Set A

1. (12 points)

(a) Write the series

$$-\frac{3}{1}+\frac{5}{4}-\frac{7}{9}+\frac{9}{16}-\frac{11}{25}+\frac{13}{36}-\frac{15}{49}+\cdots$$

in sigma notation. (You don't have to investigate convergence or divergence.)

(Any one answer is acceptable)

$$\sum_{n=1}^{\infty} \frac{(-1)^n \cdot (2n+1)}{n^2} \quad \text{or} \quad \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \cdot (2n+3)}{(n+1)^2}$$

$$\left(\text{or} \sum_{n=2} \frac{(-1)^{n+1} \cdot (2n-1)}{(n-1)^2} \right)$$

(b) Determine whether the series

$$\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{2^{3n}}$$

converges or diverges, and compute the sum if it is convergent. Show all work.

Since $\frac{(-3)^{n+1}}{2^{3n}} = \frac{(-3)\cdot(-3)^n}{(2^3)^n} = (-3)\cdot\left(\frac{-3}{8}\right)^n,$

this is a geometric series with initial term $\frac{9}{8}$ and ratio $\frac{-3}{8}$.

Since $\left|\frac{-3}{8}\right| < 1$, the series converges, and the sum is $\frac{9/8}{1-(-3/8)} = \boxed{\frac{9}{11}}$.

2. (24 points) Determine whether the following series are convergent or divergent. Indicate clearly which tests you use and what conclusions you draw from them.

(a) $\sum_{n=1}^{\infty} \frac{n+5}{5^n}$

Ratio test: $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$= \lim_{n\to\infty} \left(\frac{n+6}{5^{n+1}}\right) \Big/ \left(\frac{n+5}{5^n}\right)$$

$$= \lim_{n\to\infty} \frac{n+6}{5^{n+1}} \cdot \frac{5^n}{n+5}$$

$$= \lim_{n\to\infty} \frac{5^{n+1}}{5^n} \cdot \frac{n+6}{n+5}$$

$$= \lim_{n\to\infty} \frac{1}{5} \cdot \left(\frac{1+6/n}{1+5/n}\right)$$

$$= \frac{1}{5} \cdot \frac{1+0}{1+0} = \underline{\underline{\frac{1}{5} < 1}},$$

so the series converges.

(b) $\sum_{n=1}^{\infty}(-1)^n 2^{1/n}$

Test for Divergence: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n 2^{1/n}$ does not exist,

because as $n\to\infty$, the value $2^{1/n}$ approaches $2^0 = 1$,

but the sign factor $(-1)^n$ alternates, so $(-1)^n 2^{1/n}$

is alternately close to -1 or 1; there is no single limit value.

Thus, by the Test for Divergence, this series must be divergent.

(c) $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$

Ratio Test: $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left(\frac{3^{n+1}\cdot(n+1)^2}{(n+1)!} \right) \Big/ \left(\frac{3^n n^2}{n!} \right)$

$$= \lim_{n\to\infty} \frac{3^{n+1}\cdot(n+1)^2}{(n+1)!} \cdot \frac{n!}{3^n n^2}$$

$$= \lim_{n\to\infty} \frac{3^{n+1}}{3^n} \cdot \frac{n!}{(n+1)!} \cdot \frac{(n+1)^2}{n^2}$$

$$= \lim_{n\to\infty} 3 \cdot \frac{n!}{(n+1)\cdot n!} \cdot \left(\frac{n+1}{n}\right)^2$$

$$= 3 \cdot \lim_{n\to\infty} \frac{1}{n+1} \cdot \left(\frac{1+1/n}{1}\right)^2$$

$$= 3 \cdot 0 = \underline{\underline{0 < 1}},$$

so the series converges.

(d) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

Let $f(x) = \frac{1}{x\sqrt{\ln x}}$. Since $a_n = f(n)$ for $n \geq 2$,

we examine the improper integral $\int_2^{\infty} f(x)\,dx$.

Since $\int_2^{\infty} f(x)dx = \lim_{N\to\infty} \int_2^N \frac{1}{x\sqrt{\ln x}}\,dx = \lim_{N\to\infty} \int_{\ln 2}^{\ln N} \frac{1}{\sqrt{u}}\,du$ $\left(\begin{array}{c}\text{using:}\\ u = \ln x\\ du = \frac{1}{x}dx\end{array}\right)$

$$= \lim_{N\to\infty} \int_{\ln 2}^{\ln N} u^{-1/2}\,du$$

$$= \lim_{N\to\infty} 2u^{1/2}\Big]_{\ln 2}^{\ln N}$$

$$= \lim_{N\to\infty} 2\sqrt{\ln N} - 2\sqrt{\ln 2}$$

$$= \infty,$$

the improper integral is divergent.

By the Integral Test, this implies that the series $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ is also divergent.

3. (20 points) Find, with complete justification, the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{n+1}{n^3}(3x+2)^n$$

Ratio Test : $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\left(\frac{n+2}{(n+1)^3}(3x+2)^{n+1}\right) \cdot \left(\frac{n^3}{n+1} \cdot \frac{1}{(3x+2)^n}\right)\right|$

$$= \lim_{n\to\infty} \left| \frac{n+2}{n+1} \cdot \frac{n^3}{(n+1)^3} \cdot \frac{(3x+2)^{n+1}}{(3x+2)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{1+2/n}{1+1/n} \cdot \left(\frac{n}{n+1}\right)^3 \cdot (3x+2) \right|$$

$$= |3x+2| \cdot \lim_{n\to\infty} \left| \frac{1+2/n}{1+1/n} \cdot \left(\frac{1}{1+1/n}\right)^3 \right|$$

$$= |3x+2| \cdot 1 = |3x+2| .$$

The case $L<1$ gives convergence; this corresponds to $|3x+2|<1$,

so $-1<3x+2<1$, or

$-3 < 3x < -1$, which means $-1<x<-1/3$.

But since $L=1$ gives an inconclusive result, we must also check these values of x, i.e. where $|3x+2|=1$, meaning the endpoints $x=-1$ and $x=-1/3$.

Case $x=-1$: Series becomes $\sum_{n=1}^{\infty} \frac{n+1}{n^3} \cdot (-1)^n$, which is an alternating series.

- Decreasing term size? Yes, because $\frac{n+1}{n^3} = \frac{n}{n^3} + \frac{1}{n^3} = \frac{1}{n^2} + \frac{1}{n^3}$, and both terms are decreasing functions of n;
- Terms go to zero? Yes, because $\lim_{n\to\infty} \frac{n+1}{n^3} = \lim_{n\to\infty} \frac{1/n^2 + 1/n^3}{1} = 0$.

Thus, all conditions of Alternating Series Test are satisfied, so series converges.

(Use this page for any additional work on Problem 3.)

Case $x=-\frac{1}{3}$: Series becomes $\sum_{n=1}^{\infty} \frac{n+1}{n^3} \cdot (1)^n = \sum_{n=1}^{\infty} \frac{n+1}{n^3} = \sum_{n=1}^{\infty}\left(\frac{1}{n^2}+\frac{1}{n^3}\right)$.

Since both $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ are p-series that converge (since $p=2$ and $p=3$, respectively, both greater than 1), the series whose terms are $\frac{1}{n^2}+\frac{1}{n^3}$ also converges.

Conclusion: We combine the interval found via the Ratio test with the endpoint values that were found to give convergent series, to give $\{-1 \leq x \leq -\frac{1}{3}\}$, or $\boxed{[-1, -\frac{1}{3}]}$, as the interval of convergence.

4. (12 points) For this problem, we consider the series

$$s = \sum_{n=1}^{\infty} \frac{1}{n^5} = 1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \cdots$$

(Interesting aside: one reason why we'd care about s is that it is the value of the so-called *Riemann Zeta Function* at 5: this function plays an important role in the field of number theory, which concerns among other things the behavior of prime numbers, and surprisingly has applications to things like secure Internet communication.)

(a) Explain why this is a convergent series; that is, explain why the number s is defined.

The series is convergent because it is a p-series with $p = 5 > 1$.

(Alternatively, one could use the Integral Test to prove convergence. Since the function $\frac{1}{x^5}$ is continuous, positive, and decreasing, and our terms $a_n = \frac{1}{n^5}$, we may examine the integral $\int_1^\infty \frac{1}{x^5} dx$.)

(Try it!)

(b) If the first 10 terms of the series were used to approximate s, determine the accuracy of this approximation. State your conclusion in a complete sentence, and be as quantitatively precise as you can (but you do not need to simplify any expressions).

First 10 terms: $S_{10} = 1 + \frac{1}{2^5} + \cdots + \frac{1}{10^5}$. (Note, this is certainly an underestimate.)

The accuracy is measured by size of error (remainder) $R_{10} = S - S_{10}$.

By the Integral Test's Remainder Estimate, $\int_{11}^{\infty} \frac{1}{x^5}dx \leq R_{10} \leq \int_{10}^{\infty} \frac{1}{x^5}dx$.
(See previous page to recall why Integral Test applies.)

