Math 42 — Chapter 7 Practice Problems — Set A

1. Show that $y = \dfrac{\ln t + 3}{t}$ is the solution of the initial value problem: $t^2 y' + ty = 1, \quad y(1) = 3.$

2. Show that $y = \dfrac{1}{\sqrt{5 - t^2}}$ is the solution of the initial value problem: $y' = ty^3, \quad y(1) = \dfrac{1}{2}.$

3. Find the solution of the initial value problem: $y' = \dfrac{e^{-y^2}(t+1)}{yt^2}, \quad y(1) = 2.$

4. Find the solution of the initial value problem: $y' = te^{-t}y^2, \quad y(0) = 2.$

5. In a certain country the population grows according to natural growth with relative growth rate $k = \frac{1}{10}$ per year, but crowding also encourages a certain number of people to leave. Suppose that people are leaving the country at a rate $\frac{1}{10}\sqrt{P}$ million people/year, where P is the population in millions.

   (a) Write a differential equation which models the growth of the population.

   (b) What are the equilibrium values of the population?

   (c) What will happen in the long term if there are initially half a million, one million, or four million people?

   (d) Suppose there are initially $P_0 = 4$ million people. Use Euler's method with $h = 5$ to estimate what the population will be in 10 years.

   (e) Use the differential equation to find an exact expression for the population after t years if the initial population is 4 million.

6. A certain population of animals is dependent on a seasonally varying food supply. The rate at which the population grows is proportional to both the current population size P and to $\cos(\frac{\pi}{6}t)$; i.e., it is proportional to their product. (Here t is the time measured in months.)

   Suppose the initial relative growth rate (i.e., $\frac{1}{P}P'$ when $t = 0$) is $\frac{1}{6}$ per month, and the initial population is 1000.

   (a) Write a differential equation which models the growth of this population.

   (b) Suppose the initial population is 1000. Use Euler's method with $h = 3$ to estimate the population after 12 months.

   (c) Solve the differential equation to find an exact expression for the population after t months.

   (d) Find the exact population after 12 months, and compare to the Euler's method estimate.

7. Suppose the growth of a population is modeled by the modified logistic equation

$$\frac{dP}{dt} = \frac{P}{10}\left(1 - \frac{P}{1000}\right)\left(1 - \frac{200}{P}\right).$$

   (a) What are the equilibrium values of the population?

   (b) If $P(0) = 400$, use Euler's method with $h = 5$ to estimate the population at time $t = 10$.

8. Suppose a population of fish in a lake grows according to a logistic model with carrying capacity 1000 and proportionality constant $k = 1/10$, but in addition 50 fish are caught per unit time.

   (a) Write a differential equation that models the growth of this population of fish.

   (b) Suppose the population at time $t = 0$ is 400. Use Euler's Method with $h = 2$ to estimate the population at time $t = 4$.

9. True / False: (You do not need to justify your answer.)

   (a) All solutions of $y' = -3 - y^2$ are decreasing functions.

   (b) The logistic equation has exactly one nonzero equilibrium solution.

   (c) All solutions of the differential equation $y' = 2 + \cos(ty)$ are increasing functions.

   (d) If y is the solution of the initial value problem

   $$y' = 3y\left(1 - \frac{y}{20}\right), \quad y(0) = 4$$

   then $\lim_{t\to\infty} y(t) = 20$.

   (e) The function $f(t) = \dfrac{\ln t}{t}$ is a solution of $t^2 y' + ty = 1$.

10. Solve the initial value problem

    $$\frac{dy}{dt} = 3y - 2ty, \qquad y(0) = 5.$$

    Show all of your work, with full mathematical justification.

11. The population of a species of elk on an Alaskan island has been observed to be closely predicted by a logistic model. The following specific observations have also been made:

    - When the elk population was 600, the population was growing at a rate of 20 percent per year, or 120 elk per year.
    - When the population was 800, the growth was 10 percent per year; i.e., 80 elk per year.

    Find the carrying capacity of the population, giving complete mathematical justification.

