ subintervals.

First, $\Delta x = \frac{2-0}{8} = \frac{1}{4}$. Have $x_0 = 0,\ x_1 = \frac{1}{4},\ x_2 = \frac{1}{2},\ \ldots,\ x_8 = 2$.

So, $$S_8 = \frac{\Delta x}{3}\Big(h(x_0)+4h(x_1)+2h(x_2)+4h(x_3)+2h(x_4)+4h(x_5)+2h(x_6)+4h(x_7)+h(x_8)\Big)$$
$$= \frac{1}{12}\Big(h(0)+4h(\tfrac{1}{4})+2h(\tfrac{1}{2})+4h(\tfrac{3}{4})+2h(1)+4h(\tfrac{5}{4})+2h(\tfrac{3}{2})+4h(\tfrac{7}{8})+h(2)\Big).$$

(c) Suppose you know that for all x on the interval $[0, 2]$,

$$|h''(x)| \le 48 \qquad \text{and} \qquad |h^{(4)}(x)| \le 18000.$$

Which approximation rule, *using how many subintervals* n, would you use to approximate $\int_0^2 h(x)\,dx$ to be sure that you are accurate to within 10^{-6} units? Explain completely; simplify your answer as much as possible.

Let's choose Simpson's Rule, and see how many subintervals are needed. (other solutions can use other rules, this is only a sample.)

Need $|E_S| < 10^{-6}$, and so if we choose n so that $\frac{K_4 \cdot (b-a)^5}{180\, n^4} < 10^{-6}$,

we'll have that $|E_S| \le \frac{K_4 (b-a)^5}{180\, n^4} < 10^{-6}$ as desired.

By above we can take $K_4 = 18000$. Since $(b-a)^5 = 2^5 = 32$, we have

$$\frac{18000 \cdot 32}{180 \cdot n^4} < 10^{-6} \Rightarrow \frac{n^4}{3200} > 10^6$$

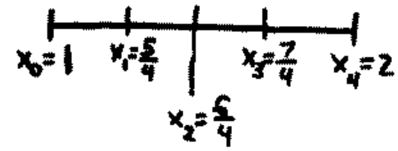
$$\Rightarrow n^4 > 32 \cdot 10^8 \Rightarrow n > \sqrt[4]{32} \cdot 10^2 = 200 \cdot \sqrt[4]{2}.$$

So we can take any even number n greater than $200 \cdot \sqrt[4]{2}$ (for example, take $n = 300$, since certainly $1.5 > \sqrt[4]{2}$).

$\boxed{3.}$ (16 points) Let $J = \int_1^2 \frac{dx}{x}$. ✪ Note: J is a number. It's the area under $f(x) = \frac{1}{x}$ from $x=1$ to $x=2$.

(a) Write a sum of numbers that estimates the value of J using the Trapezoidal Rule with $n = 4$. (You do not have to simplify this sum.)

$\Delta x = \frac{2-1}{4} = \frac{1}{4}$. Interval $[1,2]$ divided into 4 parts:

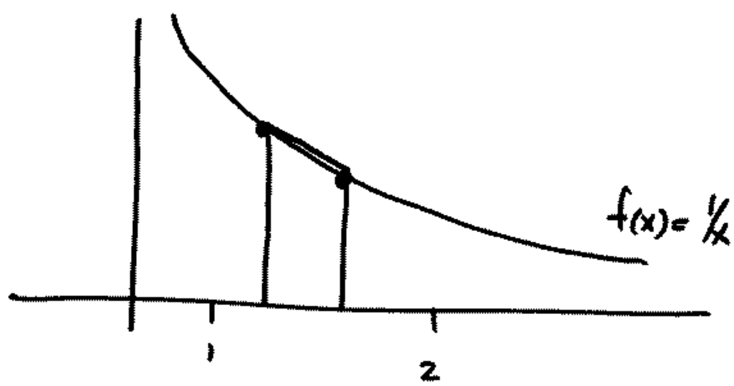


$$\Rightarrow T_4 = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$

$$= \frac{1}{8}\left[\frac{1}{1} + 2\cdot\frac{1}{5/4} + 2\cdot\frac{1}{6/4} + 2\cdot\frac{1}{7/4} + \frac{1}{2}\right]$$

$$= \frac{1}{8}\left[1 + \frac{8}{5} + \frac{8}{6} + \frac{8}{7} + \frac{1}{2}\right] \quad \text{(or equivalent)}$$

(b) Is the approximation in part (a) an overestimate or an underestimate of the true value of J? Explain. (It might help to draw a picture, but this alone is not sufficient justification.)



Any trapezoid drawn with top edge connecting two points of the curve $y = 1/x$ [i.e., any trapezoid used in the Trapezoidal Approximation] will have area larger than the corresponding area under the curve, since the top edge will always lie above the curve. The reason for this is that the curve $y = 1/x$ is concave up. This results in a Trapezoidal approximation that is always an overestimate of J.

(c) Calculate the value of the "error bound" associated to the approximation in part (a), and explain its significance in a sentence. Show all of your reasoning.

Trapezoidal error bound fmla: $|E_T| \le \frac{K_2(b-a)^3}{12n^2}$

Need value of K_2, the max. value of $|f''(x)|$ on interval $[1,2]$. (Recall $f(x)=\frac{1}{x}$.)

$f'(x) = -\frac{1}{x^2}$, so $f''(x) = 2x^{-3}$. This is an inverse-power function, positive and decreasing on positive x, so its max occurs at the left endpt $x=1$.

$\Rightarrow K_2 = f''(1) = \frac{2}{1^3} = 2.$

Thus the error bound value is $\frac{K_2\cdot(b-a)^3}{12n^2} = \frac{2\cdot(2-1)^3}{12\cdot 4^2} = \frac{2}{12\cdot 16} = \frac{1}{6\cdot 16} = \frac{1}{96}$.

This means that the actual value of J differs from the part (a) approximation by no more than $\frac{1}{96}$.

(d) Now suppose you want to make an approximation of J using the Trapezoidal Rule that is within 10^{-10} of the true value. How high must you make n? Explain.

We want the error (difference between actual value and approx. value) to be less than or equal to 10^{-10}; this is achieved if we can find n such that $\frac{K_2\cdot(b-a)^3}{12n^2} = 10^{-10}$, according to the error bound rule.

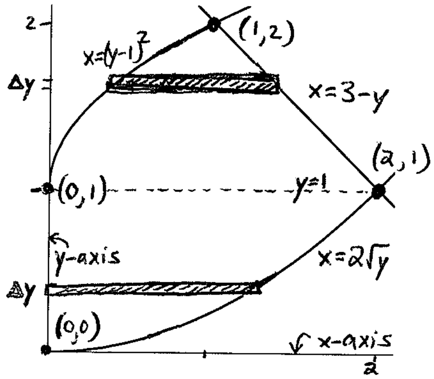
Solving: $\frac{2\cdot(2-1)^3}{12n^2} = 10^{-10} \Rightarrow n = \sqrt{\frac{10^{10}}{6}}$.

But n needs to be a whole number, and the above is a non-whole #.

Thus, must make $\boxed{n > \sqrt{\frac{10^{10}}{6}}}$ (any whole # bigger than $\sqrt{\frac{10^{10}}{6}}$ will do), because in this case $\frac{K_2\cdot(b-a)^3}{12n^2} < 10^{-10}$, and then

$|E_T| \le \frac{K_2\cdot(b-a)^3}{12n^2} < 10^{-10}$, which is a re-statement of the idea that the error must be $\le 10^{-10}$.

1. Consider the first-quadrant region, partially depicted below, which is bounded by the y-axis, the curve $x = 2\sqrt{y}$, the curve $x = (y-1)^2$, and the line $x = 3 - y$.



(a) Determine the intersection points of the curves. In the figure, label the appropriate curves and intersection points with their equations or coordinates.

(b) Using any technique you like, find the area of the region. Any integral(s) you use should be justified by the drawing of an appropriate approximating rectangle(s) on the figure above. (Hint: think about this region in two parts, and consider each part separately.)

(a) See above curves: • Curves $x=(y-1)^2$ and $x=3-y$ intersect when

$(y-1)^2 = 3-y$, so $y^2-y-2=0$, i.e.

$(y-2)(y+1)=0$. $(y=2,-1)$

The point $\underline{y=2}$ corresponds to $\underline{x=1}$; the point $y=-1$ not depicted.