Evaluating integrals: $\int_a^{\infty} \frac{1}{x^5}dx = \lim_{N\to\infty} \int_a^N \frac{1}{x^5}dx = \lim_{N\to\infty} \left.\frac{x^{-4}}{-4}\right]_a^N = \lim_{N\to\infty} -\frac{1}{4}\left(\frac{1}{N^4} - \frac{1}{a^4}\right) = \frac{1}{4a^4}$, for any a,

so $\boxed{\frac{1}{4\cdot 11^4} \leq R_{10} \leq \frac{1}{4\cdot 10^4}}$.

In other words, the 10th partial sum (our approximation) is no worse than $\frac{1}{4\cdot 10^4}$ less than the actual sum, but is at least $\frac{1}{4\cdot 11^4}$ too small.

(c) It turns out that the sum of the first 10 terms of the series is the value 1.0369073413... Use your reasoning from part (b) to obtain a more accurate approximation of s, *without* having to consider any more terms from the series. Your answer does not need to be simplified (or fully evaluated in decimal form).

Since 1.0369073413... (also known as S_{10}) is at least $\frac{1}{4\cdot 11^4}$ too small, it makes sense that we should "add on" at least this much more. The best estimate we can make along these lines, since we also know that we should not "add on" more than $\frac{1}{4\cdot 10^4}$, is to average these two values, and "add on" the result: so,

$$S \approx S_{10} + \frac{\frac{1}{4\cdot 11^4} + \frac{1}{4\cdot 10^4}}{2}$$

$$= \boxed{1.0369073413\ldots + \frac{1}{8}\left(\frac{1}{11^4} + \frac{1}{10^4}\right)}.$$

(This is known to be accurate to within $\frac{1}{2}\left(\frac{1}{4\cdot 10^4} - \frac{1}{4\cdot 11^4}\right)$ in either direction, much better than S_{10}.)

5. (12 points) Compute the Taylor series for $\ln(1+x)$ about 0.

Computing coefficients for $f(x)=\ln(1+x)$, about $a=0$:

$$f(x)=\ln(1+x) \Rightarrow f(0)=\ln(1+0)=0 \Rightarrow c_0=\frac{f(0)}{0!}=0,$$

$$f'(x)=\frac{1}{1+x} \Rightarrow f'(0)=\frac{1}{1+0}=1 \Rightarrow c_1=\frac{f'(0)}{1!}=1,$$

$$f''(x)=-\frac{1}{(1+x)^2} \Rightarrow f''(0)=-\frac{1}{(1+0)^2}=-1 \Rightarrow c_2=\frac{f''(0)}{2!}=-\frac{1}{2},$$

$$f'''(x)=\frac{2}{(1+x)^3} \Rightarrow f'''(0)=\frac{2}{(1+0)^3}=2 \Rightarrow c_3=\frac{f'''(0)}{3!}=\frac{2}{6}=\frac{1}{3},$$

$$f''''(x)=\frac{-3\cdot 2}{(1+x)^4} \Rightarrow f''''(0)=\frac{-3\cdot 2}{(1+0)^4}=-3! \Rightarrow c_4=\frac{f''''(0)}{4!}=-\frac{3!}{4!}=-\frac{1}{4},$$

$$f^{(5)}(x)=\frac{4\cdot 3\cdot 2}{(1+x)^5} \Rightarrow f^{(5)}(0)=\frac{4\cdot 3\cdot 2}{(1+0)^5}=4! \Rightarrow c_5=\frac{f^{(5)}(0)}{5!}=\frac{4!}{5!}=\frac{1}{5}.$$

The pattern emerging is that $f^{(n)}(x)=(-1)^{n+1}\frac{(n-1)!}{(1+x)^n}$, for $n\geq 1$,

so that $$c_n=\frac{f^{(n)}(0)}{n!}=\frac{(-1)^{n+1}(n-1)!}{n!\,(1+0)^n}=\frac{(-1)^{n+1}}{n}.$$

Thus the Taylor series is $\boxed{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot x^n} = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\dots\dots$

(Note $c_0=0$)

6. (20 points) Let $f(x) = \sqrt{x}$.

(a) Find both the degree-1 and degree-2 Taylor polynomials for f about 100. (These functions are also called, respectively, the linear and quadratic approximations for f at 100.)

Computing Taylor coefficients for $f(x) = x^{1/2}$, about $a = 100$:

$$\begin{array}{lll} f(x) = x^{1/2} & f(100) = \sqrt{100} = 10 & c_0 = \frac{f(100)}{0!} = 10 \\ f'(x) = \frac{1}{2}x^{-1/2} & f'(100) = \frac{1}{2} \cdot \frac{1}{\sqrt{100}} = \frac{1}{20} & c_1 = \frac{f'(100)}{1!} = \frac{1}{20} \\ f''(x) = -\frac{1}{4}x^{-3/2} & f''(100) = -\frac{1}{4} \cdot \frac{1}{(100)^{3/2}} = -\frac{1}{4000} & c_2 = \frac{f''(100)}{2!} = \frac{-1}{8000} \end{array}$$

Thus $T_1(x) = c_0 + c_1(x-100) = \boxed{10 + \frac{1}{20}(x-100)}$,

and $T_2(x) = c_0 + c_1(x-100) + c_2(x-100)^2 = \boxed{10 + \frac{1}{20}(x-100) - \frac{1}{8000}(x-100)^2}$.

(b) Use the polynomials from part (a) to obtain two different approximations for $\sqrt{97}$.

Using T_1, we have $\sqrt{97} = f(97) \approx T_1(97) = 10 + \frac{1}{20}(97-100)$

$$= 10 - \frac{3}{20} = \boxed{9\frac{17}{20}}.$$

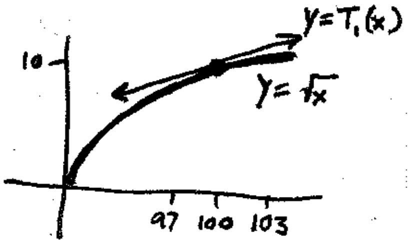
Using T_2, we have $\sqrt{97} = f(97) \approx T_2(97) = 10 + \frac{1}{20}(97-100) - \frac{1}{8000}(97-100)^2$

$$= \boxed{9\frac{17}{20} - \frac{9}{8000}}.$$

$$\left(= 9\frac{6791}{8000}\right)$$

(c) Determine the accuracy of each approximation, including whether each is an underestimate or an overestimate of the actual value. You may use any valid reasoning (and, if you wish, you may take as a fact that the Taylor series converges to $\sqrt{97}$). State your conclusions in sentence form, being as precise as you can (but without needing to simplify any arithmetic).

The linear approximation is an overestimate, which can be argued graphically:



The graph of $y=T_1(x)$ is just the tangent line at $x=100$ to the curve $y=\sqrt{x}$. Since the graph of $f(x)=\sqrt{x}$ is concave down (see prev. page where we find $f''(x)<0$), the tangent lies above curve, and value $T_1(97)$ will be greater than $\sqrt{97}$.

The precise measure of accuracy for $T_1(97)$ can be obtained through Taylor's Inequality, which measures the error (remainder) $R_1(97)=f(97)-T_1(97)$. But to do this, we must find an M such that $|f''(x)|\leq M$ for $97\leq x\leq 103$: [This interval is specified by Taylor's Inequality.]

- Since $f''(x)=-\frac{1}{4}x^{-3/2}$, we see that $|f''(x)|=\frac{1}{4x^{3/2}}$ is greatest when x is least, so we can use $M=\frac{1}{4\cdot 97^{3/2}}$ since our interval is $[97,103]$.

Thus, according to Taylor's Inequality with $n=1$, $a=100$, $x=97$, we have

$$|R_1(97)|\leq\frac{M}{2!}\cdot|97-100|^2=\frac{9}{2}\cdot\frac{1}{4\cdot 97^{3/2}}=\boxed{\frac{9}{8\cdot 97^{3/2}}}.$$

i.e., the linear approximation is accurate to within $\frac{9}{8\cdot 97^{3/2}}$ of the actual value of $\sqrt{97}$, and is an overestimate.

(Thus, an even better sentiment is that the linear approximation is no more than $\frac{9}{8\cdot 97^{3/2}}$ too large an estimate.)

(Use this page for any additional work on Problem 6.)

We can just re-apply Taylor's Inequality to determine the accuracy of $T_2(97)$, but we need to find a new value of M, a value such that $|f'''| \le M$ on $[97,103]$.

- Since $f'''(x) = \frac{3}{8}x^{-5/2}$, we see that this is greatest when x is least, so we can use $\underline{\underline{M = \frac{3}{8}\cdot 97^{-5/2}}}$ since our specified interval is $97 \le x \le 103$.

Thus, using Taylor's Inequality with $n=2$, $a=100$, $x=97$, we have

$$|f(97) - T_2(97)| = |R_2(97)| \le \frac{M}{3!}|97-100|^3 = \boxed{\frac{3^3}{3!}\cdot\frac{3}{8}\cdot 97^{-5/2}},$$

i.e., the quadratic approximation $T_2(97)$ is accurate to within $\frac{3^4}{3!\cdot 8\cdot 97^{5/2}}$ of the actual value of $\sqrt{97}$.