12. Two species, X and Y, compete with each other for the same limited resources. Their population sizes, $x(t)$ and $y(t)$, measured in the thousands and modeled as functions of time t (in years), obey the following system of differential equations:

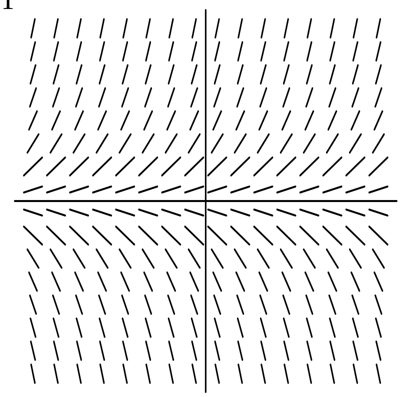
    $$\frac{dx}{dt} = 2x - xy,$$
    $$\frac{dy}{dt} = 3y - xy.$$

    (a) Find the equilibrium solutions of this system. Show your work.

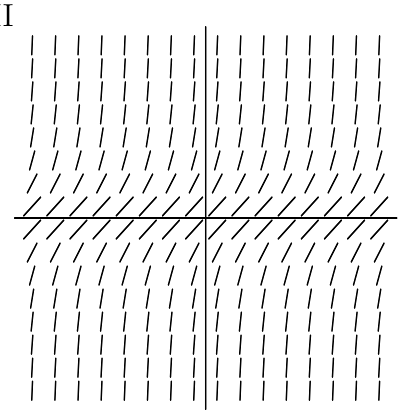
    (b) At time $t = 0$ suppose $x(0) = 2$ and $y(0) = 4$. Find the rates of change of the two populations at this moment. What will happen in the long run, as t increases? Justify your answer.

13. Match the direction fields below with their differential equations, and give your reasoning. Each field is graphed for $-5 \leq x \leq 5$, $-5 \leq y \leq 5$.

I



II



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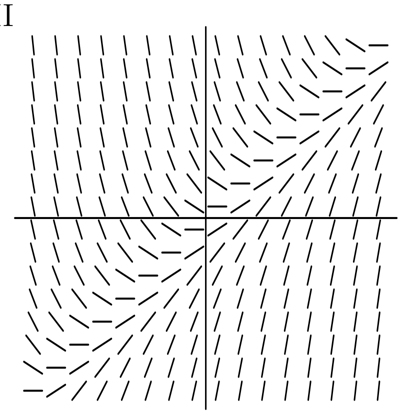

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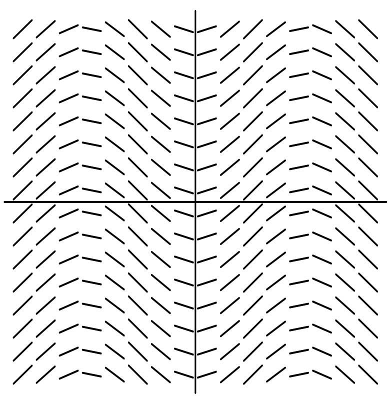
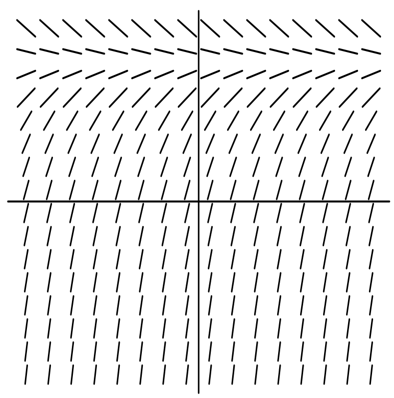
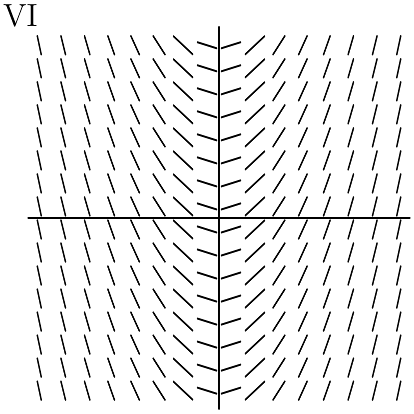


V



VI

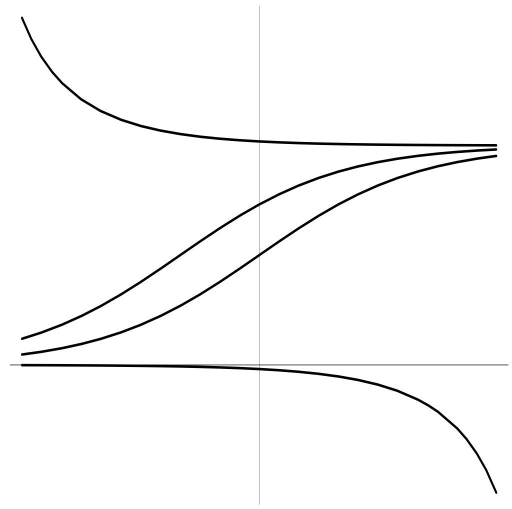


Equation	I, II, III, IV, V, or VI	Brief reason
$dy/dx = 1 + y^2$		
$dy/dx = x$		
$dy/dx = \sin x$		
$dy/dx = y$		
$dy/dx = x - y$		
$dy/dx = 4 - y$		

