

① a) Solution 1:  $u = r^2 - 4, du = 2r dr$

$$\int \frac{r}{r^2-4} dr = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln |u| + C$$

$$= \frac{1}{2} \ln |r^2 - 4| + C$$

(Note the absolute values are necessary this time!)

• Solution 2:  $\frac{r}{r^2-4} = \frac{r}{(r+2)(r-2)} = \frac{A}{r+2} + \frac{B}{r-2}$

$$r = A(r-2) + B(r+2)$$

Using  $r = -2$ :  $-2 = -4A \Rightarrow A = \frac{1}{2}$

Using  $r = 2$ :  $2 = 4B \Rightarrow B = \frac{1}{2}$

So  $\frac{r}{r^2-4} = \frac{1}{2} \left[ \frac{1}{r+2} + \frac{1}{r-2} \right]$  (Check this!)

$$\int \frac{r}{r^2-4} dr = \frac{1}{2} \int \left[ \frac{1}{r+2} + \frac{1}{r-2} \right] dr$$

$$= \frac{1}{2} \left[ \ln |r+2| + \ln |r-2| \right] + C$$

Log properties show this is the same as the first answer.

• Solution 3:  $r = 2 \sec \theta$ , so  $r^2 - 4 = 4(\sec^2 \theta - 1)$   
 $= 4 \tan^2 \theta$ ,

$$dr = 2 \sec \theta \tan \theta d\theta$$

$$\int \frac{r}{r^2 - 4} dr = \int \frac{2 \sec \theta}{4 \tan^2 \theta} 2 \sec \theta \tan \theta d\theta$$

$$= \int \frac{\sec^2 \theta}{\tan \theta} d\theta \quad u = \tan \theta \quad du = \sec^2 \theta d\theta$$

$$= \int \frac{1}{u} du = \ln |u| + C = \ln |\tan \theta| + C$$

$$= \ln \left| \tan \left( \sec^{-1} \left( \frac{r}{2} \right) \right) \right| + C.$$

It's tricky but one can show this is the same as the other answers — except this method technically only works for  $r^2 - 4 \geq 0$  (a point we didn't discuss in this class) since  $|\sec \theta| \geq 1$  for any  $\theta$ , whereas the other solutions are also fine for  $r^2 - 4 < 0$ .

Compare the three solutions and judge for yourself which are easiest and hardest. Think what this says about which methods to try first.

$$b) \frac{1}{r^2-4} = \frac{A}{r+2} + \frac{B}{r-2}$$

$$1 = A(r-2) + B(r+2)$$

$$\text{Using } r = -2: 1 = -4A \Rightarrow A = -\frac{1}{4}$$

$$\text{Using } r = 2: 1 = 4B \Rightarrow B = \frac{1}{4}$$

$$\int \frac{1}{r^2-4} dr = \frac{1}{4} \int \left[ \frac{1}{r-2} - \frac{1}{r+2} \right] dr$$

$$= \frac{1}{4} [\ln|r-2| - \ln|r+2|] + C \leftarrow$$

$$= \frac{1}{4} \ln \left| \frac{r-2}{r+2} \right| + C \quad \leftarrow \text{Either form is fine.}$$

You might also try the trig substitution used in Solution 3 of part a), but that will soon stall at  $\int \csc \theta d\theta$ , which is quite tricky.

$$c) \int_1^e x \ln x dx$$

$$u = \ln x \quad dv = x dx$$

$$du = \frac{1}{x} dx \quad v = \frac{1}{2} x^2$$

$$= \left[ \frac{1}{2} x^2 \ln x \right]_1^e - \int_1^e \frac{1}{2} x^2 \cdot \frac{1}{x} dx = \left[ \frac{1}{2} x^2 \ln x \right]_1^e - \int_1^e \frac{1}{2} x dx$$

$$= \left[ \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 \right]_1^e = \frac{1}{2} e^2 - \frac{1}{4} e^2 - \left( 0 - \frac{1}{4} \right)$$