It's harder to appeal to graphical methods to answer whether $T_2(97)$ is an over- or under-estimate. However, we can make use of the fact that the Taylor series converges to $\sqrt{97}$ to notice the following:

$$f(97) = \sqrt{97} = \sum_{n=0}^{\infty} c_n(97-100)^n = 10 + \frac{1}{20}(97-100) - \frac{1}{8000}(97-100)^2 + c_3(97-100)^3 + c_4(97-100)^4 + \cdots,$$

and a pattern that will emerge in $c_1, c_2, c_3, c_4, \ldots$ is that they will alternate in sign: $c_1 = \frac{1}{20}$, $c_2 = -\frac{1}{8000}$, $c_3 = \frac{f'''(100)}{3!} = \frac{1}{3!}\cdot\frac{3}{8}\cdot(100)^{-5/2} > 0$,

and since $f''''(x) = -\frac{5}{2}\cdot\frac{3}{8}\cdot x^{-7/2}$, we'll have $c_4 = \frac{f''''(100)}{4!} < 0$, etc.

(The negative exponent will ensure this pattern continues.) Meanwhile, though, the terms $c_n(97-100)^n$ are going to stay the same sign (negative), since $(97-100)^n$ alternates sign too. Thus, $\sqrt{97} = 10 - \frac{3}{20} - \frac{9}{8000} - \left(\substack{\text{third}\\ \text{term}}\right) - \left(\substack{\text{fourth}\\ \text{term}}\right) - \cdots$, ← (all quantities positive)

and so the value $T_2(97)$, which is simply the sum of the first three terms, is going to be larger than the value of the infinite sum. Thus $T_2(97)$ is an overestimate for $\sqrt{97}$.

$\boxed{7.}$ (12 points) The Taylor series for $\arctan x$ about 0 is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

(you do *not* have to prove this).

(a) Find the *radius* of convergence of this series, showing all reasoning.

Ratio Test: Compute $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2(n+1)+1}}{2(n+1)+1} \cdot \frac{2n+1}{(-1)^n x^{2n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x^{2n+3}}{x^{2n+1}} \right| \cdot \left(\frac{2n+1}{2n+3} \right)$$

$$= |x^2| \cdot \lim_{n\to\infty} \frac{2n+1}{2n+3}$$

$$= |x^2| \cdot \lim_{n\to\infty} \frac{2+1/n}{2+3/n} = |x^2| \cdot \frac{2+0}{2+0} = |x^2| = x^2.$$

For convergence, we require $x^2 < 1 \Rightarrow -1 < x < 1$. This means the <u>radius</u> of convergence is $\boxed{R=1}$. (Note we aren't required to investigate convergence for $x=1$ or $x=-1$...)

(b) It is a fact that the value of the series at $x = 1$ equals the value $\arctan 1 = \frac{\pi}{4}$. That is,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots,$$

a formula due to Leibniz. Find, with justification, a partial sum of this series that represents $\frac{\pi}{4}$ accurately to within 10^{-2}. (You don't need to compute the value of the partial sum.)

We should be able to use the Alternating Series Remainder Estimate; what we need to do is check that the conditions of the Alternating Series Test are satisfied with our series:

$$S = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}.$$

Our terms are of the form $(-1)^{n-1} b_n$, for $b_n = \frac{1}{2n-1}$ $(n \geq 1)$.

Checking conditions of test:

- Is $b_{n+1} \leq b_n$ for all $n \geq 1$? Yes, because $b_n = \frac{1}{2n-1}$ is a decreasing function of n (as $2n-1$ is a positive & increasing function of n); equivalently, $\frac{1}{2(n+1)-1} \leq \frac{1}{2n-1}$ because $0 < 2n-1 \leq 2n+1$.
- Is $\lim_{n\to\infty} b_n = 0$? Yes! $\left[\lim_{n\to\infty} \frac{1}{2n+1} = 0.\right]$

So we can apply the Alt. Series Remainder Estimate, which says that the nth partial sum gives an error of $|S - S_n| \leq b_{n+1} = \frac{1}{2(n+1)-1} = \frac{1}{2n+1}$.

Since we want error less than 10^{-2}, we can take n so that $\frac{1}{2n+1} < \frac{1}{100}$.

Thus $n=50$ works, so the partial sum $\boxed{S_{50} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + (-1)^{49}\frac{1}{99}}$ represents $S = \frac{\pi}{4}$ accurately to within (less than) $\frac{1}{100}$ (in fact, $\frac{1}{101}$!).

8. (12 points) Suppose you are given a function f such that its n-th derivative evaluated at 6 is

$$f^{(n)}(6) = \frac{(-1)^n n!}{4^n(n+1)}.$$

(a) Find $f(6)$, $f'(6)$, and $f''(6)$, and use these values to write down the a formula for the quadratic approximation to f at 6 (that is, the Taylor polynomial $T_2(x)$ at 6).

$$f(6) = f^{(0)}(6) = \frac{(-1)^0 \cdot 0!}{4^0 \cdot (0+1)} = 1.$$

$$f'(6) = f^{(1)}(6) = \frac{(-1)^1 \cdot 1!}{4^1 \cdot (1+1)} = -\frac{1}{8}.$$

$$f''(6) = f^{(2)}(6) = \frac{(-1)^2 \cdot 2!}{4^2 \cdot (2+1)} = \frac{2}{16 \cdot 3} = \frac{1}{24}.$$

Thus, since $T_2(x) = c_0 + c_1 \cdot (x-6) + c_2 \cdot (x-6)^2$, where $c_n = \frac{f^{(n)}(6)}{n!}$, we have

$$T_2(x) = \frac{f(6)}{0!} + \frac{f'(6)}{1!} \cdot (x-6) + \frac{f''(6)}{2!} \cdot (x-6)^2 = \boxed{1 - \frac{1}{8}(x-6) + \frac{1}{48}(x-6)^2}.$$

(b) Suppose the Taylor series of f centered at 6 converges to $f(x)$ for all x. How many terms of the Taylor series are required to approximate $f(7)$ with error less than 0.002? Write a sum of numbers (which you do not have to simplify) that represents this approximation.

The Taylor series of f centered at 6 looks like $\sum_{n=0}^{\infty} c_n (x-6)^n$, where the coefficients $c_n = \frac{f^{(n)}(6)}{n!} = \frac{(-1)^n n!}{4^n \cdot (n+1) \cdot n!} = \frac{(-1)^n}{4^n \cdot (n+1)}$.

Moreover, to use this series to find $f(7)$, we'd be plugging in $x=7$:

$$f(7) = \sum_{n=0}^{\infty} c_n \cdot (7-6)^n = \sum_{n=0}^{\infty} c_n \cdot 1^n = \sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n (n+1)}.$$

That is, $f(7)$ is the sum of the alternating series $1 - \frac{1}{8} + \frac{1}{48} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n \cdot (n+1)}$.

To find out how many terms are needed to have the error less than 0.002, let's use the Alternating Series Remainder Estimate. Conditions of test satisfied? Yes:

- $\frac{1}{4^n \cdot (n+1)}$ is a decreasing function of $n \geq 1$, since $4^n \cdot (n+1)$ is positive & increasing;
- $\lim_{n\to\infty} \frac{1}{4^n (n+1)} = 0$.

Thus by the Remainder Estimate, we'd like to find n such that $\frac{1}{4^n (n+1)} < \frac{2}{1000} = \frac{1}{500}$, and we'll then take a partial sum up to (but not including) this value of n.

If we try a couple values of n, we see $\frac{1}{4^3 \cdot (3+1)} = \frac{1}{4^3 \cdot 4} = \frac{1}{4^4} = \frac{1}{16^2} = \frac{1}{256}$, but $\frac{1}{4^4 \cdot (4+1)} = \frac{1}{4^4 \cdot 5} = \frac{1}{256 \cdot 5} < \frac{1}{500}$ for sure. Thus, if we take a partial sum that includes all terms up to and including $n=3$, the error will satisfy

$$|R_3| = |s - s_3| < b_4 = \frac{1}{4^4 \cdot (4+1)} < \frac{1}{500},$$

which is what we want. That means we should take the partial sum

$$S_3 = c_0 + c_1 + c_2 + c_3 = \boxed{1 - \frac{1}{8} + \frac{1}{48} - \frac{1}{4^3 \cdot (3+1)}}.$$

$\boxed{9.}$ (14 points)

(a) Determine whether the series

$$\sum_{n=1}^{\infty} \frac{2^{3n}}{7 \cdot 3^n}$$

converges or diverges, and compute the sum if it is convergent. Show all work.

$$a_n = \frac{2^{3n}}{7 \cdot 3^n} = \frac{1}{7} \cdot \frac{(2^3)^n}{3^n} = \frac{1}{7} \cdot \left(\frac{8}{3}\right)^n,$$

so this is a geometric series with initial term $\frac{2^3}{7 \cdot 3} = \frac{8}{21}$ and ratio $\frac{8}{3}$. Since $\frac{8}{3} > 1$, the series <u>diverges</u>.