14. Each picture below depicts a few possible solution curves to a differential equation chosen from the list at the bottom of the page. Match each equation to its sketch of solutions. Here a, b, and k are fixed *positive* constants. The scales on the axes (t is horizontal; y is vertical) have been omitted because they don't affect the answers.

I

II



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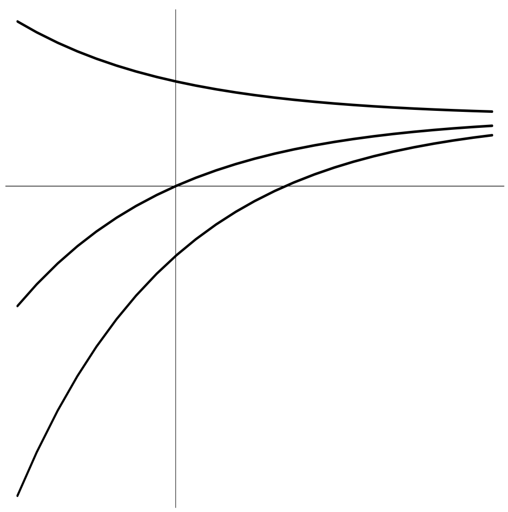

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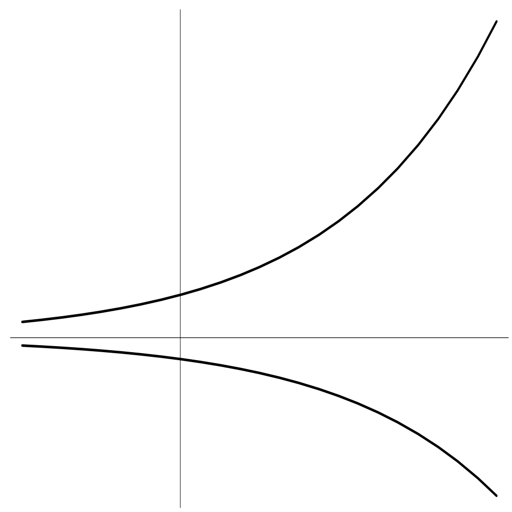
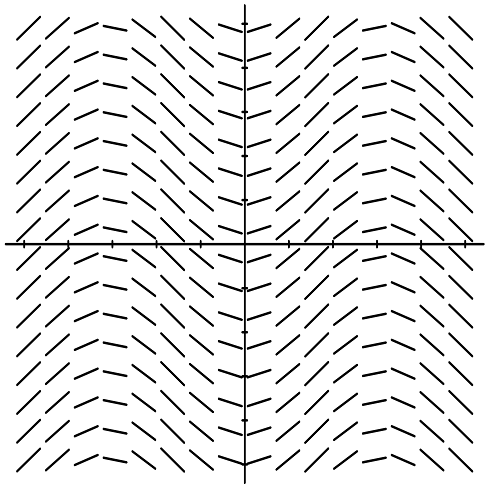


Equation	I, II, III, or IV	Brief reason
$dy/dt = ky$		
$dy/dt = (a - by)y$		
$dy/dt = a - by$		
$dy/dt = (a - bt)y$		

15. Each direction field pictured below corresponds to one of the differential equations listed at the bottom of the page. Match each equation with its direction field (or write "none," which will be necessary for one equation), and give your reasoning. Here k is a fixed positive constant, and the scales on all axes are the same. (The horizontal variable is t; the vertical is y.)