$$= \frac{1}{4} e^2 + \frac{1}{4}$$

$$\begin{aligned}
d) \int \sin^4 x \, dx &= \int (\sin^2 x)^2 \, dx \\
&= \int \left[ \frac{1}{2} (1 - \cos 2x) \right]^2 \, dx \\
&= \frac{1}{4} \int (1 - 2\cos 2x + \cos^2 2x) \, dx \\
&= \frac{1}{4} x - \frac{1}{2} \int \cos 2x \, dx + \frac{1}{4} \int \cos^2 2x \, dx \\
&= \frac{1}{4} x - \frac{1}{4} \sin 2x + \frac{1}{8} \int (1 + \cos 4x) \, dx \\
&= \frac{1}{4} x - \frac{1}{4} \sin 2x + \frac{1}{8} x + \frac{1}{32} \sin 4x + C.
\end{aligned}$$

e) This is an improper integral, since the integrand is undefined at  $x=0$ .

$$\begin{aligned}
\int_0^1 x^{-1/5} \, dx &= \lim_{t \rightarrow 0^+} \int_t^1 x^{-1/5} \, dx \\
&= \lim_{t \rightarrow 0^+} \left[ \frac{5}{4} x^{4/5} \right]_t^1 = \frac{5}{4} \lim_{t \rightarrow 0^+} (1 - t^{4/5}) \\
&= \frac{5}{4}.
\end{aligned}$$

$$f) \int_{\pi/4}^{\pi/2} \sin^2 \theta \cos \theta d\theta$$

$$u = \sin \theta$$

$$du = \cos \theta d\theta$$

$$\text{When } \theta = \pi/4, u = \frac{1}{\sqrt{2}}$$

$$\text{When } \theta = \frac{\pi}{2}, u = 1.$$

$$= \int_{1/\sqrt{2}}^1 u^2 du$$

$$= \left[ \frac{1}{3} u^3 \right]_{1/\sqrt{2}}^1 = \frac{1}{3} \left( 1 - \frac{1}{2^{3/2}} \right).$$

$$g) \text{ If } x = 3 \tan \theta, \quad x^2 + 9 = 9(\tan^2 \theta + 1) \\ = 9 \sec^2 \theta$$

$$\text{and } dx = 3 \sec^2 \theta, \text{ and } \theta = \tan^{-1} \left( \frac{x}{3} \right).$$

$$\int \frac{1}{(x^2 + 9)^{3/2}} dx = \int \frac{3 \sec^2 \theta}{(9 \sec^2 \theta)^{3/2}} d\theta = \frac{3}{3^3} \int \frac{\sec^2 \theta}{\sec^3 \theta} d\theta$$

$$= \frac{1}{9} \int \frac{1}{\sec \theta} d\theta = \frac{1}{9} \int \cos \theta d\theta$$

$$= \frac{1}{9} \sin \theta + C = \frac{1}{9} \sin \left( \tan^{-1} \left( \frac{x}{3} \right) \right) + C.$$

h)  $\frac{x^3}{x^3 - x^2 - x + 1}$  does not have lower degree on the top than on the

bottom, so we start with long division.

$$\begin{array}{r}
 1 \\
 \hline
 x^3 - x^2 - x + 1 \overline{) x^3 + 0x^2 + 0x - 0} \\
 \underline{x^3 - x^2 - x + 1} \\
 x^2 + x - 1
 \end{array}$$

So  $\frac{x^3}{x^3 - x^2 - x + 1} = 1 + \frac{x^2 + x - 1}{x^3 - x^2 - x + 1}$ .

If you're clever, you might skip the long division just by writing

$$\begin{aligned}
 \frac{x^3}{x^3 - x^2 - x + 1} &= \frac{x^3 - x^2 - x + 1 + x^2 + x - 1}{x^3 - x^2 - x + 1} \\
 &= 1 + \frac{x^2 + x - 1}{x^3 - x^2 - x + 1}.
 \end{aligned}$$

Now  $\frac{x^2 + x - 1}{x^3 - x^2 - x + 1} = \frac{x^2 + x - 1}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$ .

$$x^2 + x - 1 = A(x-1)^2 + B(x+1)(x-1) + C(x+1).$$

Using  $x = -1$ :  $-1 = 4A \Rightarrow A = -\frac{1}{4}$ .

