Math 42 — Chapter 7 Practice Problems — Set B

1. Which of the following functions is a solution of the differential equation $\frac{dy}{dx} = 4xy$?

   (a) $y = e^{-4x}$ (c) $y = e^{2x^2}$ (e) $y = e^{2x}$ (g) $y = 4e^{2x^2}$

   (b) $y = 4x$ (d) $y = -4x$ (f) $y = 2x^2$ (h) $y = 2e^{4x}$

2. Which of the following functions are solutions of the differential equation $y'' + y' = 6y$? Show your work.

   (a) $y = e^{-4x}$ (c) $y = e^{-2x}$ (e) $y = e^{2x}$ (g) $y = 2\sin(2x)$ (i) $y = 3e^{-3x}$

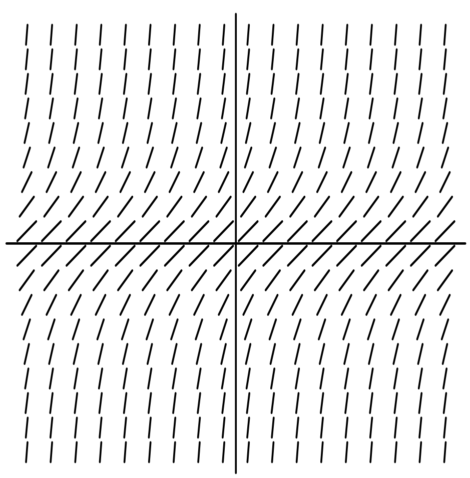
   (b) $y = e^{-3x}$ (d) $y = -4e^{-2x^2}$ (f) $y = 2x^2$ (h) $y = 4e^{-4x}$ (j) $y = 3e^{2x}$

Direction Fields

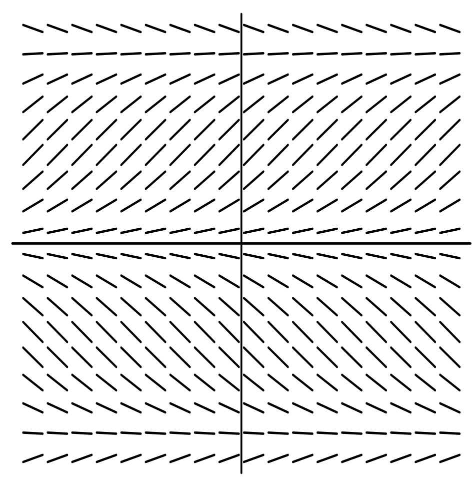
3. Match each of the slope fields below with exactly one of the differential equations. (The scales on the x- and y-axes are the same.) Also, provide enough explanation to show why no other matches are possible.

   (i) $y' = xy + 1$ (ii) $y' = \sin x$ (iii) $y' = xe^{-y}$ (iv) $y' = y^2 + 1$ (v) $y' = \sin y$

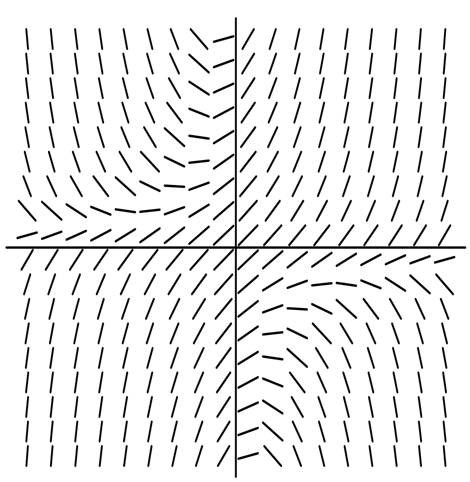
(a)