• Curves $x=2\sqrt{y}$ and $x=3-y$ intersect when $2\sqrt{y}=3-y$, so $4y=(3-y)^2$, which means $0=y^2-10y+9=(y-1)(y-9)$. Thus $y=1$ is a solution, but you can check that $y=9$ is $\underline{\text{not}}$. The point $\underline{y=1}$ corresponds to $\underline{x=2}$.

(b) All curves express x in terms of y, so we could choose y as our integrating variable, i.e. make horizontal slices from $y=0$ to $y=2$. However, the slices have distinct behavior depending whether $y \le 1$ or $y \ge 1$ (see above):

If $0 \le y \le 1$, slice has length $x_{right} - x_{left} = 2\sqrt{y} - 0 = 2\sqrt{y}$;

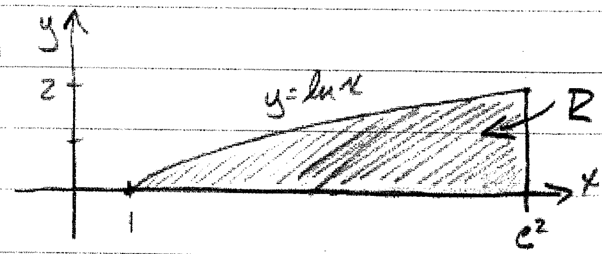
If $1 \le y \le 2$, slice has length $x_{right} - x_{left} = 3-y-(y-1)^2$.

Thus since Area $= \int_0^2 (\text{length}(y))\, dy$, we have

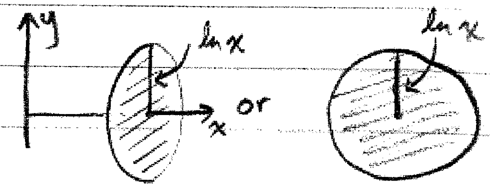
$$A = \int_0^1 2\sqrt{y}\,dy + \int_1^2 3-y-(y-1)^2\,dy = \frac{4}{3}y^{3/2}\Big]_0^1 + \left(3y - \frac{y^2}{2} - \frac{(y-1)^3}{3}\right)\Big]_1^2 = \boxed{\frac{5}{2}}.$$

(Doing this all with an integrating variable x is not much easier or harder.)

2. a)



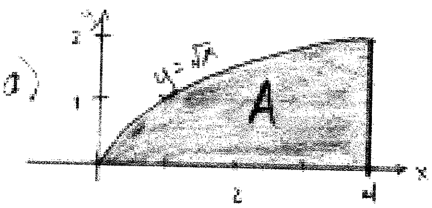
b)



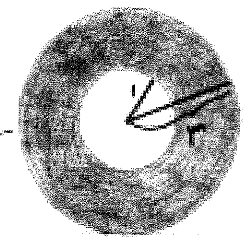
c) $A(x) = \underline{\pi(\ln x)^2}$

d) Volume of S $= \underline{\int_1^{e^2} \pi(\ln x)^2\,dx}$

3. a)

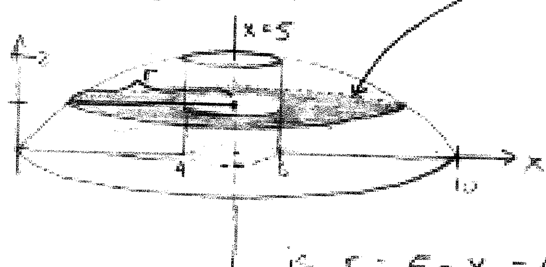


b)



r depends on the height of the horizontal slice

c)

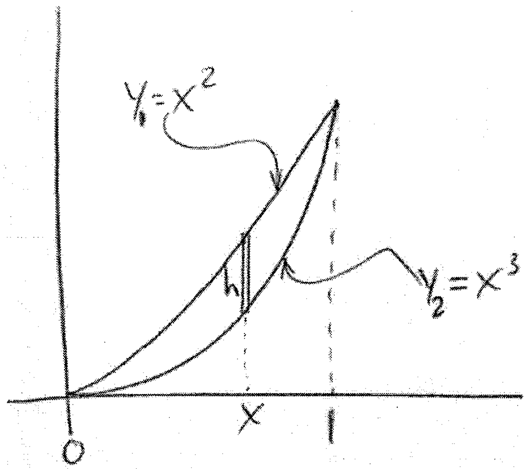


For a slice at height y, the outer radius of the annulus is $r = 5 - x = 5 - y^2$. So the area of a slice at height y is $\pi(5-y^2)^2 - \pi(1)^2 = \pi[25 - 10y^2 + y^4 - 1] = \underline{\pi(y^4 - 10y^2 + 24)}$.

d) Volume $= \int_0^2 \pi(y^4 - 10y^2 + 24)\,dy = \underline{\pi\int_0^2 (y^4 - 10y^2 + 24)\,dy}$

e) Volume $= \pi\left[\frac{y^5}{5} - \frac{10y^3}{3} + 24y\right]\Big|_0^2 = \underline{\pi\left[\frac{32}{5} - \frac{80}{3} + 48\right]}$

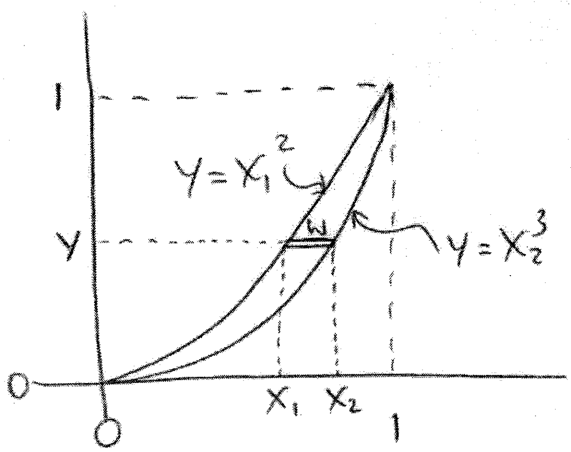
4. (a)



$$A = \int dA \quad \text{(= area of vertical slice)}$$

$$= \int_0^1 h\,dx = \int_0^1 (y_1 - y_2)\,dx$$

$$\boxed{= \int_0^1 (x^2 - x^3)\,dx}$$

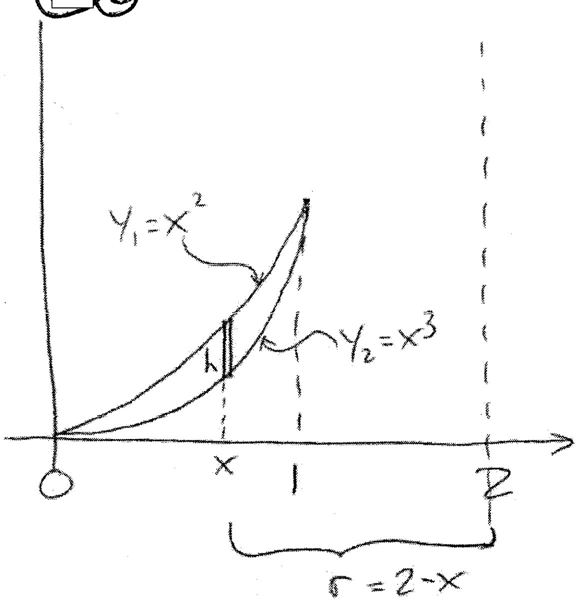


$$A = \int dA \quad \text{(= area of horiz. slice)}$$

$$= \int_0^1 w\,dy = \int_0^1 (x_2 - x_1)\,dy$$

$$\boxed{= \int_0^1 \left(y^{1/3} - y^{1/2}\right) dy}$$

4. (b)