Using  $x = 1$ :  $1 = 2C \Rightarrow C = \frac{1}{2}$ .

Using  $x = 0$ :  $-1 = A - B + C = \frac{1}{4} - B + \frac{1}{2}$

$$\Rightarrow B = 1 - \frac{1}{4} + \frac{1}{2} = \frac{5}{4}.$$

$$\int \frac{x^3}{(x+1)(x-1)^2} dx = \int \left[ 1 + \left(-\frac{1}{4}\right) \frac{1}{x+1} + \frac{5}{4} \frac{1}{x-1} + \frac{1}{2} \frac{1}{(x-1)^2} \right] dx$$

$$= x - \frac{1}{4} \ln|x+1| + \frac{5}{4} \ln|x-1| - \frac{1}{2} \frac{1}{(x-1)} + C.$$

(i)  $\int_1^4 (at+b)\sqrt{t} dt = \int_1^4 (at^{3/2} + bt^{1/2}) dt$

$$= \left[ a \frac{2}{5} t^{5/2} + b \frac{2}{3} t^{3/2} \right]_1^4$$

$$= a \frac{2}{5} 2^5 + b \frac{2}{3} 2^3 - a \frac{2}{5} - b \cdot \frac{2}{3}$$

$$= \frac{2}{5} a (32-1) + \frac{2}{3} b (8-1)$$

$$= \frac{62}{5} a + \frac{14}{3} b.$$

② If  $x \geq 3 > e$ , then  $\ln x > 1$ , so

$\frac{\ln x}{\sqrt{x}} > \frac{1}{\sqrt{x}} > 0$ , so we check if  $\int_3^{\infty} \frac{1}{\sqrt{x}} dx$  diverges.

$$\int_3^{\infty} \frac{1}{\sqrt{x}} dx = \lim_{t \rightarrow \infty} \int_3^t x^{-1/2} dx = \lim_{t \rightarrow \infty} \left[ 2x^{1/2} \right]_3^t$$

$$= 2 \lim_{t \rightarrow \infty} (t^{1/2} - \sqrt{3}) = \infty.$$

Since  $\int_3^{\infty} \frac{1}{\sqrt{x}} dx$  diverges, and  $\frac{\ln x}{\sqrt{x}} > \frac{1}{\sqrt{x}} > 0$

for  $x \geq 3$ , the Comparison Theorem tells us

that  $\int_3^{\infty} \frac{\ln x}{\sqrt{x}} dx$  diverges also.

③ If  $r(t)$  = rate of growth in year  $t$ , then the total population growth is  $\int_{1980}^{2000} r(t) dt$ .

Simpson's Rule approximates this by

$$\frac{5}{3} [r(1980) + 4r(1985) + 2r(1990)$$

$$+ 4r(1995) + r(2000)]$$

$$= \frac{5}{3} [1 + (-4) + 0 + 4 + 2] = 5.$$

④<sub>a</sub>)  $f(x) = \sin(x^2)$       $f'(x) = 2x \cos(x^2)$

$$f''(x) = 2 \cos(x^2) - 4x \sin(x^2).$$

$$|f''(x)| = |2 \cos(x^2) - 4x \sin(x^2)|$$

$$\leq 2 |\cos(x^2)| + 4 |x| |\sin(x^2)|$$

$$\leq 2 \cdot 1 + 4 \cdot 1 \cdot 1 = 6$$

using  $|\sin \theta| \leq 1$ ,  $|\cos \theta| \leq 1$ , and  $|x| \leq 1$  for  $0 \leq x \leq 1$ .



So we can use  $K=6$ . (A smaller value of  $K$  is possible, but it's hard to come by.)

