Techniques of Integration

1. Evaluate the following definite integrals.

   (a) $\displaystyle\int_0^2 x^2 e^{x^3}\,dx$ (b) $\displaystyle\int_0^1 x^2 e^{3x}\,dx$

2. Evaluate the following integrals, showing all of your work.

   (a) $\displaystyle\int \frac{x^3}{\sqrt{1-x^2}}\,dx$ (b) $\displaystyle\int \sin^3 x\cos^3 x\,dx$

   (c) $\displaystyle\int x^2(\ln x)^2\,dx$ (d) $\displaystyle\int \sin^4 x\,dx$

3. Evaluate the following integrals, showing all of your work.

   (a) $\displaystyle\int_4^{12} \frac{x}{\sqrt{1+2x}}\,dx$ (b) $\displaystyle\int \frac{\ln x}{x^{1/3}}\,dx$

   (c) $\displaystyle\int \sqrt{t}\,e^{\sqrt{t}}\,dt$ (d) $\displaystyle\int_{-\pi/2}^{\pi/2} x\cos^2 x\,dx$

4. Find each of the following, showing all of your work.

   (a) $\displaystyle\int \tan x\,dx \quad \left(\text{Hint: } \tan x = \frac{\sin x}{\cos x}\right)$ (b) $\displaystyle\int x\sqrt{4+x}\,dx$

   (c) $\displaystyle\int (\sin^2 x - \cos^2 x + \sec^2 x)\,dx$ (d) $\displaystyle\int \frac{\ln x}{x^2}\,dx$

5. Evaluate the following integrals. Use whatever method you like, but be sure to show all work.

   (a) $\displaystyle\int \frac{dx}{3x(1-\frac{x}{20})}$

   (b) $\displaystyle\int \frac{t(t^{10}+1)}{t^4+5}\,dt$

6. Evaluate the following integrals. Use whatever method you like, but be sure to show all work.

   (a) $\displaystyle\int \frac{dx}{2x^2-6x}$

   (b) $\displaystyle\int \frac{dx}{x^3\sqrt{9-x^4}}$

7. See problem 10(c) in the "Volumes" section below to practice another substitution-style integral.

Approximate Integration

1. Consider the function f, whose formula and derivatives are given below:

$$f(x) = \frac{1}{1+x^2} \qquad f'(x) = \frac{-2x}{(1+x^2)^2} \qquad f''(x) = \frac{-2+6x^2}{(1+x^2)^3} \qquad f'''(x) = \frac{-24x(x^2-1)}{(1+x^2)^4}$$

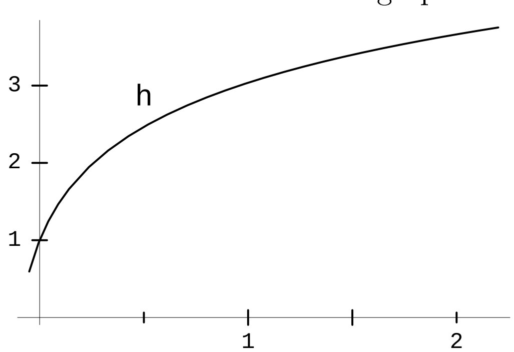
   (a) Let $J = \int_0^1 f(x)\,dx$. Write an expression involving only numbers that estimates the value of J using the Trapezoidal Rule with $n = 6$ subintervals. (You do *not* have to simplify this expression.)

   (b) Calculate the value of the "error bound" associated with the approximation above, and explain its significance in a complete sentence. Be as mathematically precise as you can in your reasoning; however, you don't have to simplify all your arithmetic. (You may find the derivatives of f provided above to be helpful.)

   (c) Now suppose you want to make a Trapezoidal Rule approximation of our $J = \int_0^1 f(x)\,dx$ that is sure to be within 10^{-8} of the true value. How many subintervals would you use? Explain completely; simplify your answer as much as possible.

   (d) Now consider an arbitrary function $g(x)$. How does the graph of g affect whether an approximation by the Trapezoidal Rule is an overestimate or an underestimate? Explain why this is so. (It might help to draw a picture, but this alone is not sufficient justification.)

2. Let h be the function graphed below.



Four students (I, II, III, and IV) approximated the area under the graph of h from $x = 0$ to $x = 2$. They all used the same number of subintervals, but they each used a different method among the ones listed below. Here are their results:

I: 5.4386 II: 5.70486 III: 5.73442 IV: 5.97112

   (a) Which result corresponds to which method? Explain.

Method	I, II, III, or IV?	Brief reason
Left Endpoint Rule		
Right Endpoint Rule		
Midpoint Rule		
Trapezoidal Rule		

(b) Write an expression, involving h evaluated at specific numbers, that represents the Simpson's Rule approximation to the area $\int_0^2 h(x)\,dx$ using $n = 8$ subintervals.