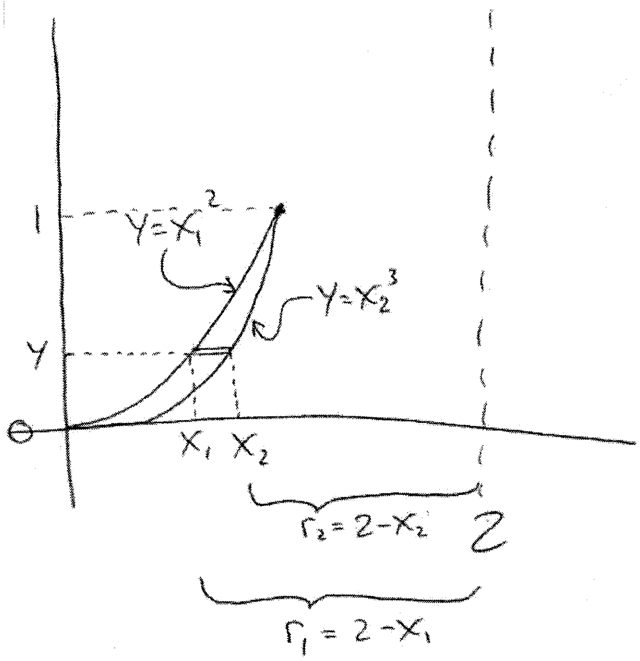
$$V = \int dV \quad \text{(= vol. of cyl. shell obtained by rotating vertical slice of area)}$$

$$= \int_0^1 (2\pi r)(h)\,dx$$

$$\boxed{= \int_0^1 2\pi(2-x)(x^2 - x^3)\,dx}$$

(over)

4. (b) continued



$$V = \int dV$$

(= vol of washer obtained by rotating horiz. slice of area)

$$= \int_0^1 \pi r_1^2 - \pi r_2^2 \, dy$$

$$= \pi \int_0^1 (2-x_1)^2 - (2-x_2)^2 \, dy$$

$$\boxed{= \pi \int_0^1 (2-y^{1/2})^2 - (2-y^{1/3})^2 \, dy}$$

Math 21 – Spring 2006 – Midterm #1 Solutions

5. (18 points)

(a) Set up, but do not evaluate, an integral representing the area of the region bounded by the curves $y = -\frac{1}{2}x^2 + 5x - 4$ and $2y - x = 0$. As justification, draw a picture with a sample slice (i.e., approximating rectangle) labeled.

First find the intersection points by combining equations:

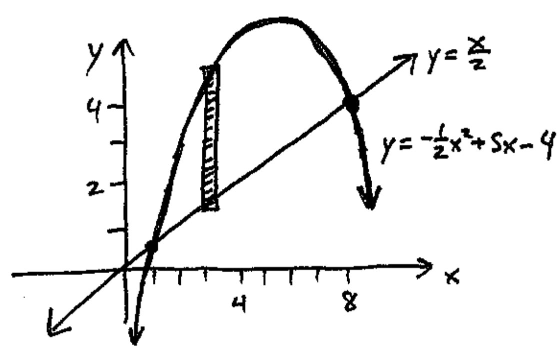
$$\frac{x}{2} = y = -\frac{1}{2}x^2 + 5x - 4$$

$$\Rightarrow \quad -\frac{1}{2}x^2 + \frac{9}{2}x - 4 = 0$$

$$\Rightarrow \quad x^2 - 9x + 8 = 0 \quad \Rightarrow \quad (x-8)(x-1) = 0, \quad \text{so } x = 1, 8.$$

If $x=1$, then $y = \frac{1}{2}$; if $x = 8$, then $y = 4$.

Sketch of curves & region:



Height of rectangle at coord x: $\quad y_{top} - y_{bottom} = \left(-\frac{1}{2}x^2 + 5x - 4\right) - \left(\frac{x}{2}\right)$

$$= -\frac{1}{2}x^2 + \frac{9}{2}x - 4.$$

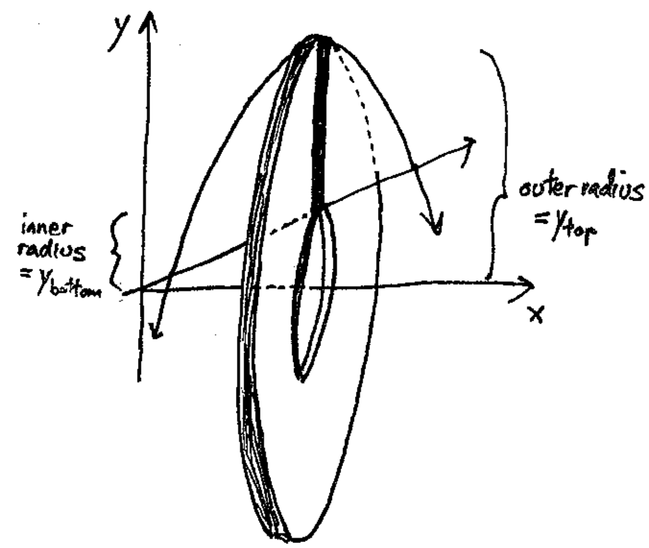
$$\Rightarrow \quad \text{Area} = \int_1^8 (y_{top} - y_{bottom})\,dx$$

$$= \boxed{\int_1^8 \left(-\frac{1}{2}x^2 + \frac{9}{2}x - 4\right) dx}$$

(b) Set up an integral representing the volume obtained by rotating the region from part (a) around the x-axis. Use the *disk/washer method*; make sure you justify your answer (draw and label a diagram). Again, don't evaluate the integral.

Disk/washer method (i.e. cross-section method): make slices perp. to rotation axis, i.e. <u>vertical</u> slices.

What matters is how the vertical slice of the region (a rectangle) creates a washer:

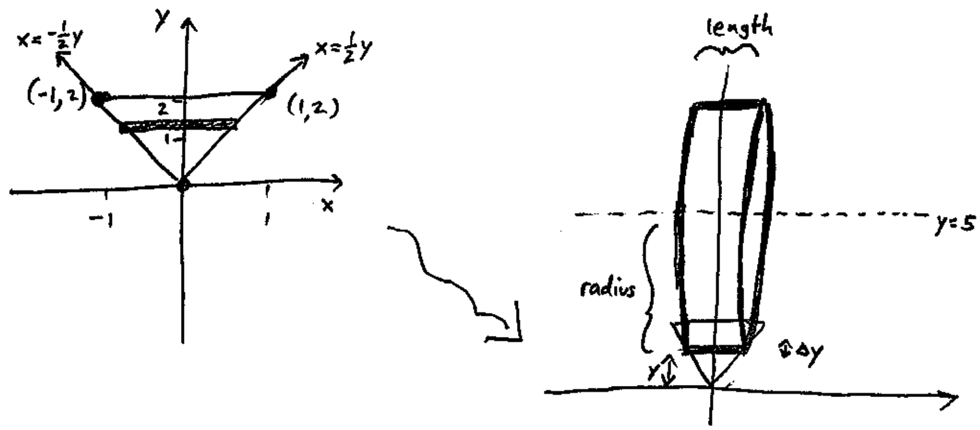


$$A(x) = \underset{\text{at coordinate } x}{\text{Area of washer}} = \pi r_{outer}^2 - \pi r_{inner}^2 = \pi y_{top}^2 - \pi y_{bottom}^2$$

$$= \pi\left(-\tfrac{1}{2}x^2 + 5x - 4\right)^2 - \pi\left(\tfrac{x}{2}\right)^2.$$

$$\Rightarrow \quad \text{Volume} = \int_{x_{min}}^{x_{max}} A(x)\,dx = \boxed{\int_1^8 \pi\left(-\tfrac{1}{2}x^2 + 5x - 4\right)^2 - \pi\left(\tfrac{x}{2}\right)^2 dx}$$

6. (10 points) Let R be the triangle in the xy-plane whose vertices are the points $(0,0)$, $(-1,2)$, and $(1,2)$. Consider the solid formed by rotating R about the line $y = 5$. Set up an integral representing the volume of this solid; use the method of *cylindrical shells*. Justify your answer by drawing and labelling a picture showing an appropriate slice.



Cylindrical shells method: make slices parallel to rotation axis, i.e. horizontal slices.

Ask: given a single slice at coordinate y, thickness Δy,

what are radius & length of shell created by this slice?