(b) Find the value of q such that

$$1 + q + q^2 + q^3 + \cdots = 5.$$

Series is geometric with initial term 1 and ratio q, so (assuming $|q| < 1$) its sum is $\frac{1}{1-q}$.

Solving $\frac{1}{1-q} = 5$, we find $1 - q = \frac{1}{5}$, so $\boxed{q = \frac{4}{5}}$ (which is okay since $\left|\frac{4}{5}\right| < 1$).

$\boxed{10.}$ (20 points) Determine whether the following series are convergent or divergent. Indicate clearly which tests you use and what conclusions you draw from them.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+\ln n}$

Since $\displaystyle\lim_{n\to\infty} \frac{n}{n+\ln n} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}$ (applied L'Hôpital's rule since $\frac{\infty}{\infty}$ form)

$$= \frac{1}{1+0} = 1 \neq 0,$$

the series <u>diverges</u> by the <u>Test for Divergence</u>.

(b) $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}+1}{n^2}$

Since $\frac{\sqrt[3]{n+1}}{n^2} = \frac{\sqrt[3]{n}}{n^2} + \frac{1}{n^2} = \frac{n^{1/3}}{n^2} + \frac{1}{n^2}$

$= n^{-5/3} + n^{-2}$,

and since the series

$$\sum_{n=1}^{\infty} n^{-5/3} \quad \text{and} \quad \sum_{n=1}^{\infty} n^{-2}$$

are both p-series with

$$p = \frac{5}{3} > 1 \quad \text{and} \quad p = 2 > 1$$

respectively, so that both are convergent,

we have that $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n+1}}{n^2} = \sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n^2}$

is convergent as well.

(Integral Test would also lead to a successful conclusion, but Ratio Test would be inconclusive here.)

(c) $\sum_{n=1}^{\infty} 3ne^{-n}$

We'll apply the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{3(n+1)e^{-(n+1)}}{3ne^{-n}}$$

$$= \lim_{n\to\infty}\frac{n+1}{n}\cdot\frac{e^n}{e^{n+1}}$$

$$= \frac{1}{e}\cdot\lim_{n\to\infty}\frac{1+1/n}{1}$$

$$= \frac{1}{e}\cdot\frac{1+0}{1} = \frac{1}{e} < 1,$$

and so $\sum_{n=1}^{\infty} 3ne^{-n}$ converges.

(Integral Test would also work here, but it requires integration by parts.)

$\boxed{11.}$ (20 points) Find, with complete justification, the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{3^n(x-1)^n}{n+1}.$$

To determine the radius of convergence, we apply the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}(x-1)^{n+1}}{n+2}\cdot\frac{n+1}{3^n(x-1)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n+1}{n+2}\cdot\frac{3^{n+1}}{3^n}\cdot\frac{(x-1)^{n+1}}{(x-1)^n}\right|$$

$$= 3|x-1|\cdot\lim_{n\to\infty}\frac{1+1/n}{1+2/n} = 3|x-1|,$$

so we require $3|x-1|<1$, which leads to

$-1 < 3(x-1) < 1$, and so

$-\frac{1}{3} < x-1 < \frac{1}{3}$, and therefore

$\frac{2}{3} < x < \frac{4}{3}$. (Endpoints not yet accounted for)

(Thus the radius of convergence is $\frac{1}{3}$.)

We must also separately check endpoints $x=\frac{2}{3}$ and $x=\frac{4}{3}$:

- Case $x=\frac{2}{3}$: Series becomes $\sum_{n=0}^{\infty}\frac{3^n(-1/3)^n}{n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}$, which is an alternating series.

  Since: (1) The values $\frac{1}{n+1}$ decrease as n increases, and

  (2) $\lim_{n\to\infty}\frac{1}{n+1} = 0$,

  we can apply the alternating series test to conclude $\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}$ converges.

- Case $x=\frac{4}{3}$: Series becomes $\sum_{n=0}^{\infty}\frac{3^n(\frac{1}{3})^n}{n+1} = \sum_{n=0}^{\infty}\frac{1}{n+1} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots = \sum_{n=1}^{\infty}\frac{1}{n}$,

  which is a divergent series because it is a p-series with $p=1$.

Conclusion: Interval of Convergence is $\boxed{\left\{\frac{2}{3}\le x<\frac{4}{3}\right\};\ \text{i.e. } \left[\frac{2}{3},\frac{4}{3}\right)}$.

12. (20 points)

(a) Compute the Taylor series for $\cos x$ about 0.

Computing Taylor series coefficients $c_n = \dfrac{f^{(n)}(0)}{n!}$:

$f(x) = \cos x \qquad f(0) = \cos 0 = 1 \qquad c_0 = \frac{1}{0!} = 1$

$f'(x) = -\sin x \qquad f'(0) = -\sin 0 = 0 \qquad c_1 = 0$

$f''(x) = -\cos x \qquad f''(0) = -\cos 0 = -1 \qquad c_2 = \frac{-1}{2!} = -\frac{1}{2!}$

$f'''(x) = \sin x \qquad f'''(0) = \sin 0 = 0 \qquad c_3 = 0$

$f^{(4)}(x) = \cos x \qquad f^{(4)}(0) = \cos 0 = 1 \qquad c_4 = \frac{1}{4!}$

$f^{(5)}(x) = -\sin x \qquad f^{(5)}(0) = -\sin 0 = 0 \qquad c_5 = 0$

$\vdots$ (repeats every four terms)

Thus odd coefficients are 0, and the $(2n)^{th}$ coeff will be $\dfrac{(-1)^n}{(2n)!}$.

$$\Rightarrow \text{Series is } \boxed{1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}}$$

(b) It is a fact that the Taylor series of part (a) indeed converges to the value of $\cos x$. Use the Taylor series to write an expression for $\cos(0.1)$ as an infinite series.

Plug in $x=0.1$:

$$\boxed{\cos(0.1) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (0.1)^{2n}}{(2n)!}} \quad \left(= 1 - \frac{(0.1)^2}{2!} + \frac{(0.1)^4}{4!} - \frac{(0.1)^6}{6!} + \cdots \right)$$

(c) Suppose we use the first four nonzero terms of the sum in part (b) as an approximation to the value of $\cos(0.1)$. Determine the accuracy of this approximation. State your conclusion in a complete sentence, and be as quantitatively precise as you can (but you do not need to simplify any expressions).

Solution #1: The infinite alternating series above for $\cos(0.1)$ satisfies the conditions of the Alternating Series Test, because:

- $$\frac{(0.1)^{2(n+1)}}{(2(n+1))!} < \frac{(0.1)^{2n}}{(2n)!}$$ due to the fact that $$\underset{\substack{\uparrow \\ (n+1)\text{st} \\ \text{term}}}{\frac{(0.1)^{2(n+1)}}{(2(n+1))!}} = \underset{\substack{\uparrow \\ n^{th} \\ \text{term}}}{\frac{(0.1)^{2n}}{(2n)!}} \cdot \underset{\substack{\uparrow \\ \text{clearly less} \\ \text{than } 1}}{\frac{(0.1)^2}{(2n+1)(2n+2)}}$$

and

- $$\lim_{n\to\infty} \frac{(0.1)^{2n}}{(2n)!} = 0$$ due to fact that numerator approaches 0 & denominator approaches ∞;

thus, by the Remainder Estimate in the Alternating Series Test, the error in the four-term approximation is no more than the fifth term in the series, which is $\frac{(0.1)^8}{8!}$. (I.e., the sum $1 - \frac{(0.1)^2}{2!} + \frac{(0.1)^4}{4!} - \frac{(0.1)^6}{6!}$ approximates $\cos(0.1)$ to within $\frac{(0.1)^8}{8!}$.)

Solution #2 uses Taylor's Inequality instead – see next page.

Solution #2 - There are two ways to use Taylor's inequality, actually:

Note that $1-\frac{(0.1)^2}{2!}+\frac{(0.1)^4}{4!}-\frac{(0.1)^6}{6!}$ is the value of the 6th-degree Taylor polynomial $T_6(x)$ at $x=0.1$. But, in this special situation where the coefficient of every odd power in the series is zero, it happens that the above sum is also the value of the 7th degree Taylor polynomial $T_7(x)$ at $x=0.1$. Thus, two slightly different estimates using Taylor's Inequality are acceptable.

Option 2A: Since we are approximating $f(0.1)=\cos(0.1)$ using $T_6(0.1)$, the error is $R_6(0.1)=\cos(0.1)-T_6(0.1)$. Taylor's Inequality requires that we find M such that $|f^{(7)}|\leq M$ on the interval $[-0.1,0.1]$.
Since $f^{(7)}(x)=\sin x$, which is increasing on this interval (and symmetric about the origin), we may take $M=\sin(0.1)$.
Thus, by Taylor's ineq, $|R_6(0.1)|\leq\frac{M}{7!}\cdot(0.1-0)^7=\frac{\sin(0.1)}{7!}\cdot(0.1)^7$.
That is, the four-term sum at top approximates $\cos(0.1)$ to within $\frac{\sin(0.1)\cdot(0.1)^7}{7!}$.