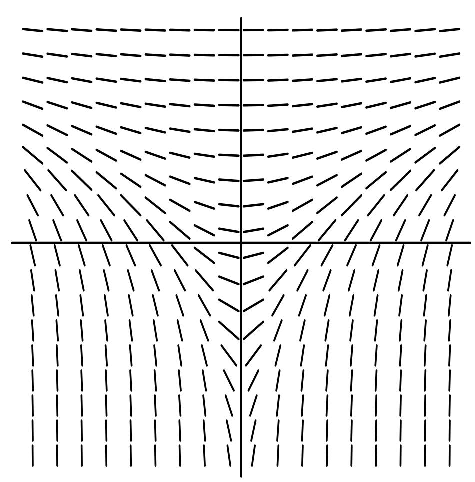
(c)



(b)



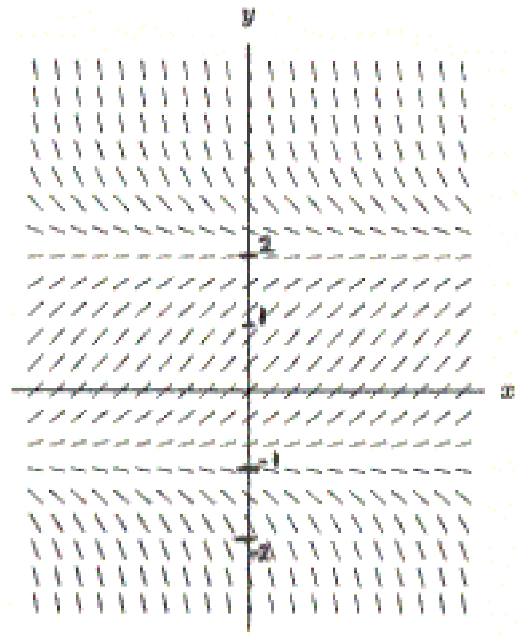
(d)



4. The slope field for the differential equation

$$y' = 0.5(1+y)(2-y)$$

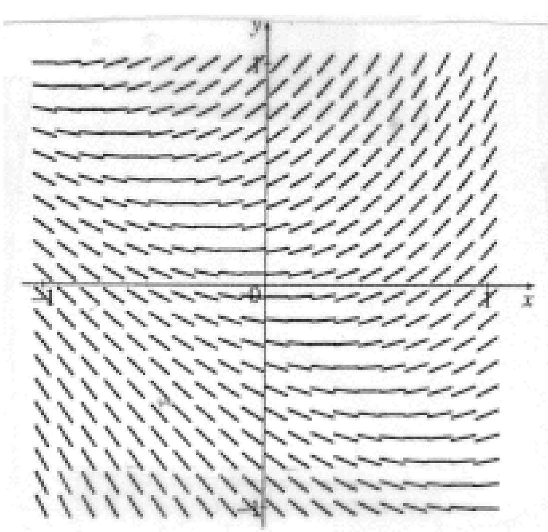
is shown below. (The scales on the x- and y-axes are the same, and some of the values from -2 to 2 are marked on the y-axis.)



(a) For which regions are all solution curves increasing? Justify your answer using the differential equation.

(b) For which regions do the solution curves tend toward a finite y-value as $x \to \infty$? Justify your answer using the differential equation.

5. A direction field is given in the picture below. Which of the following represents its differential equation? Explain why each of the other differential equations is **not** represented by the direction field.



(a) $y' = y - x$

(b) $y' = y^2 - x^2$

(c) $y' = y + x$

(d) $y' = y^2 + x^2$

(e) $y' = y - x^2$

(f) $y' = x - y^2$

Separation of Variables

6. Solve the differential equation $\dfrac{dy}{dx} = \dfrac{x + \sin x}{3y^2}$.

7. Solve the initial value problem

$$\frac{dy}{dx} = 4 - 7y, \qquad y(0) = 3.$$

   Show all of your work, with full mathematical justification.

8. In this problem, we will solve the differential equation $xy' + 2y = \cos(x^2)$, even though it is not a separable equation.

   (a) Suppose $y(x)$ satisfies the above equation (for $x \neq 0$). Verify that the new function $z(x) = x^2 y(x)$ satisfies $z' = x\cos(x^2)$.