Answer: Radius = vertical dist. from slice to rotation axis

$= 5-y$

Length = length of rectangle = horiz. dist between lines $x=\frac{1}{2}y$ and $x=-\frac{1}{2}y$

$= \frac{1}{2}y - (-\frac{1}{2}y) = y.$

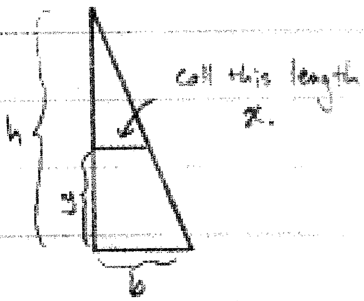
Thus an individual shell has approximate volume $= 2\pi(\text{radius})(\text{length})(\text{thickness})$

$= 2\pi(5-y)\,y\,\Delta y,$

and total volume $= \displaystyle\int_{y_{min}}^{y_{max}} 2\pi(5-y)y\,dy$

$= \boxed{\displaystyle\int_0^2 2\pi(5-y)\,y\,dy}$

7. If we slice the pyramid horizontally, then each slice is a rectangle. If a slice at height y has area $A(y)$, then the volume of the pyramid will be $Vol = \int_0^h A(y)\,dy$. To find $A(y)$, we look at a vertical slice, and use similar triangles. Then



$$\frac{h}{b} = \frac{h-y}{x}, \text{ so } hx = bh - by, \text{ and}$$

$$x = b - \frac{by}{h} = b\left(1 - \frac{y}{h}\right)$$

The area of a rectangle at height y will be $x \cdot x$, or

$$b\left(1-\frac{y}{h}\right)\cdot b\left(1-\frac{y}{h}\right) = b^2\left(1 - \frac{2y}{h} + \frac{y^2}{h^2}\right). \text{ So}$$

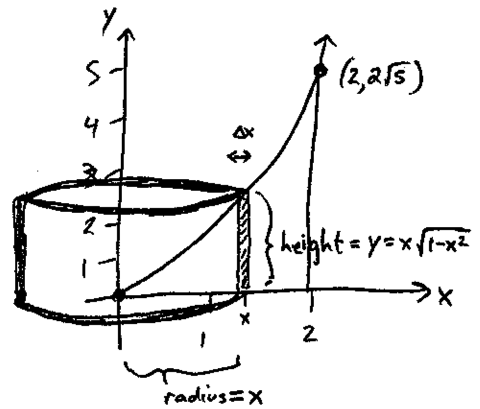
$$Vol = \int_0^h A(y)\,dy = \int_0^h b^2\left(1 - \frac{2}{h}y + \frac{1}{h^2}y^2\right)dy = b^2\int_0^h\left(1 - \frac{2}{h}y + \frac{1}{h^2}y^2\right)dy$$

$$= b^2\left[y - \frac{1}{h}y^2 + \frac{1}{3h^2}y^3\right]\Big|_0^h = b^2\left[h - \frac{1}{h}\cdot h^2 + \frac{1}{3h^2}\cdot h^3\right] = b^2\left[h - h + \frac{1}{3}h\right]$$

$$= b^2\cdot\frac{1}{3}h = \underline{\frac{1}{3}b^2h}$$

8. (22 points) Let R be the region in the first quadrant of the xy-plane bounded by the curve $y = x\sqrt{1+x^2}$, the x-axis, and the line $x = 2$.

(a) Consider the three-dimensional solid formed by rotating R about the y-axis. Using any method you choose, calculate the volume of this solid. Your answer must include the *name* or *description* of the method, plus an appropriately labelled diagram that corresponds to the setup of your integral. If you use a table entry to evaluate the integral, be sure to cite which you used.



Method of cylidrical shells: a vertical slice at coordinate x and thickness Δx

creates a shell of radius $= x$

and height $= x\sqrt{1+x^2}$,

so shell has approx. volume $2\pi \underbrace{(x)}_{\text{radius}} \underbrace{(x\sqrt{1+x^2})}_{\text{height}} \underbrace{\Delta x}_{\text{thickness}}$.

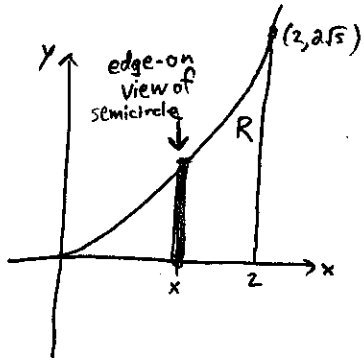
Thus total volume $= \displaystyle\int_{x_{min}}^{x_{max}} 2\pi x^2\sqrt{1+x^2}\,dx = \int_0^2 2\pi x^2\sqrt{1+x^2}\,dx$

$$\left(\begin{array}{c}\text{by Table \#2}\\ \text{with } u = x,\\ a = 1\end{array}\right) = 2\pi\left(\frac{x}{8}(1+2x^2)\sqrt{1+x^2} - \frac{1}{8}\ln\left(x+\sqrt{1+x^2}\right)\right)\Bigg]_{x=0}^{x=2}$$

$$= 2\pi\cdot\left(\frac{2}{8}(1+8)\sqrt{1+4} - \frac{1}{8}\ln\left(2+\sqrt{1+4}\right)\right)$$

$$= \boxed{\frac{\pi}{4}\left(18\sqrt{5} - \ln(2+\sqrt{5})\right)}.$$

(b) Consider the three-dimensional solid formed by building a semicircle atop each vertical cross-section of R. (In other words, the solid has the region R as its base, and every cross-section perpendicular to the x-axis is a semicircle.) Set up an integral expression that represents the volume of this solid, showing all reasoning. You do not have to evaluate this integral.



At a coordinate x, a vertical cross-section of R of thickness Δx has height $y = x\sqrt{1+x^2}$. This is also the diameter of the semicircle lying there, so the area of this semicircle is

$$A(x) = \frac{1}{2}\pi\left(\frac{x\sqrt{1+x^2}}{2}\right)^2,$$

and thus the volume of this cross-section of the solid is

$$A(x)\cdot\Delta x = \frac{1}{2}\pi\left(\frac{x\sqrt{1+x^2}}{2}\right)^2\Delta x.$$

Thus, the total volume is

$$\int_{x_{min}}^{x_{max}} A(x)\,dx = \boxed{\int_0^2 \frac{1}{2}\pi\left(\frac{x\sqrt{1+x^2}}{2}\right)^2 dx}.$$

Math21 Spring 2005 – Midterm Solutions

9. (16 points)

(a) Set up, but do not evaluate, an integral representing the area of the region bounded by the curve $x = y^2 + 1$ and the curve $x = 3 + 3y - y^2$. As justification, draw a picture with a sample slice (i.e., approximating rectangle) labeled.

Points of intersection: Set $y^2 + 1 = 3 + 3y - y^2$

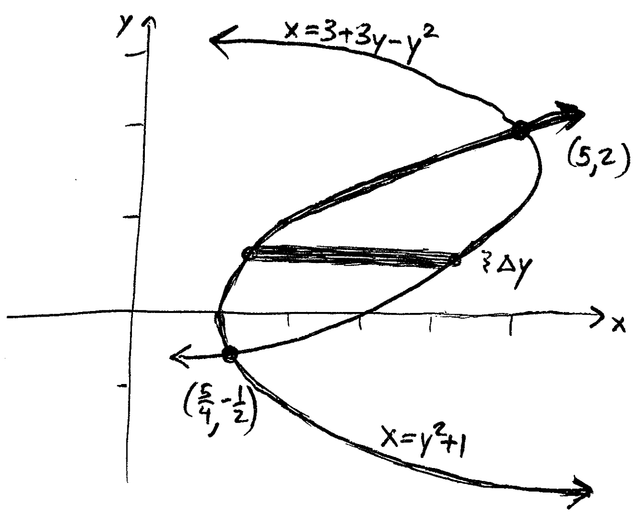
$\Rightarrow 2y^2 - 3y - 2 = 0$

$\Rightarrow (2y+1)(y-2) = 0$

$\Rightarrow y = -\frac{1}{2}, 2 \Rightarrow$ plug in these to find x; you get $y = -\frac{1}{2} \Rightarrow x = \frac{5}{4}$ and $y = 2 \Rightarrow x = 5$.

So, points are $(5,2)$ & $(\frac{5}{4}, -\frac{1}{2})$.