I



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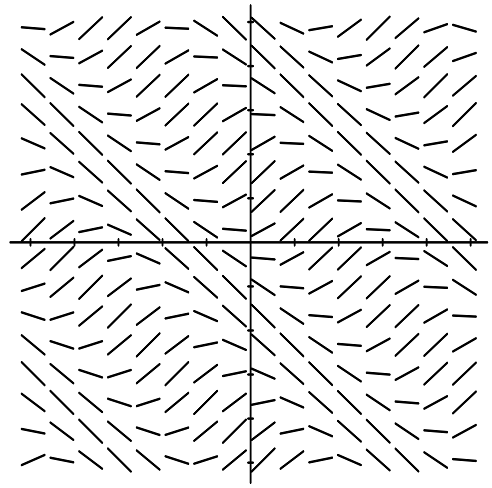

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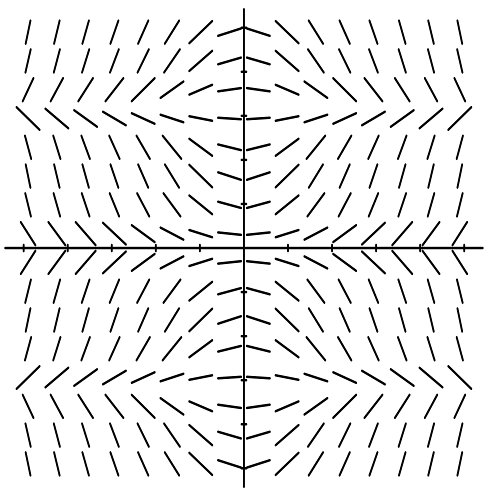

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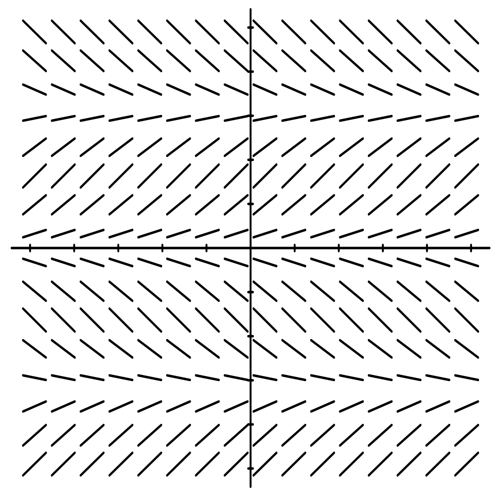

Equation	I, II, III, IV, or "none"	Brief reason
$dy/dt = k \sin y$		
$dy/dt = k \sin t$		
$dy/dt = kt \sin y$		
$dy/dt = k \sin(ty)$		
$dy/dt = k \sin(t+y)$		

16. The *Millenium Falcon*, widely believed to be the fastest spaceship in the galaxy, is trying to fly away from the Death Star without getting pulled in by the Death Star's powerful tractor beam. The beam is activated at time $t = 0$, and the distance $x(t)$, in kilometers, between the spaceship and the (stationary) Death Star after t seconds satisfies the differential equation

$$\frac{dx}{dt} = -7x(x - 100)(x - 10000).$$

   (a) What is the physical meaning of "equilibrium solution" in this context? Find all equilibrium solutions.

   (b) For what positive values of x is x increasing? decreasing? Give complete answers.

   (c) Predict the fate of the *Falcon* as t approaches infinity; your answer will likely depend on the initial distance $x_0 = x(0)$. That is, are there values of x_0 for which the ship escapes? Is sucked into the Death Star? Or some other possibility? A complete description should provide predictions that together cover every possible (positive) value of the initial distance x_0.

17. (a) The graph of a function $y = y(x)$ in the xy-plane has the property that the slope of the curve at each point is proportional to the y-coordinate of that point. Moreover, the curve passes through the point $(4, -9)$, and the value of the second derivative of the function at this point is -1. Find the function (or family of functions) with these properties.

   (b) Find the general solution $y(t)$ to

$$\frac{dy}{dt} = \frac{1}{10}(10 - 3y)(10 + 3y).$$

   (Hint: an integration technique may be helpful, but it can be avoided by instead showing that the function $P(t) = 10 + 3y(t)$ satisfies a certain Logistic differential equation involving dP/dt.)

18. On a strange tropical island, two native species, the mattababy and the henway, participate in a very predictable relationship. The mattababy population is described over time by a function $x(t)$, where t is measured in days; the henway population is described by a function $y(t)$. Naturalists coming to the island observe that the rates of growth of the two species are given by the equations

$$\frac{dx}{dt} = -4x + 0.02xy$$
$$\frac{dy}{dt} = -3y + 0.01xy$$

   (a) Is the relationship between the two species one of mutual competition, mutual cooperation, or predator-prey (and if the latter case, which plays which role)? Justify your answer by explaining how the equations show what effect each species has on the other.

   (b) Find all equilibrium solutions to the above system. Show your work.