Option 2B: We can instead view the error as $R_7(0.1)=\cos(0.1)-T_7(0.1)$, as noted above. In finding appropriate M, we note that $f^{(8)}(x)=\cos x$, and that on the interval $[-0.1,0.1]$, $|\cos x|$ takes the max. value 1 (at $x=0$). Thus $M=1$ is appropriate. By Taylor's ineq., $|R_7(0.1)|\leq\frac{M}{8!}(0.1-0)^8=\frac{(0.1)^8}{8!}$, which means that the four-term sum above approximates $\cos(0.1)$ to within $\frac{(0.1)^8}{8!}$.

13.

[✓ T] F If $\{b_n\}$ is a positive sequence such that $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ converges.

T [✓ F] If $\{b_n\}$ is a positive sequence and $\lim_{n\to\infty} b_n = 0$, but $\{b_n\}$ is not decreasing, then $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ diverges.

[✓ T] F If $\{b_n\}$ is a positive decreasing sequence but $\lim_{n\to\infty} b_n \neq 0$, then $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ diverges.

[✓ T] F If $\{a_n\}$ is a sequence such that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = -2$, then $\sum_{n=1}^{\infty} a_n$ diverges.

T [✓ F] If $\{a_n\}$ is a positive sequence and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \ln(a_n)$ converges.

[✓ T] F If $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = 2$, it converges when $x = 1$.

T [✓ F] If $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = 2$, it converges when $x = -2$.

$\boxed{14.}$ (20 points)

(a) Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{(-2)^{2n+1}}$.

Let $a_n = \dfrac{3^n}{(-2)^{2n+1}}$. Since $\dfrac{a_{n+1}}{a_n} = \dfrac{3^{n+1}}{(-2)^{2(n+1)+1}} \cdot \dfrac{(-2)^{2n+1}}{3^n}$

$$= \frac{3^{n+1}}{3^n} \cdot \frac{(-2)^{2n+1}}{(-2)^{2n+3}} = \frac{3}{4},$$

this is a geometric series with ratio $r = 3/4$ and initial term $a_0 = \dfrac{3^0}{(-2)^1} = -\dfrac{1}{2}$,

so the sum is $\dfrac{a_0}{1-r} = \dfrac{-1/2}{1-3/4} = \boxed{-2}$.

(b) Is the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ convergent or divergent? Explain.

Since $\dfrac{1}{n\sqrt{n}} = \dfrac{1}{n^{3/2}}$, this is a p-series with $p = 3/2$.

Thus, the series is <u>convergent</u> (because $p > 1$).

(c) Is the series $\sum_{n=1}^{\infty} \frac{\sqrt{n+16}}{(n+1)(n+2)}$ convergent or divergent? Explain completely.

Use Limit Comparison Test with the series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$: (note all terms are positive)

$$\lim_{n\to\infty} \frac{1/n^{3/2}}{\frac{\sqrt{n+16}}{(n+1)(n+2)}} = \lim_{n\to\infty} \frac{(n+1)(n+2)}{n^{3/2}\sqrt{n+16}} = \lim_{n\to\infty} \frac{\frac{1}{n^2}(n+1)(n+2)}{\frac{1}{n^2}\cdot n^{3/2}\sqrt{n+16}}$$

$$= \lim_{n\to\infty} \frac{(1+1/n)(1+2/n)}{\sqrt{1+16/n}} = 1,$$

So the series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ and $\sum_{n=1}^{\infty} \frac{\sqrt{n+16}}{(n+1)(n+2)}$ either both converge or both diverge;

but since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges (see prev. problem), follows that

$\sum_{n=1}^{\infty} \frac{\sqrt{n+16}}{(n+1)(n+2)}$ converges.

$\boxed{15.}$ (15 points) Determine whether the following series are convergent or divergent. Indicate clearly which tests you use and what conclusions you draw from them.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{n^5}{5^n \ln n}$

Use Ratio Test: $\displaystyle L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^5}{5^{n+1} \ln(n+1)} \cdot \frac{5^n \ln n}{n^5} \right|$

$$= \lim_{n\to\infty} \left| \left(\frac{n+1}{n}\right)^5 \cdot \frac{\ln n}{\ln(n+1)} \cdot \frac{1}{5} \right| ,$$

and since $\displaystyle\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^5 = \lim_{n\to\infty} \left(\frac{1+1/n}{1}\right)^5 = 1$ and

$\displaystyle\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = \lim_{n\to\infty} \frac{1/n}{1/(n+1)} = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \frac{1+1/n}{1} = 1$ (using L'Hôpital's Rule for the first $\frac{\infty}{\infty}$ limit),

it follows that $\displaystyle L = \left[\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^5\right] \cdot \left[\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)}\right] \cdot \frac{1}{5} = \frac{1}{5}$,

and since $L < \frac{1}{5}$, it follows that $\displaystyle\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{n^5}{5^n \ln n}$ converges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$

Use Limit Comparison Test with the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$: (note all terms are positive)

$$\lim_{n\to\infty} \frac{e^{1/n}/n^2}{1/n^2} = \lim_{n\to\infty} e^{1/n} = e^0 = 1,$$ so the two series

$\sum \frac{1}{n^2}$ and $\sum \frac{e^{1/n}}{n^2}$ either both converge or both diverge. The former series is a p-series with $p=2>1$, so it converges, and this means $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$ converges.

Alternate solution: we may use the Integral Test, since all terms are positive and the function $f(x) = \frac{e^{1/x}}{x^2}$ is decreasing for $x>0$.

(To justify, note that $f'(x) = \dfrac{-x^{-2}e^{1/x} \cdot x^2 - 2xe^{1/x}}{x^4} = -e^{1/x} \cdot \dfrac{1+2x}{x^4} < 0$ for $x>0$.)

Then since $$\int_1^{\infty} \frac{e^{1/x}}{x^2}dx = \lim_{N\to\infty} \int_1^N \frac{e^{1/x}}{x^2}dx = \lim_{N\to\infty} \left(-e^{1/x} \Big]_1^N \right)$$

$$= \lim_{N\to\infty} \left(-e^{1/N} + e^1\right) = e - 1,$$

a convergent integral, it follows that the series $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$ also converges.

$\boxed{16.}$ (20 points) Let $y(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$.

(a) Find the interval of convergence of $y(x)$.

We calculate $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{2(n+1)}}{(n+1)!} \cdot \frac{n!}{x^{2n}} \right|$

$$= \lim_{n\to\infty} \left| \frac{x^{2n+2}}{x^{2n}} \cdot \frac{n!}{(n+1)!} \right| = \lim_{n\to\infty} \left| \frac{x^2}{n+1} \right| = 0,$$

and since $L < 1$ for <u>any</u> value of x, it follows from the Ratio Test that the series $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$ is convergent for <u>all</u> values of x.

Thus, the interval of convergence of $y(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$ is $\boxed{(-\infty, \infty)}$.

(b) (For easy reference, $y(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$.) Calculate $y'(x)$. Give your answer either in summation notation, or by writing at least the first *five* nonzero terms of the power series for $y'(x)$.

Since $y(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \frac{x^{10}}{5!} + \cdots,$

we can differentiate term-by-term, or using the general formula, to obtain

$$y'(x) = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = \frac{d}{dx}\left(1 + \frac{x^2}{1} + \frac{x^4}{2} + \frac{x^6}{3\cdot 2} + \frac{x^8}{4\cdot 3\cdot 2} + \frac{x^{10}}{5!} + \cdots\right)$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(2n)\cdot x^{2n-1}}{n!} = 2x + \frac{4x^3}{2} + \frac{6x^5}{3\cdot 2} + \frac{8x^7}{4\cdot 3\cdot 2} + \frac{10x^9}{5!} + \cdots}$$

$$= \sum_{n=1}^{\infty} \frac{(2n)\cdot x^{2n-1}}{n!} = \sum_{n=1}^{\infty} \frac{2\cdot x^{2n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{n!} \quad \swarrow \left(\text{other acceptable forms}\right)$$

$$= 2x + \frac{2x^3}{1} + \frac{2x^5}{2} + \frac{2x^7}{3\cdot 2} + \frac{2x^9}{4\cdot 3\cdot 2} + \cdots$$

(c) Show that $y'(x) = 2x \cdot y(x)$, either by using summation notation or by writing out all relevant power series to at least five nonzero terms.

We can write the power series for $2x \cdot y(x)$ as

$$2x\cdot y(x) = 2x \cdot \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = 2x \cdot \left(1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \frac{x^{10}}{5!} + \cdots\right)$$

$$= \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{n!} = 2x + \frac{2x^3}{1!} + \frac{2x^5}{2!} + \frac{2x^7}{3!} + \frac{2x^9}{4!} + \frac{2x^{11}}{5!} + \cdots.$$

Then by the above, both series have the general formula $\sum_{n=0}^{\infty} \frac{2x^{2n+1}}{n!}$;

alternatively the first five terms match the ones in part ⓑ after simplification.

Set B

① Answer: ⓑ can be used; ⓐ and ⓒ cannot.