Curves are parabolas opening sideways:



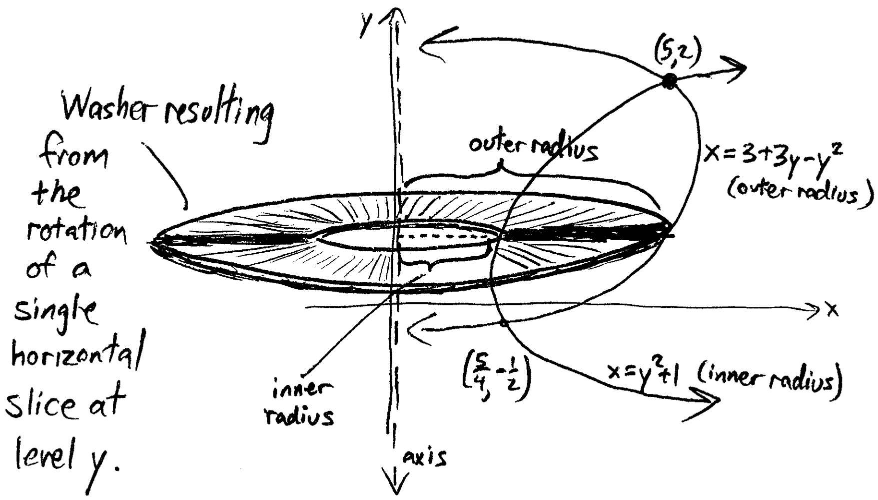
For a single horizontal slice with coordinate y,

* length $= x_{right} - x_{left}$
$= (3+3y-y^2) - (y^2+1)$,
* thickness $= \Delta y$.

Thus area $\approx (3+3y-y^2-y^2-1)\,\Delta y$.

Thus the exact area is the integral $\boxed{\int_{-1/2}^{2} (3+3y-2y^2-1)\,dy}$ (or equivalent).

(b) Set up an integral representing the volume obtained by rotating the region from part (a) around the y-axis. Use the *washer method*; make sure you justify your answer (draw and label a diagram). Again, don't evaluate the integral.



outer radius $=$ horizontal distance from axis to curve $x=3+3y-y^2$

$= x$-coord of curve $= 3+3y-y^2$.

inner radius $=$ horiz. distance from axis to curve $x=y^2+1$

$= x$-coord of curve $= y^2+1$

$A(y)$ = area of a washer created by a slice at level y

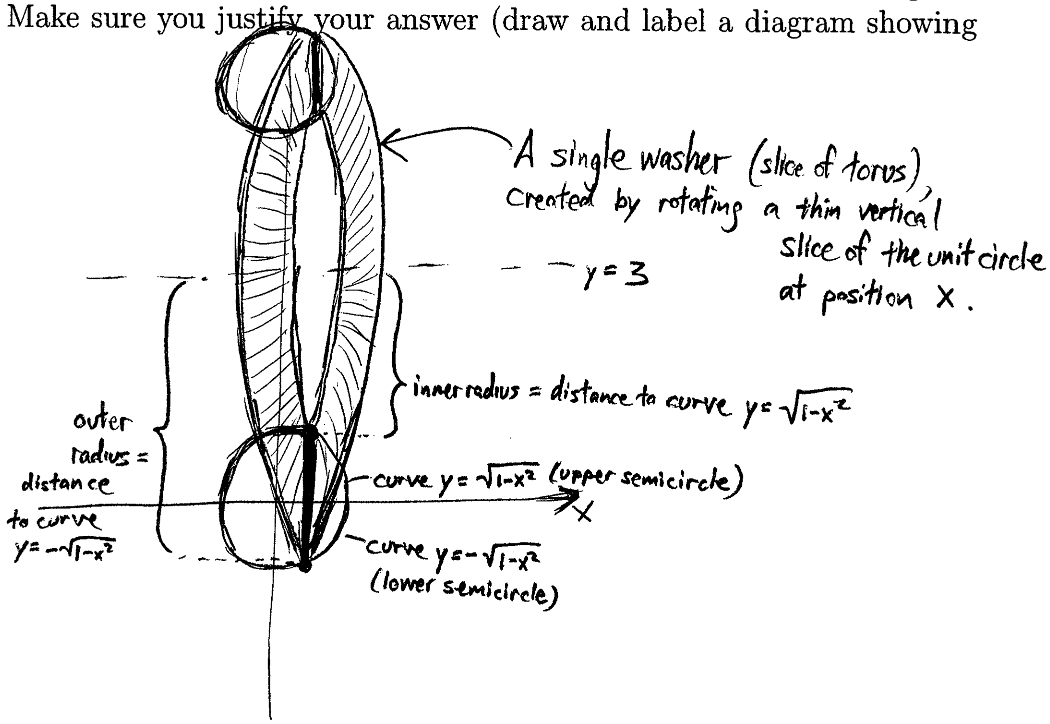
$$= \pi r_{outer}^2 - \pi r_{inner}^2$$

$$= \pi(3+3y-y^2)^2 - \pi(y^2+1)^2$$

$$\text{Volume} = \int_{-1/2}^{2} A(y)\,dy = \boxed{\int_{-1/2}^{2} \pi(3+3y-y^2)^2 - \pi(y^2+1)^2\,dy}.$$

10. (18 points) The unit circle $x^2 + y^2 = 1$ is rotated about the line $y = 3$, forming a torus (a doughnut-shaped figure).

(a) Set up, but do not evaluate, an integral representing the volume of this torus using the *washer method.* Make sure you justify your answer (draw and label a diagram showing a sample slice).



Washer method ⇒ Make slices of the unit circle perpendicular to line $y=3$

⇒ Vertical slices at each position x, $-1 \le x \le 1$.
Each creates a washer when spun:

Inner radius = Vertical distance from $y=3$ (axis) to $y=\sqrt{1-x^2}$ (upper semicircle)
$= 3-\sqrt{1-x^2}$

Outer radius = Vertical distance from $y=3$ (axis) to $y=-\sqrt{1-x^2}$ (lower semicircle)
$= 3+\sqrt{1-x^2}$

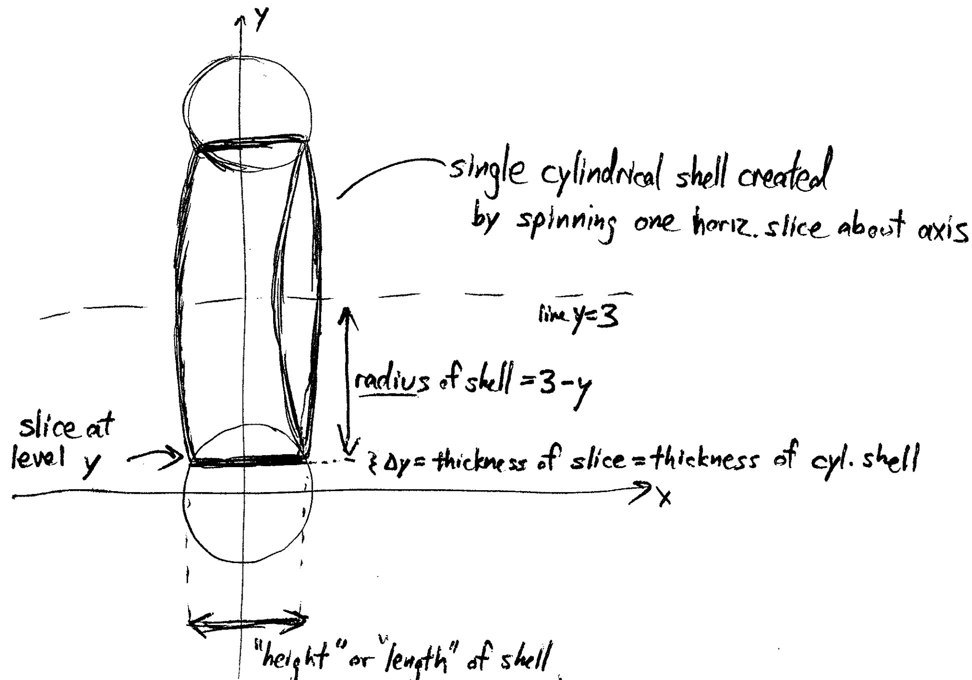
$A(x)$ = Area of washer located at coordinate x

$$= \pi r_{outer}^2 - \pi r_{inner}^2 = \pi\left(3+\sqrt{1-x^2}\right)^2 - \pi\left(3-\sqrt{1-x^2}\right)^2$$

$$\text{Volume} = \int_{-1}^{1} A(x)\,dx = \boxed{\int_{-1}^{1} \pi\left(3+\sqrt{1-x^2}\right)^2 - \pi\left(3-\sqrt{1-x^2}\right)^2\,dx}$$

(b) Set up, but do not evaluate, an integral representing the volume of this torus using the *cylindrical shells method.* Again, justify your answer by drawing and labelling an appropriate picture.

