Math 42 — Chapter 8 Practice Problems

Set A

1. (a) Write the series: $\quad -\frac{3}{1}+\frac{5}{4}-\frac{7}{9}+\frac{9}{16}-\frac{11}{25}+\frac{13}{36}-\frac{15}{49}+\cdots \quad$ in sigma notation.

   (You don't have to investigate convergence or divergence.)

   (b) Determine whether the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-3)^{n+1}}{2^{3n}}$ converges or diverges, and compute the sum if it is convergent.

2. Determine whether the following series are convergent or divergent. Indicate clearly which tests you use and what conclusions you draw from them.

   (a) $\displaystyle\sum_{n=1}^{\infty}\frac{n+5}{5^n}$

   (b) $\displaystyle\sum_{n=1}^{\infty}(-1)^n 2^{1/n}$

   (c) $\displaystyle\sum_{n=1}^{\infty}\frac{3^n n^2}{n!}$

   (d) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\sqrt{\ln n}}$

3. Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty}\frac{n+1}{n^3}(3x+2)^n$.

4. For this problem, we consider the series: $\quad s=\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^5}=1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\cdots$

   (Interesting aside: one reason why we'd care about s is that it is the value of the so-called *Riemann Zeta Function* at 5: this function plays an important role in the field of number theory, which concerns among other things the behavior of prime numbers, and surprisingly has applications to things like secure Internet communication.)

   (a) Explain why this is a convergent series; that is, explain why the number s is defined.

   (b) If the first 10 terms of the series were used to approximate s, determine the accuracy of this approximation. State your conclusion in a complete sentence, and be as quantitatively precise as you can (but you do not need to simplify any expressions).

   (c) It turns out that the sum of the first 10 terms of the series is the value 1.0369073413... Use your reasoning from part (b) to obtain a more accurate approximation of s, *without* having to consider any more terms from the series. Your answer does not need to be simplified (or fully evaluated in decimal form).

5. Compute the Taylor series for $\ln(1+x)$ about 0.

6. Let $f(x) = \sqrt{x}$.

   (a) Find both the degree-1 and degree-2 Taylor polynomials for f about 100. (These functions are also called, respectively, the linear and quadratic approximations for f at 100.)

   (b) Use the polynomials from part (a) to obtain two different approximations for $\sqrt{97}$.

   (c) Determine the accuracy of each approximation, including whether each is an underestimate or an overestimate of the actual value. You may use any valid reasoning (and, if you wish, you may take as a fact that the Taylor series converges to $\sqrt{97}$). State your conclusions in sentence form, being as precise as you can (but without needing to simplify any arithmetic).

7. The Taylor series for $\arctan x$ about 0 is: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

   (you do *not* have to prove this).

   (a) Find the *radius* of convergence of this series, showing all reasoning.

   (b) It is a fact that the value of the series at $x = 1$ equals the value $\arctan 1 = \frac{\pi}{4}$. That is,

   $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots,$$

   a formula due to Leibniz. Find, with justification, a partial sum of this series that represents $\frac{\pi}{4}$ accurately to within 10^{-2}. (You don't need to compute the value of the partial sum.)

8. Suppose you are given a function f such that its n-th derivative evaluated at 6 is

   $$f^{(n)}(6) = \frac{(-1)^n n!}{4^n(n+1)}.$$

   (a) Find $f(6)$, $f'(6)$, and $f''(6)$, and use these values to write down the a formula for the quadratic approximation to f at 6 (that is, the Taylor polynomial $T_2(x)$ at 6).

   (b) Suppose the Taylor series of f centered at 6 converges to $f(x)$ for all x. How many terms of the Taylor series are required to approximate $f(7)$ with error less than 0.002? Write a sum of numbers (which you do not have to simplify) that represents this approximation.

9. (a) Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^{3n}}{7 \cdot 3^n}$ converges or diverges, and compute the sum if it is convergent.

   (b) Find the value of q such that: $1 + q + q^2 + q^3 + \cdots = 5$.

10. Determine whether the following series are convergent or divergent. Indicate clearly which tests you use and what conclusions you draw from them.

    (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n + \ln n}$

    (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{n} + 1}{n^2}$

    (c) $\displaystyle\sum_{n=1}^{\infty} 3ne^{-n}$

11. Find, with complete justification, the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{3^n(x-1)^n}{n+1}$.

12. (a) Compute the Taylor series for $\cos x$ about 0.

    (b) It is a fact that the Taylor series of part (a) indeed converges to the value of $\cos x$. Use the Taylor series to write an expression for $\cos(0.1)$ as an infinite series.

    (c) Suppose we use the first four nonzero terms of the sum in part (b) as an approximation to the value of $\cos(0.1)$. Determine the accuracy of this approximation. State your conclusion in a complete sentence, and be as quantitatively precise as you can (but you do not need to simplify any expressions).

13. Mark each statement below as *true* or *false* by circling **T** or **F**. No justification is necessary.

T **F** If $\{b_n\}$ is a positive sequence such that $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, then $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ converges.

T **F** If $\{b_n\}$ is a positive sequence and $\displaystyle\lim_{n\to\infty} b_n = 0$, but $\{b_n\}$ is not decreasing, then $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ diverges.