   (b) Use separation of variables to find all solutions to $z' = x\cos(x^2)$.

   (c) Solve the initial value problem

$$xy' + 2y = \cos(x^2), \qquad y(\sqrt{\pi}) = 0.$$

   (Hint: remember, the function $x^2 y(x)$ is a solution to part (b).)

9. An equation used to model the growth of animal tumors is given by $y' = -ay\ln(y/b)$, where a and b are positive constants. (This is known as the Gompertz equation.)

   (a) Find any equilibrium solutions of the Gompertz equation.

   (b) If $y(0) > 0$, which one of the following values equals $\lim_{t\to\infty} y(t)$?

   (i) 0
   (ii) a
   (iii) b
   (iv) 1
   (v) $\dfrac{1}{2}$
   (vi) $a/2$
   (vii) $b/2$

   (c) Solve the Gompertz equation. Show all of your work, and be sure to find y as a function of t.

   (d) Verify that the answer you obtained in part (c) satisfies the Gompertz equation. Show all of your work.

10. A tank is constructed in the shape of a cone, with vertex pointing down, having height H and base radius R. The vertex of the cone has a valve which releases water; the rate $\frac{dV}{dt}$ of decrease in the volume of water in the tank at time t is proportional to the water's height $h(t)$ at time t. Let k be the constant of proportionality.

   (a) Find the differential equation satisfied by the height $h(t)$. (Hint: when the water in the cone has height h, the volume of this water is $V = \frac{1}{3}\pi\left(\frac{R}{H}h\right)^2 h$.)

   (b) Solve the above equation for $h(t)$, subject to the initial condition that the tank is full at time $t = 0$.

   (c) How long does it take for all the water to drain out?

Exponential Growth and Decay

11. A leaf of lettuce from one of the dining areas on campus contains approximately 99.999536% as much carbon-14 as a freshly cut lettuce leaf. (Recall that upon harvesting, the C^{14} in a living object decays with a fixed half-life; use 5730 years for the value of the half-life.)
    (a) How long ago was the lettuce leaf harvested?
    (b) How much carbon-14 would a one-week-old leaf of lettuce contain?
12. A population growing at a constant relative growth rate takes 20 years to triple. How long does it take for the same population to double?
13. In 1970, the Brown County groundhog population was 100. By 1980, there were 900 groundhogs in Brown County. If the rate of population growth of these animals is proportional to the population size, how many groundhogs might one expect to see in 1995?
14. The method of carbon-14 dating was used to trace the successive emergence of the Hawaiian islands from the oldest, Kauai, to the youngest, Hawaii. The island of Hawaii is approximately $100,000$ years old. What fraction of its original Carbon 14 does it contain? (Recall that the half-life of carbon-14 is approximately 5730 years.)

The Logistic Equation

15. A population $P(t)$ grows according to the logistic equation, with initial relative growth rate 0.05 and carrying capacity 30,000. If the population is initially equal to 5700, write down an initial value problem for the population. (You do not need to solve it.)
16. The growth of a certain rabbit population is given by the logistic equation with proportionality constant (i.e., initial relative growth rate) 0.25 when time is measured in months. Assume that at time $t = 0$, the population is equal to 1 percent of its carrying capacity K. How many months does it take the population to climb to a level of 50% of its carrying capacity?

Multi-Topical Questions

17. In this problem, we will analyze the differential equation

$$\frac{dP}{dt} = \frac{1}{2}P\left(1 - \frac{P}{50}\right) - 6.$$

    (a) Suppose $P(t)$ represents the number of mature lemons on a tree at time t, where t is measured in months. Explain the significance of the term -6. (Hint: you might suggest an interpretation for this term.)
    (b) Find any equilibrium solutions to the differential equation.
    (c) For which values of P is $P(t)$ increasing? decreasing?
    (d) Using the equation, find the population for which the increase in the population is the most rapid. Be sure to include all of the details necessary for a complete solution.