Note:
cyl. shells ⇒
We slice our circle parallel to axis of rotation, so get horizontal slices at coordinates y, for $-1 \leq y \leq 1$



same as length of horiz. slice at level y, given by $2\cdot(\sqrt{1-y^2})$ using equation of unit circle (x in terms of y)

$\Delta V =$ Volume of a single shell created by spinning the slice at level y about line $y=3$

$= 2\pi(\text{radius})(\text{length})(\text{thickness})$

$= 2\pi(3-y)(2\sqrt{1-y^2})\,\Delta y$

Thus total volume $= \boxed{\int_{-1}^{1} 2\pi(3-y)\cdot 2\sqrt{1-y^2}\,dy}$

(c) Now choose *one* of the integrals from part (a) or (b), and evaluate it to find the volume. Use whatever integration method you like, but be sure to show all work.

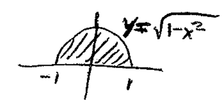
Part (a) integral: $V = \int_{-1}^{1} \pi\left(9+2\cdot 3\sqrt{1-x^2}+1-x^2\right) - \pi\left(9-2\cdot 3\sqrt{1-x^2}+1-x^2\right)\,dx$

$= \pi \int_{-1}^{1} 12\sqrt{1-x^2}\,dx$ (lots of cancellation!)

$$\left.\begin{array}{l} x=\sin\theta \\ dx=\cos\theta\,d\theta \\ x=-1 \Rightarrow \theta=-\frac{\pi}{2} \\ x=1 \Rightarrow \theta=\frac{\pi}{2} \end{array}\right\} = \pi\int_{-\pi/2}^{\pi/2} 12\sqrt{1-\sin^2\theta}\cdot\cos\theta\,d\theta = 12\pi\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta$$

$$= 12\pi\int_{-\pi/2}^{\pi/2} \tfrac{1}{2}(1+\cos 2\theta)\,d\theta = 12\pi\left(\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right)\Bigg|_{\theta=-\pi/2}^{\theta=\pi/2} = \boxed{6\pi^2}.$$

Tricky alternative: $\pi\int_{-1}^{1} 12\sqrt{1-x^2}\,dx = 12\pi\int_{-1}^{1}\sqrt{1-x^2}\,dx = (12\pi)$(Area of a semicircle of radius 1)



$= (12\pi)\left(\frac{\pi}{2}\right) = 6\pi^2$!

Part (b) integral: $V = \int_{-1}^{1} 2\pi(3-y)\cdot 2\sqrt{1-y^2}\,dy$

$= 12\pi\int_{-1}^{1}\sqrt{1-y^2}\,dy - 4\pi\int_{-1}^{1} y\sqrt{1-y^2}\,dy = 6\pi^2 - 4\pi\int_{-1}^{1} y\sqrt{1-y^2}\,dy$

by the same reasoning as above

$= 6\pi^2 - 0$

$= \boxed{6\pi^2}.$

Because the integrand above is an odd function, and so its integral from -1 to 1 will be 0!
(See below for a second computation.)

Note: Since $\int y\sqrt{1-y^2}\,dy = -\frac{1}{3}(1-y^2)^{3/2} + C$ by u-substitution, you can also directly compute the integral: $\int_{-1}^{1} y\sqrt{1-y^2}\,dy = -\frac{1}{3}(1-y^2)^{3/2}\Big]_{y=-1}^{y=1} = -\frac{1}{3}(0-0) = 0.$

Math 21 Midterm Solutions – Spring 2004

11. (20 points)

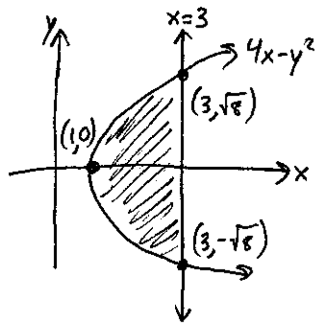
(a) Set up two distinct integrals, each in terms of a single variable, representing the area between the curve $4x - y^2 = 4$ and the line $x = 3$. For each, draw a picture with a sample slice (i.e., approximating rectangle) labeled. Don't evaluate either integral.

$4x-y^2=4$ intersects $x=3$ when: $\quad 4\cdot 3 - y^2 = 4$

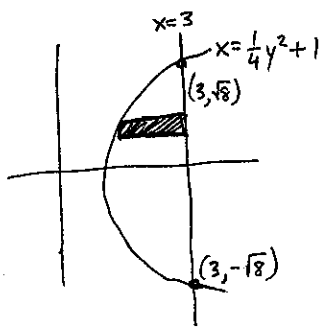
$\Rightarrow y^2 = 8$, so $y = \pm\sqrt{8}$

(and $x=3$)

Picture:



(i) Rectangles horizontal; integrate with respect to y



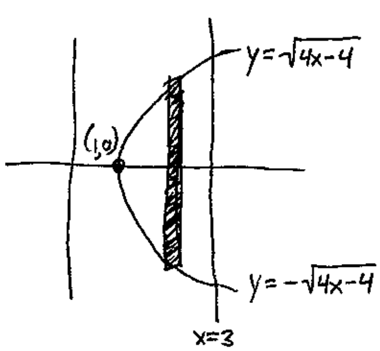
$$\text{Area} = \int_{y_{min}}^{y_{max}} (x_{Right} - x_{Left})\, dy$$

$$= \boxed{\int_{-\sqrt{8}}^{\sqrt{8}} \left(3 - \left(\tfrac{1}{4}y^2 + 1\right)\right) dy}$$

(ii) Rectangles vertical; integrate with respect to x

$4x-y^2=4 \rightarrow$ Solve for y in terms of x, get two solutions

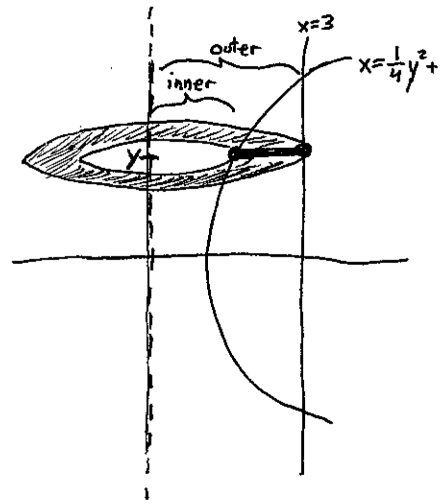
(upper & lower curves): $\quad y = \pm\sqrt{4x-4}$



$$\text{Area} = \int_{x_{min}}^{x_{max}} (y_{Top} - y_{Bottom})\, dx$$

$$= \int_1^3 \left(\sqrt{4x-4} - \left(-\sqrt{4x-4}\right)\right) dx = \boxed{\int_1^3 2\sqrt{4x-4}\, dx}$$

(b) Set up an integral representing the volume obtained by rotating the region from part (a) around the y-axis. Use the *washer method*; make sure you justify your answer (draw and label a diagram). Don't evaluate the integral.



Slice perp. to rotation axis:

Horizontal rectangle traces out a washer.

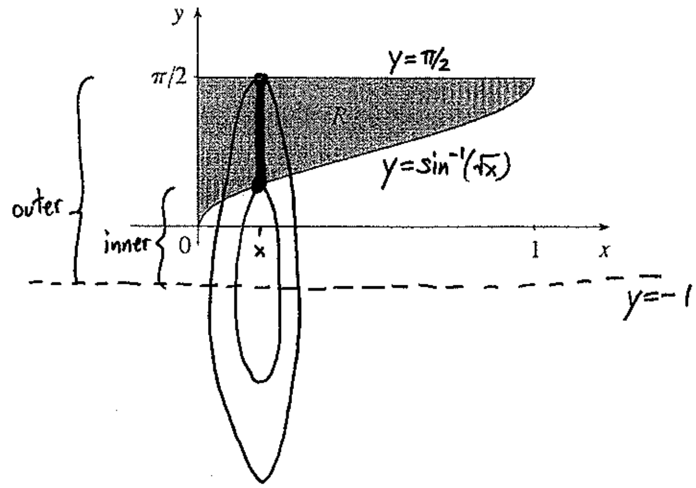
Inner & outer radii are x-distances.

$$A(y) = \text{Washer's area at a height of } y = \pi r_{outer}^2 - \pi r_{inner}^2$$

$$= \pi \cdot 3^2 - \pi \cdot \left(\tfrac{1}{4}y^2+1\right)^2$$

$$\text{Volume} = \int_{y_{min}}^{y_{max}} A(y)\,dy = \boxed{\int_{-\sqrt{8}}^{\sqrt{8}} \left(\pi \cdot 3^2 - \pi\left(\tfrac{1}{4}y^2+1\right)^2\right) dy}$$

12. (20 points) Consider the region R bounded by the curve $\sin y = \sqrt{x}$, the line $y = \pi/2$, and the y-axis, as shown below.