For ⓐ, the computation is

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{3}{(n+1)(n+3)}\cdot\frac{n(n+2)}{3} = \lim_{n\to\infty}\frac{n(n+2)}{(n+1)(n+3)}$$

$$= \lim_{n\to\infty}\frac{n^2+2n}{n^2+4n+3}$$

$$= \lim_{n\to\infty}\frac{1+2/n}{1+4/n+3/n^2} = 1,$$

which means the Ratio Test is inconclusive.

For ⓑ, we compute

$$\lim_{n\to\infty}\frac{\frac{3}{n(n+2)}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{3n^2}{n(n+2)} = \lim_{n\to\infty}\frac{3}{1+2/n} = \frac{3}{1+0} = 3.$$

Since $0 < 3 < \infty$, we know that either both series $\sum_1 \frac{3}{n(n+2)}$ and $\sum_1 \frac{1}{n^2}$ converge or both diverge. Since $\sum_1 \frac{1}{n^2}$ converges (because it is a p-series with $p=2>1$), we see that $\sum_1 \frac{3}{n(n+2)}$ converges.

For ⓒ, we note that we can't get a result that $\sum_1 \frac{3}{n(n+2)}$ converges by using Limit Comparison with $\sum_1 \frac{3}{n}$, because $\sum \frac{3}{n}$ is divergent and LC can only at best say that the two compared series are either both convergent or both divergent. ($\sum_1 \frac{3}{n}$ is divergent because it's -- or at least its partial sums are -- a multiple of the harmonic series $\sum_1 \frac{1}{n}$, which diverges as a p-series with $p=1 \leqslant 1$.) Worse yet for the case of ⓒ, an attempt to apply LC anyway leads to the computation

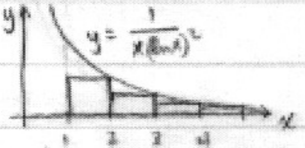
$$\lim_{n\to\infty}\frac{\frac{3}{n(n+2)}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{3n}{n(n+2)} = \lim_{n\to\infty}\frac{3}{n+2} = 0, \text{ and}$$

this value is not in the interval $(0,\infty)$, so LC is inconclusive (as we'd hope!).

② a) yes b) no c) yes d) yes e) no

The series in a) converges because it is a p-series with $p = 1.1 > 1$. The series in b) cannot converge because the terms $a_n = \frac{n^2}{2n^2+1}$ do not go to zero: $\lim\limits_{n\to\infty} \frac{n^2}{2n^2+1} = \frac{1}{2} \neq 0$. So we say the series in b) diverges by the test for divergence. Since the series in c) is an alternating series, we can use the alternating series test, with $b_n = \frac{1}{\ln(n+1)}$. Certainly $b_{n+1} \leq b_n$, since $\frac{1}{\ln(n+2)} < \frac{1}{\ln(n+1)}$. Also, $\lim\limits_{n\to\infty} \frac{1}{\ln(n+1)} = 0$. Since the two conditions of the alternating series test are satisfied, the series converges.

We will use the integral test for the series in d). First we compute $\int_2^\infty \frac{1}{x(\ln x)^3}\,dx$. Let $u = \ln x$; then $du = \frac{1}{x}dx$, and $\int \frac{1}{x(\ln x)^2}dx = \int \frac{1}{u^2}du = \int u^{-2}du$ $= -u^{-1} + C = -\frac{1}{\ln x} + C$. So $\int_2^\infty \frac{1}{x(\ln x)^3}\,dx = -\frac{1}{\ln x}\Big|_2^\infty = \left(\lim\limits_{t\to\infty} -\frac{1}{\ln t}\right) - \left(-\frac{1}{\ln 2}\right)$ $= 0 + \frac{1}{\ln 2} = \frac{1}{\ln 2}$. Since this is finite, we will try to show that the series converges.



From the picture, we see that $\sum\limits_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \leq \int_2^\infty \frac{1}{x(\ln x)^2}\,dx$.

Hence, by the integral test, the series converges.

Since the series in e) contains an exponential and a factorial, we expect that the ratio test will be useful. ☺ So we compute $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!}{e^{n+1}} \cdot \frac{e^n}{n!}\right| = \left|\frac{(n+1)}{e}\right| = \frac{n+1}{e}$. Then $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \frac{n+1}{e} = \infty$. So, by the ratio test, the series $\sum\limits_{n=1}^{\infty} \frac{n!}{e^n}$ diverges.

③ⓐ The series $\sum_{n=1}^\infty (-1)^{n-1}\frac{1}{\sqrt{n}}$ is not absolutely convergent since $\sum_{n=1}^\infty \frac{1}{\sqrt{n}}$ diverges (p-series with $p = \frac{1}{2}$). But by the Alternating Series Test, the series converges:

$$\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}, \text{ and } \frac{1}{\sqrt{n}} \to 0.$$

Thus, the series is convergent but not absolutely convergent.

ⓑ We will use the Ratio Test (often a good idea when factorials are present):

$$\left|\frac{(-1)^{n+1}e^{n+1}}{(n+1)!}\frac{n!}{(-1)^n e^n}\right| = \frac{e}{n+1} \to 0 < 1,$$

so the series $\sum_{n=0}^\infty (-1)^n \frac{e^n}{n!}$ is absolutely convergent.

③ⓒ Since ln n is less than n, and $\frac{n}{n^3-2}$ is approximately $\frac{n}{n^3}=\frac{1}{n^2}$ for n large, we expect the series to converge. To be precise, we will emply Limit Comparison -- but carefully, since this test is sensitive to powers and logs. Thus we compare using $\sum \frac{\ln n}{n^3}$:

$$\lim_{n\to\infty} \frac{\frac{\ln n}{n^3-2}}{\frac{\ln n}{n^3}} = \lim_{n\to\infty} \frac{\ln n}{n^3-2}\cdot\frac{n^3}{\ln n} = \lim_{n\to\infty}\frac{n^3}{n^3-2}$$

$$= \lim_{n\to\infty}\frac{1}{1-2/n^3} = \frac{1}{1-0} = 1.$$

Since $0<1<\infty$, we can conclude by LC that $\sum \frac{\ln n}{n^3-2}$ and $\sum \frac{\ln n}{n^3}$ either both converge (the result we anticipate) or both diverge.

Now although not quite a p-series, the series $\sum \frac{\ln n}{n^3}$ can be analyzed with the Integral Test :

* Conditions of I.T. : If $f(x)=\frac{\ln x}{x^3}$, then certainly f is positive when $x \geq 2$, and it is decreasing here because:

$$f'(x) = \frac{(1/x)(x^3)-3x^2\ln x}{x^6} = -\frac{3\ln x - 1}{x^4} < 0$$

* Calculate improper integral :

$$\int_1^\infty f(x)dx = \int_1^\infty x^{-3}\ln x = \lim_{N\to\infty}\left[\frac{x^{-2}}{4}(2\ln x-1)\right]_1^N$$

↖ (using an integral table, say: line #101)

$$= \lim_{N\to\infty}\left(\frac{\ln N}{2N^2}-\frac{1}{4N^2}+\frac{1}{2}\right) = \frac{1}{2}-0+\frac{1}{2}\lim_{N\to\infty}\frac{\ln N}{N^2},$$

which converges so long as the indeterminate $\left(\frac{\infty}{\infty}\right)$ limit $\lim_{N\to\infty}\frac{\ln N}{N^2}$ exists. But by L'Hôpital's rule, $\lim_{N\to\infty}\frac{\ln N}{N^2} = \lim_{N\to\infty}\frac{1/N}{2N} = \lim_{N\to\infty}\frac{1}{2N^2} = 0$, so the integral converges.

Thus, by the Integral Test, the series $\sum \frac{\ln n}{n^3}$ converges, and so because of our use of Limit Comparison, we conclude that the series $\sum \frac{\ln n}{n^3-2}$ converges.

④ⓐ First check whether the nth term goes to 0:

$$\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) = \lim_{n\to\infty}\frac{\sin(\frac{1}{n})}{\frac{1}{n}} = \lim_{x\to\infty}\frac{\sin(\frac{1}{x})}{\frac{1}{x}} = \lim_{y\to 0}\frac{\sin y}{y} = 1$$

by l'Hospital's Rule. Since the nth term does not have limit 0, the series diverges.