$b-a=1$ , and  $n=5$ , so

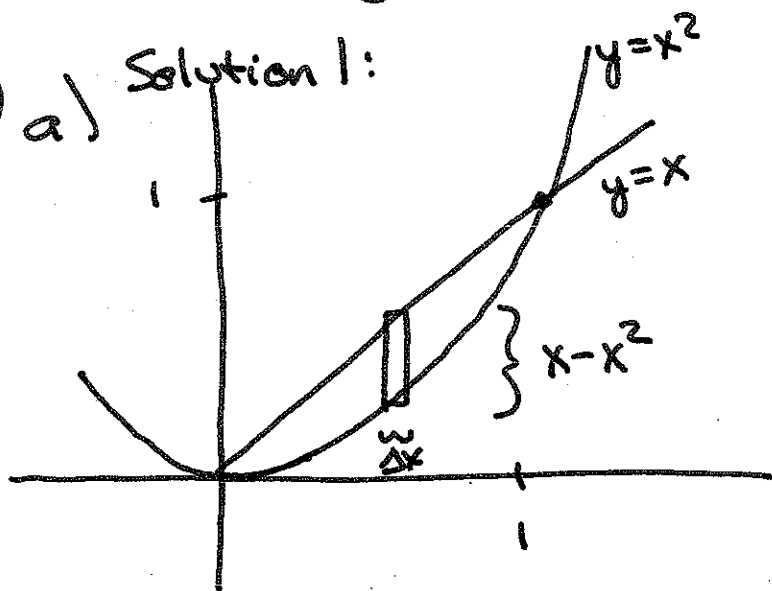
$$|E_T| \leq \frac{6 \cdot 1^3}{12(6^2)} = \frac{1}{2 \cdot 25} = \frac{1}{50}$$

b) Using  $K=6$  as above, we want

$$\frac{6 \cdot 1^3}{12 n^2} \leq \frac{1}{200}, \text{ so } n^2 \geq 100.$$

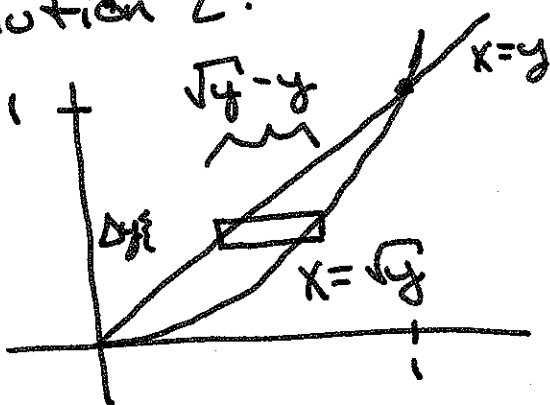
So it's good enough to have  $n=10$ .

⑤ a) Solution 1:

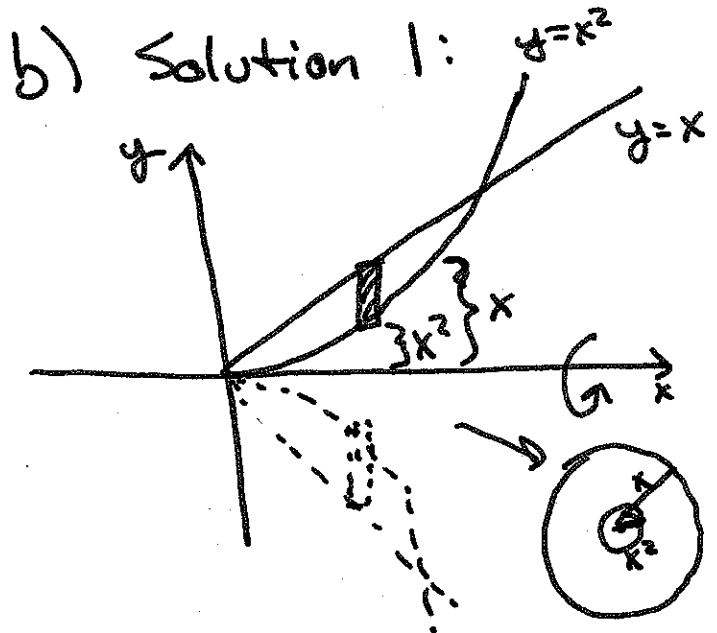


$$\begin{aligned} \Delta A &\approx (x-x^2) \Delta x, \\ \text{so } A &= \int_0^1 (x-x^2) dx \\ &= \left[ \frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_0^1 \\ &= \frac{1}{2} - \frac{1}{3} - 0 = \frac{1}{6}. \end{aligned}$$