(a) Suppose we rotate this region about the line $y = -1$ to make a solid S. Set up, but do not evaluate, an integral representing the volume of S using *washers*. Show all of your reasoning.

Slice perp. to rotation axis $\Rightarrow$ Vertical rectangle traces out a washer; Inner & outer radii are y-distances.

(Curve $\sin y = \sqrt{x}$ same as $y = \sin^{-1}(\sqrt{x})$.)

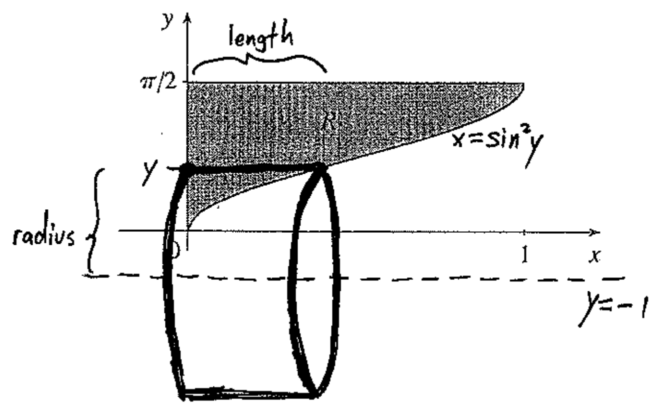
$A(x)$ = area of washer at position x

$$= \pi \cdot r_{outer}^2 - \pi \cdot r_{inner}^2$$

$$= \pi\left(\frac{\pi}{2}+1\right)^2 - \pi\left(\sin^{-1}(\sqrt{x})+1\right)^2$$

$$Vol = \int_{x_{min}}^{x_{max}} A(x)\,dx = \boxed{\int_0^1 \left(\pi\left(\frac{\pi}{2}+1\right)^2 - \pi\left(\sin^{-1}(\sqrt{x})+1\right)^2\right)dx}$$

(b) Set up, but do not evaluate, an integral to find the volume of S using *cylindrical shells.* Show all reasoning.



Slice parallel to rotation axis at height y:

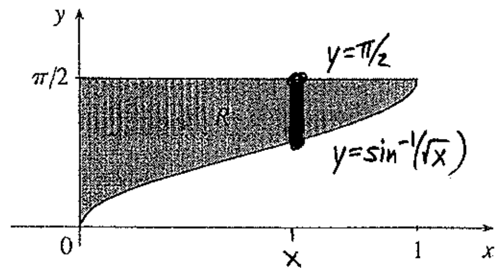
Horizontal rectangle traces out cylindrical shell, with

radius $= y+1$

length $= \sin^2 y$ (x-coord. of curve $\sin y = \sqrt{x}$)

$$\text{Volume} = \int_{y_{min}}^{y_{max}} 2\pi(\text{radius})(\text{length})(\text{thickness}) = \boxed{\int_0^{\pi/2} 2\pi(y+1)\sin^2 y\, dy}$$

(c) Now suppose that we form a solid T by constructing a square atop each vertical cross-section of R. (In other words, the solid T has the region R as its base, and every cross-section perpendicular to the x-axis is a square.) Set up, but do not evaluate, an integral to find the volume of T, showing all reasoning.



Slice perpendicular to x-axis: At a position x,

side length of square cross-section of T = height of rectangular slice of R

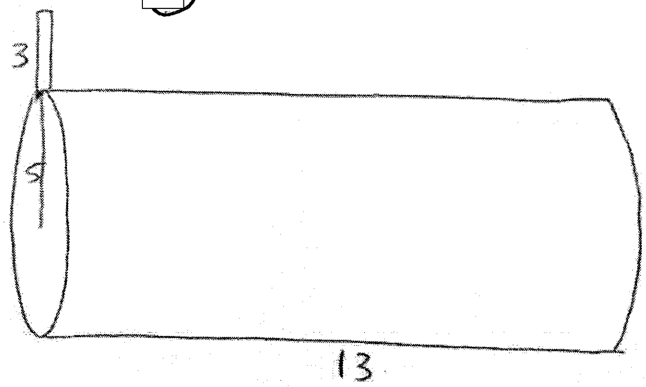
$= y_{top} - y_{bottom}$

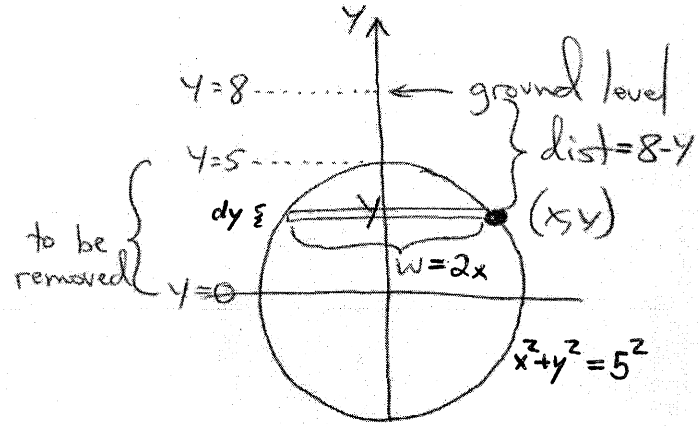
$= \frac{\pi}{2} - \sin^{-1}(\sqrt{x})$.

$\Rightarrow$ $A(x)$ = area of square cross-section of T $= (\text{side})^2 = \left(\frac{\pi}{2} - \sin^{-1}(\sqrt{x})\right)^2$.

$$\text{Volume} = \int_{x_{min}}^{x_{max}} A(x)\,dx = \boxed{\int_0^1 \left(\frac{\pi}{2} - \sin^{-1}(\sqrt{x})\right)^2 dx}$$

1.





Slice the cylinder horizontally and calculate the work needed to pump <u>a single slice</u> to ground level: [see above diagram]

$$\text{Work} = \text{Force}\cdot\text{Distance} = \text{Weight}\cdot\text{Distance} = (30\cdot\text{Volume})\cdot\text{Distance},$$

and $\quad \text{Volume of one slice} = (\text{Area})(\text{Thickness})$

$$= (\text{Width})(\text{Length})(\text{Thickness}) = (2x)(13)(dy)$$

$$= 2\sqrt{25-y^2}\cdot 13\,dy$$

while $\quad \text{Distance to lift slice} = 8-y,$

so $\quad \text{Work per slice} = 30\cdot\left(26\sqrt{25-y^2}\,dy\right)(8-y)$

$$= 30\cdot 26\cdot(8-y)\sqrt{25-y^2}\,dy,$$

and $\quad \text{Total Work} = \boxed{\int_0^5 30\cdot 26\cdot(8-y)\sqrt{25-y^2}\,dy}.$

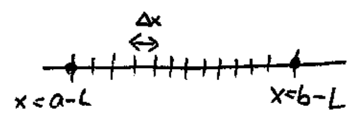
2. (8 points) A spring is put through various tests in order to determine its properties. It is found to obey Hooke's Law, and it is determined that the work required to stretch the spring from a length of 11cm to a length of 15cm is equal to five times the work required to stretch the spring from 9cm to 11cm. What is the natural length of the spring?

(*Hooke's Law* states that the force required to hold a spring in a given position is proportional to the distance that the spring is stretched from its natural length; that is, if x represents this latter amount, then the force $F = kx$ for some constant k.)

Let L = natural length of spring.

If spring is stretched from a length of $\underline{a}$ to a length of $\underline{b}$, then its "amount stretched" changes from $x = a - L$ to $x = b - L$, and over any <u>tiny</u> amount of distance Δx moved from x to $x + \Delta x$, the work required is $\approx$ (force)(dist) $= kx \cdot \Delta x$.

Thus the total work equals $\displaystyle\int_{a-L}^{b-L} kx\,dx$.