(e) Use the equation to show what happens to P in the long run. Be sure to consider the cases for various initial values of P.

(f) Solve the initial value problem

$$\frac{dP}{dt} = \frac{1}{2}P\left(1 - \frac{P}{50}\right) - 6, \qquad P(0) = 25.$$

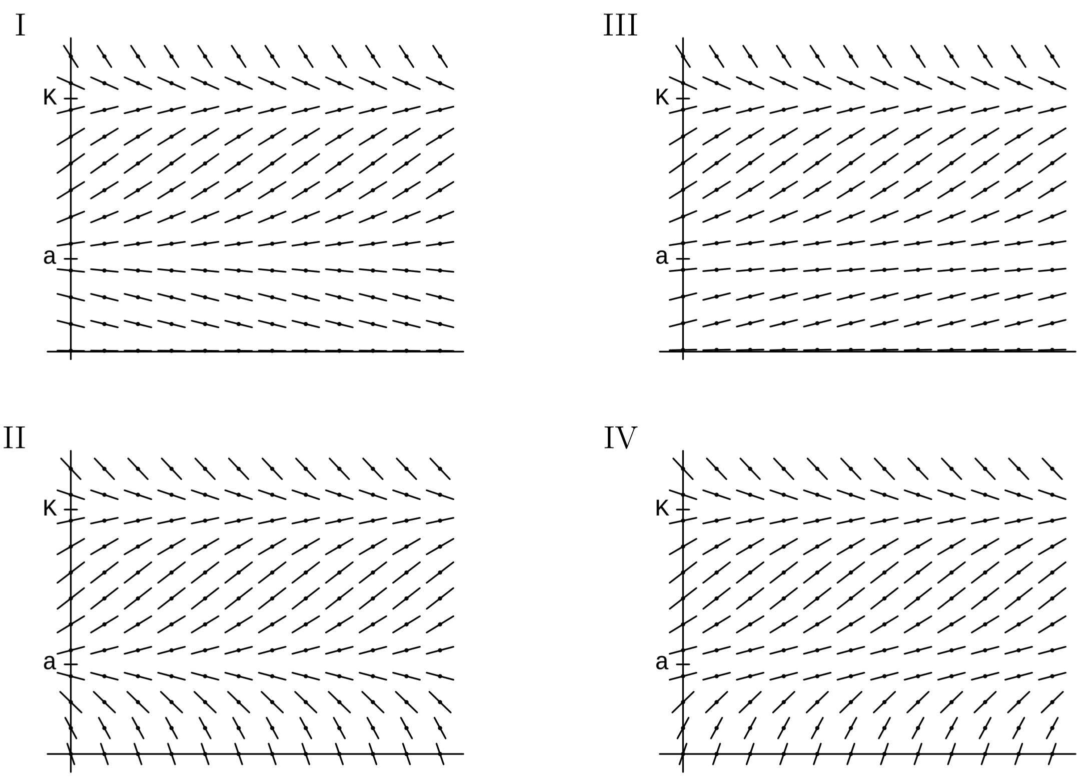
18. In many species, if the population density falls below a certain level, then the population may experience very low reproduction rates, due to an inability to find mates. (This is called an Allee effect – see *Animal Aggressions: A Study in General Sociology*, by W. C. Allee, University of Chicago Press, 1931.) This can be modeled with an extension of the logistic equation as follows. Let $N = N(t)$ be the density of a population at time t. Then

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)\left(1 - \frac{a}{N}\right),$$

where r, a and K are positive constants. We assume that $0 < a < K$, where K is the carrying capacity. The constant a is called the threshold population size.

(a) Find *all* equilibrium solutions of the differential equation.

(b) Which of the following best depicts the direction field of the above equation?



(c) If a solution curve (other than an equilibrium solution) approaches an equilibrium value as $t \to \infty$, such a limiting value is called a *stable equilibrium*. Determine which (if any) of the equilibrium solutions from part (a) are stable. Give full mathematical justification of your answers, using the differential equation.