Solution 2:



$$\begin{aligned} \Delta A &\approx (\sqrt{y}-y) \Delta y \\ A &= \int_0^1 (y^{1/2}-y) dy \\ &= \left[ \frac{2}{3}y^{3/2} - \frac{1}{2}y^2 \right]_0^1 = \frac{2}{3} - \frac{1}{2} - 0 \\ &= \frac{1}{6}. \end{aligned}$$



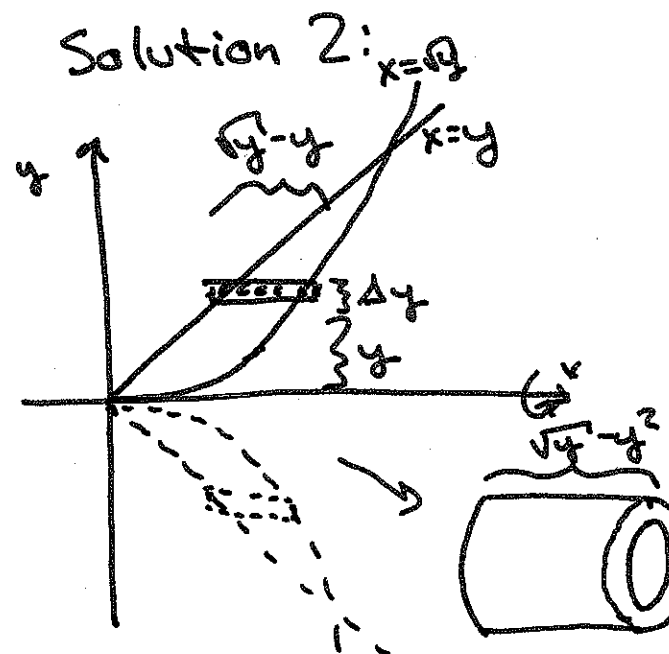
Get fat washers:

$$\Delta V \approx (\pi x^2 - \pi(x^2)^2) \Delta x$$

$$V = \pi \int_0^1 (x^2 - x^4) dx$$

$$= \pi \left( \frac{1}{3} x^3 - \frac{1}{5} x^5 \right) \Big|_0^1$$

$$= \pi \left( \frac{1}{3} - \frac{1}{5} \right) - 0 = \frac{2\pi}{15}$$



Get cylindrical shells

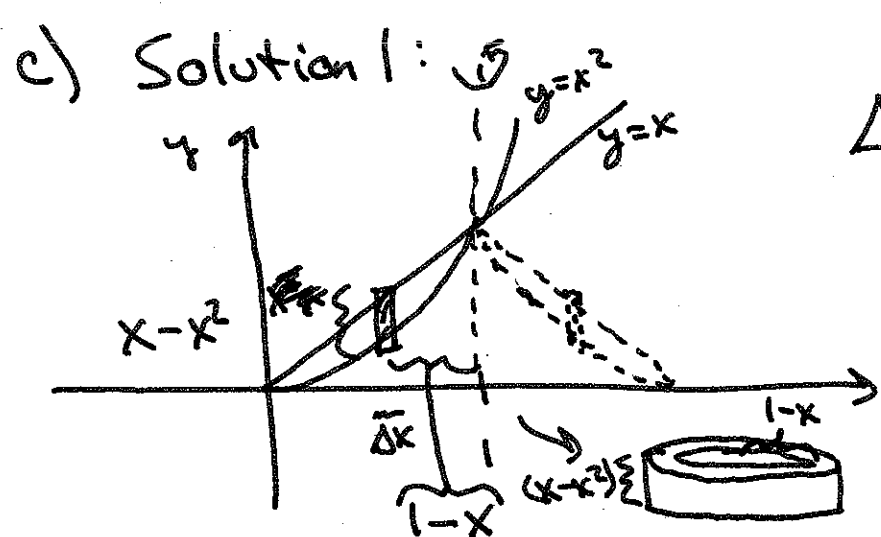
$$\Delta V \approx 2\pi y (\sqrt{y} - y) \Delta y$$