It follows from the given information that

$$\int_{11-L}^{15-L} kx\,dx = 5\int_{9-L}^{11-L} kx\,dx \; ; \quad \text{that is,}$$

$$\frac{1}{2}kx^2\Big]_{x=11-L}^{x=15-L} = \frac{5k}{2}x^2\Big]_{x=9-L}^{x=11-L}, \quad \text{and so}$$

$$\frac{1}{2}k\left((15-L)^2-(11-L)^2\right) = \frac{5k}{2}\left((11-L)^2-(9-L)^2\right).$$

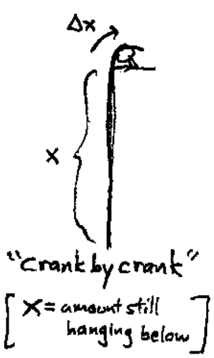
Dividing by $\frac{k}{2}$ and expanding, we find that all L^2 terms cancel:

$$225 - 30L + \cancel{L^2} - 121 + 22L - \cancel{L^2} = 5(121 - 22L + \cancel{L^2} - 81 + 18L - \cancel{L^2}),$$

and so $\quad 104 - 8L = 5(40 - 4L)$, so that $L = \frac{96}{12} = \boxed{8\text{cm}}$.

3. (12 points) A 900-foot-long cable that weighs 0.5 lb/ft hangs from the top of a mineshaft that is only 700 feet deep. At the floor of the shaft, a container of coal weighing 500lb is attached to the bottom end of the cable (and, note that the bottom 200-foot portion of the cable also lies piled on the ground). How much work is required to lift the cable, and all the coal, to the top of the mineshaft?

- Coal container moves a uniform distance, so work to lift it equals (weight)(distance lifted) $= (500)(700) = 350000$ ft·lb.
- Any part of the cable initially lying on ground is lifted a uniform distance, so this portion of cable may be considered as a single unit, and work to lift it = (weight)(distance lifted)
$$= \left(0.5\tfrac{lb}{ft}\right)(200\,ft)(700\,ft)$$
$$= (100\,lb)(700\,ft) = 70000\ ft{\cdot}lb.$$
- Remaining work: we must calculate how much work is required to take 700ft of hanging cable and hoist it all to top.

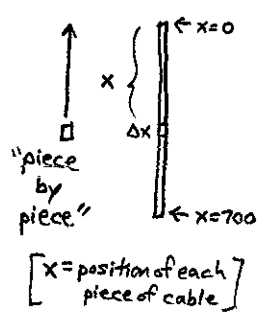


Method #1: When x feet remain hanging, a single "crank" up by a distance of Δx feet will require work equal to

(weight of remaing cable) (distance of this "crank")

$$= (0.5x)\,\Delta x.$$

We must crank our way from x = 700 at begin to x = 0 at the finish, so total work is $\int_0^{700} 0.5x\,dx = \frac{490000}{4}$ lb·ft.



Method #2: Instead ask how much work is required to take a small piece of the cable, length Δx, located x feet from the top, and hoist it all the way up: answer is (weight)(distance hoisted)

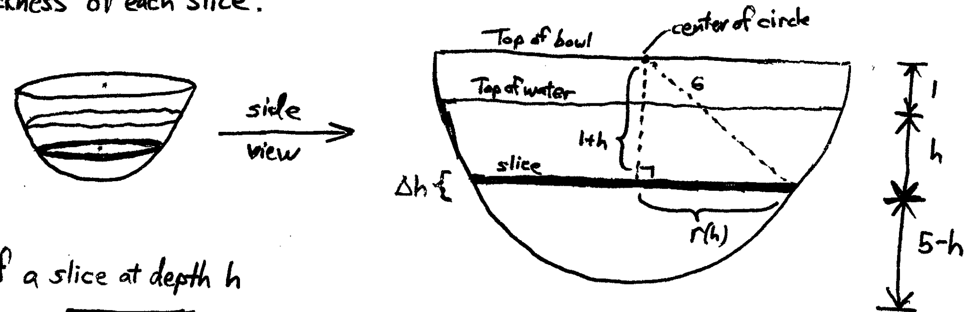
$$= (0.5\Delta x)\cdot(x),$$

so total work is again $\int_0^{700} 0.5x\,dx = \frac{1}{4}\cdot 490000$ ft·lb.

Either way, the total work is $350000 + 70000 + \frac{490000}{4} = \boxed{542500\ ft{\cdot}lb}$.

$\boxed{4.}$ (16 points) A giant hemispherical bowl, 6 feet in radius, holds muddy water. The bowl is filled to a depth of 5 feet. Due to the varying levels of mud in the water, the weight density of the water at a depth of h feet below the water's surface is $60 + 10h$ lb/ft^3. Set up, but do not evaluate, an expression representing the work required to pump all of the muddy water up to the top of the bowl. Justify your answer completely.

Slice the region of water into thin horizontal cross-sections;
let h stand for the depth of a section below the water's surface (so that h ranges from 0 to 5), and let Δh be the thickness of each slice.



By Pythagoras, the radius of a slice at depth h has the formula $r(h) = \sqrt{6^2 - (1+h)^2}$. (Alternatively, r can be found using the equation of a circle.)

Then ΔW = work required to lift a single slice to top

= (weight of slice)(distance to lift)

= (volume of slice)(weight density)(distance to lift)

= π(radius)^2(thickness)(weight-density)(distance to lift)

= $\pi(6^2 - (1+h)^2)\,\Delta h \cdot (60+10h) \cdot (h+1)$,

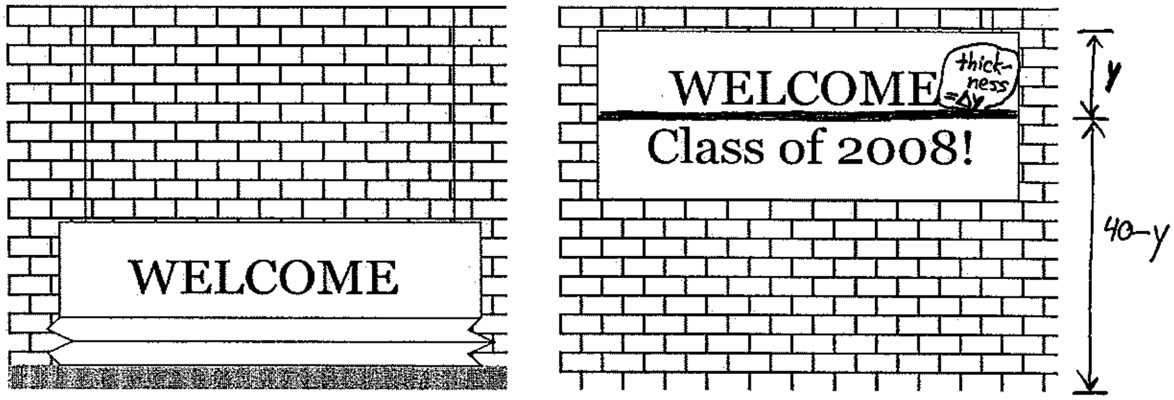
So $$W = \int_{h=0}^{h=5} dW = \boxed{\int_0^5 \pi(6^2-(1+h)^2)(60+10h)(h+1)\,dh}.$$

Note: depending on the choice of how to label the diagram, other equivalent integrals are possible. These include

$$\int_1^6 \pi(6^2 - u^2)(60 + 10(u-1))\,u\,du$$ (u stands for the depth of a slice below the bowl's top edge)

and $$\int_0^5 \pi(6^2 - (6-z)^2)(60+10(6-z))(6-z)\,dz.$$ (z stands for the height of a slice above the bowl's base)

5. (20 points) In honor of visiting Prospectives, students design a huge canvas sign, 20 feet wide and 10 feet tall, that weighs 1/2 lb. per square foot. They lay the whole sign in a pile on the ground and haul it up the side of their residence hall, so that the top of the sign is 40 feet above the ground. How much work do the students do to hang the sign?



(Ignore the ropes; we'll assume they are massless and therefore require no work.)

Let y be measured from the top of the sign downwards. Then a horizontal strip of material with height Δy weighs $(\frac{1}{2})(20)(\Delta y)$ pounds. This strip must be brought to a height of $40-y$ feet when the sign is fully raised. Thus the incremental amount of work is

$$10\cdot(40-y)\,\Delta y,$$

and total work $W = \int_0^{10} 10\cdot(40-y)\,dy = 10\left(40y - \frac{y^2}{2}\right)\Big|_0^{10}$

$$= 10(400-50) = \boxed{3500 \text{ foot-pounds}}.$$

(Another approach would be to measure y as height of a strip after the sign is hung. Then the weight of a strip of material would be the same, but the distance traveled would be y and the integral would be

$W = 10\int_{30}^{40} y\,dy = 3500$ foot-lbs. There are other approaches, too....!)