ⓑ The series is a convergent geometric series, since

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{10^n}3^{2n} = \sum_{n=1}^{\infty}\left(-\frac{9}{10}\right)^n,$$

with the absolute value of the ratio $\left|-\frac{9}{10}\right| < 1$. The series starts with $n = 1$, so the sum is

$$\frac{-\frac{9}{10}}{1-(-\frac{9}{10})} = -\frac{9}{19}.$$

---

⑤ S = sum of shaded areas = $\frac{1}{2}\cdot\pi\cdot(1)^2 + \frac{1}{2}\cdot\pi(\frac{1}{2})^2 + \frac{1}{2}\cdot\pi(\frac{1}{4})^2 + \cdots$

since the diameter of a small circle is equal to $\frac{1}{2}$ the diameter of the next biggest circle. (And so the radius of a small circle is equal to $\frac{1}{2}$ the radius of the next biggest circle.) So

$$S = \frac{\pi}{2}\left(1^2 + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{4}\right)^2 + \left(\tfrac{1}{8}\right)^2 + \cdots\right) = \frac{\pi}{2}\left(\left(\frac{1}{2^0}\right)^2 + \left(\frac{1}{2^1}\right)^2 + \left(\frac{1}{2^2}\right)^2 + \left(\frac{1}{2^3}\right)^2 + \cdots\right)$$

$$= \frac{\pi}{2}\left(\frac{1}{2^0} + \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots\right).$$

This is a geometric series with another factor of $\frac{1}{2^2}$ in each successive term. So the initial term is $a = \frac{\pi}{2}$, the ratio $r = \frac{1}{2^2} = \frac{1}{4}$, and the sum is

$$S = \frac{a}{1-r} = \frac{\pi/2}{1-\frac{1}{4}} = \frac{\pi/2}{3/4} = \pi/2 \cdot 4/3 = \underline{\frac{2\pi}{3}}.$$

---

⑥ a) **No.** The terms don't even go to zero, so there's no hope!

$$\lim_{n\to\infty}\sqrt{\frac{n}{n+1}} = \sqrt{\lim_{n\to\infty}\frac{n}{n+1}} \overset{H}{=} \sqrt{\lim_{n\to\infty}\frac{1}{1}} = \sqrt{1} = 1.$$

⑥b) Yes. Note that $\cos(\pi \cdot 1) = -1$, $\cos(\pi \cdot 2) = 1$, $\cos(\pi \cdot 3) = -1$, etc., so this is an alternating series. (Sneaky, eh? ☺) So we can use the alternating series test, with $b_n = \frac{1}{n}$. The condition $b_{n+1} \le b_n$ is satisfied since $\frac{1}{n+1} \le \frac{1}{n}$, and the condition $\lim_{n\to\infty} b_n = 0$ is satisfied since $\lim_{n\to\infty} \frac{1}{n} = 0$. So the series converges.

⑥ⓒ Yes, converges. The Ratio Test can be applied, with $a_n = \frac{2^n+1}{2\cdot 3^n - 1}$:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|a_{n+1} \cdot \frac{1}{a_n}\right| = \lim_{n\to\infty} \frac{2^{n+1}+1}{2\cdot 3^{n+1}-1} \cdot \frac{2\cdot 3^n - 1}{2^n+1}$$

$$= \lim_{n\to\infty} \frac{2\cdot 2^n + 1}{2^n+1} \cdot \frac{2\cdot 3^n - 1}{6\cdot 3^n - 1}$$

$$= \lim_{n\to\infty} \frac{2 + 1/2^n}{1 + 1/2^n} \cdot \frac{2 - 1/3^n}{6 - 1/3^n}$$

$$= \frac{2+0}{1+0} \cdot \frac{2-0}{6-0} = \boxed{2/3}.$$

Since $2/3 < 1$, the series converges.

⑥ⓓ) Yes. Since we see n in the exponent, and n factorial ($n!$), we think to ourselves, "Oh boy! I bet the ratio test will work!". So we compute

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}}\right| = \lim_{n\to\infty} \frac{3^{\cancel{n}+1}}{(n+1)!} \cdot \frac{n!}{\cancel{3^n}}$$

$$= \lim_{n\to\infty} \frac{3\,\cancel{(n)(n-1)(n-2)\cdots 3\cdot 2\cdot 1}}{(n+1)\cancel{(n)(n-1)(n-2)\cdots 3\cdot 2\cdot 1}} = \lim_{n\to\infty} \frac{3}{n+1} = 0 = L$$

Since $L<1$, the series converges.

⑦ⓐ $\sum_{n=0}^{\infty} \frac{2^n 3^{n+1}}{5^{2n}} = \sum_{n=0}^{\infty} a_n$

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}3^{n+2}/5^{2n+2}}{2^n 3^{n+1}/5^{2n}} = \frac{2\cdot 3}{5^2} = \frac{6}{25} = \underline{\underline{\text{constant}}}$$

This is a geometric series with $|r| < 1$, so

$$\sum = \frac{a_0}{1-r} = \frac{3}{1-\frac{6}{25}} = \boxed{\frac{75}{19}}$$

---

⑦ⓑ <u>Converges</u>. We can use the Limit Comparison test with the positive-term series $\sum \frac{1}{n^2}$:

$$\lim_{n\to\infty} \frac{\frac{1}{n^2-1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2-1} = \lim_{n\to\infty} \frac{1}{1-1/n^2} = \frac{1}{1-0} = 1.$$

Since $0<1<\infty$, the Limit Comp. Test implies that both series $\sum \frac{1}{n^2-1}$ and $\sum \frac{1}{n^2}$ converge or they both diverge. But $\sum \frac{1}{n^2}$ is a p-series with $p=2>1$, so it converges. Thus, $\sum \frac{1}{n^2-1}$ converges.

---

⑦ⓒ $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

This series satisfies the three conditions for the Alternating Series Test:

(i) signs alternate
(ii) terms are decreasing in abs. value
(iii) terms approach zero.

So, the series <u>converges</u>.

⑧ⓐ $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$, which is an infinite geometric series with common ratio x and initial term 1. Thus, for $|x| < 1$, the sum is convergent, and is equal to $\frac{1}{1-x}$. (For $|x| \geq 1$, the sum is not convergent.) Thus $\boxed{f(x) = \frac{1}{1-x} \quad \text{for} \quad -1 < x < 1}$.

ⓑ First, $\int_0^x \frac{1}{1-t}\,dt = \Big[-\ln|1-t|\Big]_0^x = -\ln|1-x| + \ln|1-0|$

$$= -\ln|1-x|$$

$$= \boxed{-\ln(1-x)}. \quad \text{(because } 1-x > 0 \text{ when } x < 1\text{)}$$

Next, $\int_0^x (1+t+t^2+t^3+\cdots)\,dt = \Big[t + \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{4} + \cdots\Big]_0^x$

$$= \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right) - \left(0 + \frac{0}{2} + \frac{0}{3} + \cdots\right)$$

$$= \boxed{x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots}.$$

(Alternatively, $\int_0^x \sum_{n=0}^{\infty} t^n\,dt = \left[\sum_{n=0}^{\infty} \frac{t^{n+1}}{n+1}\right]_0^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$.)

ⓒ By part ⓑ, we have $-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$ for $-1 < x < 1$. The series given is equal to -1 times the above power series with $x = 1/2$ plugged in. Thus, plug in $x = 1/2$ to both sides:

$$-\ln\left(1 - \tfrac{1}{2}\right) = \frac{1}{2} + \frac{(1/2)^2}{2} + \frac{(1/2)^3}{3} + \frac{(1/2)^4}{4} + \cdots$$

$$= \frac{1}{2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \frac{1}{4 \cdot 2^4} + \cdots$$

$$\Rightarrow \quad \ln\left(1 - \tfrac{1}{2}\right) = -\left(\frac{1}{2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \frac{1}{4 \cdot 2^4} + \cdots\right)$$

$$= -\frac{1}{2} - \frac{1}{2 \cdot 2^2} - \frac{1}{3 \cdot 2^3} - \frac{1}{4 \cdot 2^4} - \cdots,$$

so the sum we seek is equal to $\ln(1 - \frac{1}{2})$, i.e. $\ln(\frac{1}{2})$ or $\boxed{-\ln 2}$.

⑨ Ratio test to find radius of convergence: if $a_n = \frac{x^n}{\sqrt{n^2+n}}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{\sqrt{(n+1)^2+(n+1)}}\right|\cdot\left|\frac{\sqrt{n^2+n}}{x^n}\right|$$

$$= \lim_{n\to\infty} |x|\cdot\frac{\sqrt{n^2+n}}{\sqrt{n^2+2n+1+n+1}} = \lim_{n\to\infty}|x|\cdot\sqrt{\frac{n^2+n}{n^2+3n+2}}$$

$$= |x|\cdot\lim_{n\to\infty}\sqrt{\frac{1+1/n}{1+3/n+2/n^2}} = |x|.$$

Convergence if $|x|<1$; thus the $\boxed{\text{radius} = 1}$.

Interval of convergence is $\{-1<x<1\}$ along with possibly $x=1$, $x=-1$.

Test these endpoints:

$\underline{x=1}$: Series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^2+n}}$. (We'd expect divergence: if n large, these terms are approx. $\frac{1}{n}$.)

Limit comparison with the divergent harmonic series $\sum \frac{1}{n}$:

$$\lim_{n\to\infty}\frac{\frac{1}{\sqrt{n^2+n}}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{\sqrt{n^2+n}} = \lim_{n\to\infty}\frac{\sqrt{n^2}}{\sqrt{n^2+n}}$$

$$= \lim_{n\to\infty}\sqrt{\frac{n^2}{n^2+n}} = \lim_{n\to\infty}\sqrt{\frac{1}{1+1/n}} = 1.$$

Since $0<1<\infty$, Limit Comp. Test implies (using divergence of harmonic series) that $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^2+n}}$ <u>diverges</u>.

$\underline{x=-1}$: Series $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n^2+n}}$. (No chance of absolute convergence, but check with alternating series test.)