Systems of Differential Equations

19. The differential equations shown below form a predator-prey system, with populations $x(t)$ and $y(t)$ and positive constants a, b, c, d.

$$\frac{dx}{dt} = ax - bxy$$
$$\frac{dy}{dt} = -cy + dxy$$

   (a) Which population is the predator and which is the prey? Explain/justify your answer using the differential equations.

   (b) Which constant should be larger, b or d? Give as thorough an argument as possible to support your answer.

20. Assume that we have an environment populated by five species, whose population sizes are given by the functions $v(t)$, $w(t)$, $x(t)$, $y(t)$, and $z(t)$. Suppose that x, y, and z satisfy the system of differential equations:

$$x' = 0.1x - 0.005xz$$
$$y' = -0.004y + 0.00008yz$$
$$z' = -0.03z + 0.0002xz - 0.0005yz$$

   Furthermore, v and w satisfy the system

$$v' = -0.04v + 0.0005vw$$
$$w' = -0.1w + 0.003vw$$

   Discuss which species are predators upon which species, which species cooperate with which species, and/or which species compete with which species.

21. Suppose we have an environment populated by rabbits and wolves, and that we use the usual system of differential equations to model their populations:

$$\frac{dR}{dt} = kR - aRW$$
$$\frac{dW}{dt} = -rW + bRW$$

   where $k = 2.4$, $a = 0.4$, $r = 10$, and $b = 0.2$.

   (a) Find all equilibrium solutions (R, W) where both the rabbit and wolf populations are constant.

   (b) Suppose that a sudden natural disaster instantly reduces each of the rabbit and wolf populations to 50% of their equilibrium sizes. Indicate the *rate* and *direction* of change that each population will initially experience after this point in time.

   (c) Suppose that the wolf population drops to zero at a point where the rabbit population is 60. Find an explicit formula for the rabbit population as a function of time, as indicated by this system of equations.

22. Two companies share the market for a new technology. Let $x(t)$ be the net worth of one company (in millions of dollars) and let $y(t)$ be the net worth of the other company (in millions of dollars), at time t months. Suppose x and y satisfy the differential equations

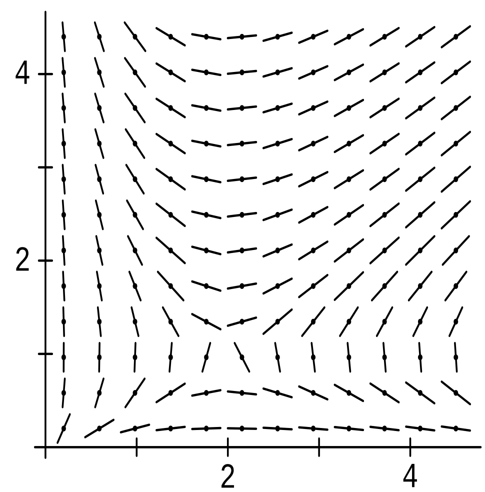
$$x' = x - xy$$
$$y' = 2y - xy$$

(a) Find the equilibrium solutions for this system of differential equations.

(b) Find an expression for $\frac{dy}{dx}$.

(c) (For this and part (d), use the system's slope field, provided below. As usual, "x" is plotted on the horizontal axis, and "y" on the vertical. Note that arrows have *not* been drawn on the slope lines.)

Explain what happens to the net worths of the companies in the long run if $x(0) = 4$ and $y(0) = 3/2$. Also, draw a trajectory (that is, a solution curve) on the slope field below that starts at $x(0) = 4$ and $y(0) = 3/2$. Include a direction arrow on your trajectory.



(d) Explain what happens in the long run if initially $x = 4$ and $y = 3/2$, but because of venture capitalist investments, very soon after $t = 0$, y is increased by 2. Also, draw a trajectory on the slope field below that shows this scenario, starting at $x(0) = 4$ and $y(0) = 3/2$. Include a direction arrow on your trajectory.