(c) Suppose you know that for all x on the interval $[0, 2]$,

$$|h''(x)| \leq 48 \qquad \text{and} \qquad |h^{(4)}(x)| \leq 18000.$$

Which approximation rule, *using how many subintervals* n, would you use to approximate $\int_0^2 h(x)\,dx$ to be sure that you are accurate to within 10^{-6} units? Explain completely; simplify your answer as much as possible.

3. Let $J = \displaystyle\int_1^2 \frac{dx}{x}$.

(a) Write a sum of numbers that estimates the value of J using the Trapezoidal Rule with $n = 4$. (You do not have to simplify this sum.)

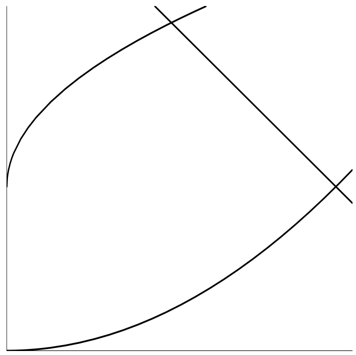
(b) Is the approximation in part (a) an overestimate or an underestimate of the true value of J? Explain. (It might help to draw a picture, but this alone is not sufficient justification.)

(c) Calculate the value of the "error bound" associated to the approximation in part (a), and explain its significance in a sentence. Show all of your reasoning.

(d) Now suppose you want to make an approximation of J using the Trapezoidal Rule that is within 10^{-10} of the true value. How high must you make n? Explain.

Area and Volume

1. Consider the first-quadrant region, partially depicted below, which is bounded by the y-axis, the curve $x = 2\sqrt{y}$, the curve $x = (y-1)^2$, and the line $x = 3 - y$.



   (a) Determine the intersection points of the curves. In the figure, label the appropriate curves and intersection points with their equations or coordinates.

   (b) Using any technique you like, find the area of the region. Any integral(s) you use should be justified by the drawing of an appropriate approximating rectangle(s) on the figure above. (Hint: think about this region in two parts, and consider each part separately.)

2. A region R is bounded by $y = \ln x$ and the x-axis for the values $1 \le x \le e^2$. If the region R is rotated about the x-axis, then the solid S is obtained.

   (a) Draw a rough sketch of the region R in the xy-plane.

   (b) Draw a rough sketch of a vertical cross-section of the solid S where the horizontal coordinate is x.

   (c) Find a formula for the area, $A(x)$, of the cross-section from part (b).

   (d) Write down an integral that represents the volume of the solid S. You do not have to evaluate this integral.

3. A region A is bounded by $y = \sqrt{x}$ and the x-axis for the values $0 \le x \le 4$, and is rotated about the line $x = 5$ to obtain a solid.

   (a) Draw a rough sketch of the region A in the xy-plane.

   (b) Draw a rough sketch of a horizontal cross-section of the solid.

   (c) Find a formula for the area of a cross-section taken at a height of y.

   (d) Write down an integral that represents the volume of the solid.

   (e) Evaluate the integral you obtained in part (d).

4. (a) Set up two distinct integrals, each in terms of a single variable, representing the area between the curves $y = x^2$ and $y = x^3$ over the interval from 0 to 1. For each, make sure you justify your answer (draw a picture and mark a sample slice). Don't evaluate either integral.

   (b) Set up two distinct integrals, each in terms of a single variable, representing the volume obtained by rotating the area from part (a) around the line $x = 2$. For each, make sure you justify your answer (draw a picture, label a sample slice, and cite the method used). Don't evaluate either integral.

5. (a) Set up, but do not evaluate, an integral representing the area of the region bounded by the curves $y = -\frac{1}{2}x^2 + 5x - 4$ and $2y - x = 0$. As justification, draw a picture with a sample slice (i.e., approximating rectangle) labeled.

   (b) Set up an integral representing the volume obtained by rotating the region from part (a) around the x-axis. Use the *disk/washer method*; make sure you justify your answer (draw and label a diagram). Again, don't evaluate the integral.

6. Let R be the triangle in the xy-plane whose vertices are the points $(0, 0)$, $(-1, 2)$, and $(1, 2)$. Consider the solid formed by rotating R about the line $y = 5$. Set up an integral representing the volume of this solid; use the method of *cylindrical shells*. Justify your answer by drawing and labelling a picture showing an appropriate slice.

7. Use calculus to find the volume of a pyramid with height h and square base with side length b. Give a complete mathematical justification, showing as much work as possible.

8. Let R be the region in the first quadrant of the xy-plane bounded by the curve $y = x\sqrt{1 + x^2}$, the x-axis, and the line $x = 2$.

   (a) Consider the three-dimensional solid formed by rotating R about the y-axis. Using any method you choose, calculate the volume of this solid. Your answer must include the *name* or *description* of the method, plus an appropriately labelled diagram that corresponds to the setup of your integral. If you use a table entry to evaluate the integral, be sure to cite which you used.

   (b) Consider the three-dimensional solid formed by building a semicircle atop each vertical cross-section of R. (In other words, the solid has the region R as its base, and every cross-section perpendicular to the x-axis is a semicircle.) Set up an integral expression that represents the volume of this solid, showing all reasoning. You do not have to evaluate this integral.