Conditions of Alt. Series Test: (i) Alternating ✓

(ii) $\frac{1}{\sqrt{n^2+n}}$ is decreasing ✓ (e.g. take deriv.)

(iii) $\lim_{n\to\infty}\frac{1}{\sqrt{n^2+n}} = 0$ ✓

Thus, by the Alternating Series Test, this <u>converges</u>.

Summary: Interval of Convergence is $\boxed{\{-1\le x<1\}}$ or $[-1,1)$.

⑩ Begin with Ratio Test (for radius of convergence): $a_n = \frac{n}{n+1} x^n$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|a_{n+1} \cdot \frac{1}{a_n}\right| = \lim_{n\to\infty} \left| \frac{n+1}{n+2} \cdot x^{n+1} \cdot \frac{n+1}{n} \cdot \frac{1}{x^n} \right|$$

$$= \lim_{n\to\infty} |x| \cdot \left(\frac{(n+1)(n+1)}{(n+2)(n)}\right)$$

$$= |x| \cdot \lim_{n\to\infty} \frac{n^2+2n+1}{n^2+2n} = |x| \cdot \lim_{n\to\infty} \frac{1+2/n+1/n^2}{1+2/n}$$

$$= |x|.$$

Convergence if this $< 1 \Longrightarrow$ Radius of convergence $= 1$.

Interval of convergence: $\{-1 < x < 1\}$, plus possibly $x = 1$, $x = -1$.

Testing endpoints:

$\underline{x=1}$. Series is $\sum_{n=0}^{\infty} \frac{n}{n+1}$. This $\underline{\text{diverges}}$, because the terms do not go to 0 as $n \to \infty$: in fact, $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1$.

$\underline{x=-1}$. Series is $\sum_{n=0}^{\infty} \frac{(-1)^n n}{n+1}$. This $\underline{\text{diverges}}$, because the terms don't go to 0 as $n \to \infty$: $\lim_{n\to\infty} \frac{(-1)^n n}{n+1}$ does not exist! (oscillating)

So, the interval of convergence is just $\boxed{\{-1 < x < 1\}}$.

⑪ Since $\sum_{n=0}^{\infty} \frac{(-3x)^{n+1}}{3n+1}$ has the variable x in it, it is a power series, so we will use the ratio test. $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-3x)^{n+1+1}}{3(n+1)+1} \cdot \frac{(3n+1)}{(-3x)^{n+1}}\right| = \left|\frac{(-3x)(3n+1)}{3n+4}\right|$. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{3(3n+1)}{(3n+4)}|x| = 3|x|$, since $\lim_{n\to\infty}\frac{(3n+1)}{(3n+4)} = 1$. So the power series will converge whenever $3|x| < 1$, or whenever $|x| < \frac{1}{3}$. The interval of convergence is thus $(-\frac{1}{3}, \frac{1}{3})$, along with possibly $-\frac{1}{3}$, $\frac{1}{3}$. Check these:

$\underline{x = -\frac{1}{3}}$: Series is $\sum_{n=0}^{\infty} \frac{1}{3n+1}$. (Expect divergence: harmonic)

Use Limit Comparison with the divergent harmonic series $\sum \frac{1}{n}$:

$$\lim_{n\to\infty} \frac{\frac{1}{3n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{3n+1} = \lim_{n\to\infty} \frac{1}{3+\frac{1}{n}} = \frac{1}{3+0} = \frac{1}{3}.$$

Since $0 < \frac{1}{3} < \infty$, the Lim. Comparison Test implies that both $\sum_{n=0} \frac{1}{3n+1}$ and $\sum \frac{1}{n}$ <u>diverge</u> (since we already know $\sum \frac{1}{n}$ diverges).

$\underline{x = \frac{1}{3}}$: Series is $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{3n+1}$. (No chance for absolute convergence, try applying Alternating Series Test.)

Conditions of Alt. Series Test: (i) Alternating Sign ✓

(ii) $\frac{1}{3n+1}$ decreasing ✓, (since $3n+1$ increasing)

(iii) $\lim_{n\to\infty} \frac{1}{3n+1} = 0$ ✓

Thus, by the Alternating Series Test, the series <u>converges</u>.

<u>Conclusion</u>: Interval of convergence is $\boxed{\{-\frac{1}{3} < x \leq \frac{1}{3}\}}$.

⑫ Let $a_n = \frac{n^3}{2^{n+1}}$, and apply the Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^3 x^{n+1}}{2^{n+2}} \frac{2^{n+1}}{n^3 x^n}\right| = \frac{1}{2}(1+\frac{1}{n})^3 |x| \to \frac{1}{2}|x|$$

as $n \to \infty$. So the power series converges for $|x| < 2$ and diverges for $|x| > 2$, and then the radius of convergence is 2. The Ratio Test is inconclusive on the boundary points -2 and 2, so we need to check those separately. For $x = 2$, the series diverges since it becomes $\frac{1}{2}\sum_{n=1}^{\infty} n^3$. And for $x = -2$, the series also diverges, since $\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n n^3$ diverges. So the interval of convergence is $(-2, 2)$.

13(a). We need to compute up to the third derivative of f:

$$\begin{aligned} f(x) &= x^{1/3} & f(27) &= 27^{1/3} = 3 \\ f'(x) &= \frac{1}{3}x^{-2/3} & f'(27) &= \frac{1}{3}\cdot 27^{-2/3} = \frac{1}{3}\cdot\frac{1}{9} = \frac{1}{27} \\ f''(x) &= -\frac{2}{9}x^{-5/3} & f''(27) &= -\frac{2}{9}\cdot 27^{-5/3} = -\frac{2}{9}\cdot\frac{1}{3^5} = -\frac{2}{3^7} \\ f'''(x) &= \frac{10}{27}x^{-8/3} & f'''(27) &= \frac{10}{27}\cdot 27^{-8/3} = \frac{10}{27}\cdot\frac{1}{3^8} = \frac{10}{3^{11}} \end{aligned}$$

Thus, the Taylor series for f begins

$$\begin{aligned} f(x) &\sim f(27) + \frac{f'(27)}{1!}\cdot(x-27) + \frac{f''(27)}{2!}\cdot(x-27)^2 + \frac{f'''(27)}{3!}\cdot(x-27)^3 + \cdots \\ &= \boxed{3 + \frac{1}{27}(x-27) - \frac{2}{3^7}\cdot\frac{1}{2}(x-27)^2 + \frac{10}{3^{11}}\cdot\frac{1}{6}(x-27)^3 + \cdots} \end{aligned}$$

(b). The second-degree Taylor polynomial is

$$T_2(x) = 3 + \frac{1}{27}(x-27) - \frac{1}{3^7}(x-27)^2,$$

so

$$\sqrt[3]{28} = f(28) \approx T_2(28) = 3 + \frac{1}{27}\cdot 1 - \frac{1}{3^7}\cdot 1^2 = \boxed{3 + \frac{1}{27} - \frac{1}{3^7}}.$$

The error in this approximation is the quantity $R_2(28) = \sqrt[3]{28} - T_2(28)$. To find a bound on this error by Taylor's Inequality, we must consider the Taylor polynomial $T_2(x)$ as approximating $f(x) = \sqrt[3]{x}$ on the interval $26 \le x \le 28$, and so we compute M, the maximum value of $|f'''(x)| = \frac{10}{27}x^{-8/3}$ on this interval. Since $\frac{10}{27}x^{-8/3}$ is decreasing on this interval, the maximum occurs at the left endpoint $x = 26$, so we can take $M = \dfrac{10}{27}\cdot 26^{-8/3}$. Then by Taylor's Inequality,

$$|R_2(28)| \le \frac{M}{3!}|28-27|^3 = \frac{1}{6}\cdot\frac{10}{27}\cdot 26^{-8/3}\cdot 1^3 = \frac{5}{81}\cdot 26^{-8/3},$$

meaning that the error in the above approximation is no more than $\dfrac{5}{81\cdot 26^{8/3}}$, which is quite small.

14(a). Since $f^{(n)}(x) = f(x) = e^x$ for all n, it follows that $f^{(n)}(0) = 1$ for all n, and so the Taylor series for f can be written

$$f(x) \sim \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}(x-0)^n = \boxed{\sum_{n=0}^{\infty}\frac{x^n}{n!}}.$$

(b). The third-degree Taylor polynomial is

$$T_3(x) = \sum_{n=0}^{3}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6},$$

so

$$e = f(1) \approx T_3(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} = \boxed{\frac{8}{3}}.$$

The error in this approximation is written $R_3(1) = e - T_3(1)$. To estimate this error by Taylor's Inequality, we must consider the Taylor polynomial $T_3(x)$ as approximating $f(x) = e^x$ on the interval $-1 \le x \le 1$, and so we compute M, the maximum value of $|f^{(4)}(x)| = e^x$ on this interval. Since e^x is increasing on this interval, the maximum occurs at the right endpoint $x = 1$, so we can take $M = e^1 = e$. Then by Taylor's Inequality,

$$|R_3(1)| \le \frac{M}{4!}|1-0|^4 = \frac{e}{24}$$

(that is, the error in the above approximation for e is no larger than $e/24$).