$$V = 2\pi \int_0^1 (y^{3/2} - y^2) dy$$

$$= 2\pi \left[ \frac{2}{5} y^{5/2} - \frac{1}{3} y^3 \right]_0^1$$

$$= 2\pi \left( \frac{2}{5} - \frac{1}{3} \right) - 0$$

$$= \frac{2\pi}{15}$$



$$\Delta V \approx 2\pi(1-x)(x-x^2) \Delta x$$

circumference  $2\pi(1-x)$ .

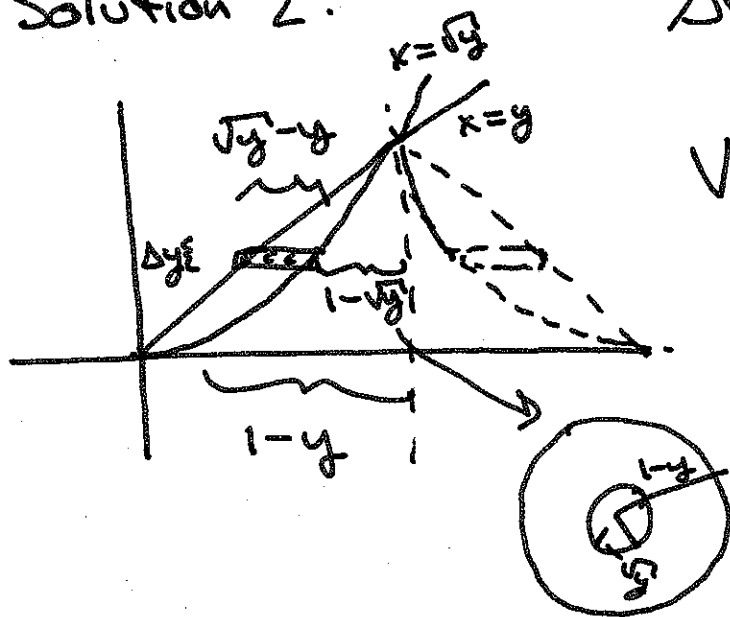
$$V = 2\pi \int_0^1 (1-x)(x-x^2) dx \quad (\text{Watch the bounds!})$$

$$= 2\pi \int_0^1 (x - 2x^2 + x^3) dx$$

$$= 2\pi \left[ \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4 \right]_0^1 = 2\pi \left[ \frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right] - 0$$

$$= \frac{\pi}{6}.$$

Solution 2:



$$\Delta V \approx [\pi(1-y)^2 - \pi(1-\sqrt{y})^2] \Delta y$$

$$V = \pi \int_0^1 [1 - 2y + y^2 - (1 - 2\sqrt{y} + y)] dy$$

$$= \pi \int_0^1 (-3y + y^2 + 2\sqrt{y}) dy$$

$$= \pi \left[ -\frac{3}{2}y^2 + \frac{1}{3}y^3 + \frac{4}{3}y^{3/2} \right]_0^1$$

$$= \pi \left( -\frac{3}{2} + \frac{1}{3} + \frac{4}{3} \right) - 0$$

$$= \frac{\pi}{6}.$$

6) a) TRUE

b) FALSE: this is backwards, and one also needs to know both functions are  $\geq 0$  to use the Comparison Theorem.

c) FALSE:

$$\frac{d}{dx} \left( \frac{1}{3} \sin^3 x + C \right) = \sin^2(x) \cdot \cos x.$$

d) TRUE: this can be done with a combination of long division, partial fractions, completing the square, the power rule, logarithms, and arctangents.

e) FALSE: Consider the case

$$f(x) = x, \quad g(x) = -x.$$