9. (a) Set up, but do not evaluate, an integral representing the area of the region bounded by the curve $x = y^2 + 1$ and the curve $x = 3 + 3y - y^2$. As justification, draw a picture with a sample slice (i.e., approximating rectangle) labeled.

   (b) Set up an integral representing the volume obtained by rotating the region from part (a) around the y-axis. Use the *washer method*; make sure you justify your answer (draw and label a diagram). Again, don't evaluate the integral.

10. The unit circle $x^2 + y^2 = 1$ is rotated about the line $y = 3$, forming a torus (a doughnut-shaped figure).

    (a) Set up, but do not evaluate, an integral representing the volume of this torus using the *washer method*. Make sure you justify your answer (draw and label a diagram showing a sample slice).

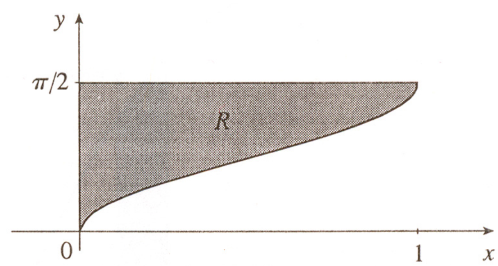
    (b) Set up, but do not evaluate, an integral representing the volume of this torus using the *cylindrical shells method*. Again, justify your answer by drawing and labelling an appropriate picture.

    (c) Now choose *one* of the integrals from part (a) or (b), and evaluate it to find the volume. Use whatever integration method you like, but be sure to show all work.

11. (a) Set up two distinct integrals, each in terms of a single variable, representing the area between the curve $4x - y^2 = 4$ and the line $x = 3$. For each, draw a picture with a sample slice (i.e., approximating rectangle) labeled. Don't evaluate either integral.

    (b) Set up an integral representing the volume obtained by rotating the region from part (a) around the y-axis. Use the *washer method*; make sure you justify your answer (draw and label a diagram). Don't evaluate the integral.

12. Consider the region R bounded by the curve $\sin y = \sqrt{x}$, the line $y = \pi/2$, and the y-axis, as shown below.



    (a) Suppose we rotate this region about the line $y = -1$ to make a solid S. Set up, but do not evaluate, an integral representing the volume of S using *washers*. Show all of your reasoning.

    (b) Set up, but do not evaluate, an integral to find the volume of S using *cylindrical shells*. Show all reasoning.

    (c) Now suppose that we form a solid T by constructing a square atop each vertical cross-section of R. (In other words, the solid T has the region R as its base, and every cross-section perpendicular to the x-axis is a square.) Set up, but do not evaluate, an integral to find the volume of T, showing all reasoning.

Work

1. At the neighborhood gas station, the gasoline is stored in a cylindrical tank buried underground, laying on its side so that the axis of symmetry is horizontal. The radius of the circular base is 5 ft, the length is 13 ft, and the top part of the tank is 3 ft beneath the surface. Assume that one cubic foot of gasoline weighs 30 pounds.

   Assuming that the tank begins completely full of gasoline, write down an integral in terms of a single variable that represents the amount of work that it takes to pump only half of the gasoline out of the tank and up to ground level. Make sure to clearly justify your answer, but do not evaluate the integral.

2. A spring is put through various tests in order to determine its properties. It is found to obey Hooke's Law, and it is determined that the work required to stretch the spring from a length of 11cm to a length of 15cm is equal to five times the work required to stretch the spring from 9cm to 11cm. What is the natural length of the spring?

   (*Hooke's Law* states that the force required to hold a spring in a given position is proportional to the distance that the spring is stretched from its natural length; that is, if x represents this latter amount, then the force $F = kx$ for some constant k.)

3. A 900-foot-long cable that weighs 0.5 lb/ft hangs from the top of a mineshaft that is only 700 feet deep. At the floor of the shaft, a container of coal weighing 500lb is attached to the bottom end of the cable (and, note that the bottom 200-foot portion of the cable also lies piled on the ground). How much work is required to lift the cable, and all the coal, to the top of the mineshaft?

4. A giant hemispherical bowl, 6 feet in radius, holds muddy water. The bowl is filled to a depth of 5 feet. Due to the varying levels of mud in the water, the weight density of the water at a depth of h feet below the water's surface is $60 + 10h$ lb/ft^3. Set up, but do not evaluate, an expression representing the work required to pump all of the muddy water up to the top of the bowl. Justify your answer completely.

5. In honor of visiting Prospectives, students design a huge canvas sign, 20 feet wide and 10 feet tall, that weighs 1/2 lb. per square foot. They lay the whole sign in a pile on the ground and haul it up the side of their residence hall, so that the top of the sign is 40 feet above the ground. How much work do the students do to hang the sign? (Ignore the ropes; we'll assume they are massless and therefore require no work.)