T **F** If $\{b_n\}$ is a positive decreasing sequence but $\displaystyle\lim_{n\to\infty} b_n \neq 0$, then $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ diverges.

T **F** If $\{a_n\}$ is a sequence such that $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = -2$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

T **F** If $\{a_n\}$ is a positive sequence and $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} \ln(a_n)$ converges.

T **F** If $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ converges when $x = 2$, it converges when $x = 1$.

T **F** If $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ converges when $x = 2$, it converges when $x = -2$.

14. (a) Find the sum of the series $\sum_{n=0}^{\infty} \frac{3^n}{(-2)^{2n+1}}$.

   (b) Is the series $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ convergent or divergent? Explain.

   (c) Is the series $\sum_{n=1}^{\infty} \frac{\sqrt{n+16}}{(n+1)(n+2)}$ convergent or divergent? Explain completely.

15. Determine whether the following series are convergent or divergent. Indicate clearly which tests you use and what conclusions you draw from them.

   (a) $\sum_{n=2}^{\infty} \frac{n^5}{5^n \ln n}$

   (b) $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$

16. Let $y(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$.

   (a) Find the interval of convergence of $y(x)$.

   (b) Calculate $y'(x)$. Give your answer either in summation notation, or by writing at least the first *five* nonzero terms of the power series for $y'(x)$.

   (c) Show that $y'(x) = 2x \cdot y(x)$, either by using summation notation or by writing out all relevant power series to at least five nonzero terms.

Set B

1. Which of the following three tests can be used to show that the series $\sum_{n=1}^{\infty} \frac{3}{n(n+2)}$ converges? Explain. (Note: your answer could be any combination of the three tests, including none of them or even all three of them!)

   (a) Ratio Test

   (b) Limit Comparison Test with $\sum_{n=1}^{\infty} n^{-2}$

   (c) Limit Comparison Test with $\sum_{n=1}^{\infty} 3n^{-1}$

2. Which of the following series converge? Explain.

   (a) $\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$

   (b) $\sum_{n=1}^{\infty} \frac{n^2}{2n^2+1}$

   (c) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln(n+1)}$

   (d) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

   (e) $\sum_{n=1}^{\infty} \frac{n!}{e^n}$

3. For each of the following series, determine whether it converges or diverges. Show all work.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{n}}$

(b) $\displaystyle\sum_{n=0}^{\infty}(-1)^{n}\frac{e^n}{n!}$

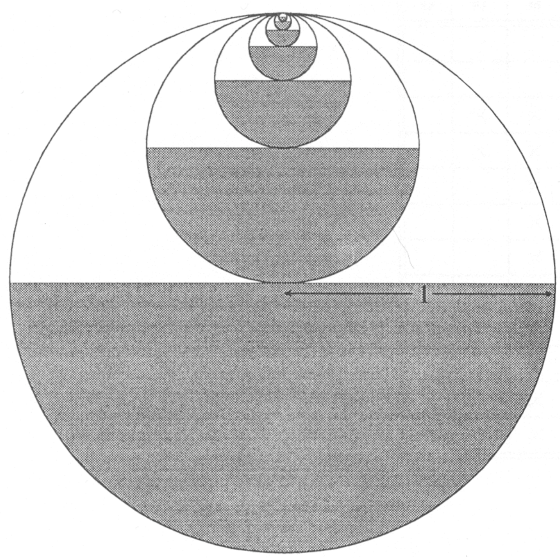
(c) $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n^3-2}$

4. Determine whether each of the following series converges or diverges. Compute the sum for any that are convergent.

(a) $\displaystyle\sum_{n=1}^{\infty} n\sin(\frac{1}{n})$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{10^n}3^{2n}$

5. Find the sum of the shaded areas, showing all of your work. Assume that there are infinitely many shaded areas, all continuing in the pattern shown, and that the radius of the outermost circle is 1. (Hint: according to the pattern, the *diameter* of each successive smaller circle equals the *radius* of the preceding circle.)



6. Which of the following series converge? Explain.

(a) $\displaystyle\sum_{n=1}^{\infty}\sqrt{\frac{n}{n+1}}$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(\pi n)}{n}$

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{2^n+1}{2\cdot 3^n-1}$

(d) $\displaystyle\sum_{n=1}^{\infty}\frac{3^n}{n!}$

7. (a) Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{2^n 3^{n+1}}{5^{2n}}$.

   (b) Does the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ converge or diverge? Explain.

   (c) Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converge or diverge? Explain.

8. Let $f(x)$ be the power series $\displaystyle\sum_{n=0}^{\infty} x^n$.

   (a) Find a formula for $f(x)$ that does not use an infinite sum. For what values of x is this formula valid?

   (b) Find two different expressions for $\int_0^x f(t)dt$, one by using the formula in part (a), and the other by using the definition of f as an infinite power series.

   (c) Use part (b) to find the sum of the series

$$-\frac{1}{1 \cdot 2^1} - \frac{1}{2 \cdot 2^2} - \frac{1}{3 \cdot 2^3} - \frac{1}{4 \cdot 2^4} - \frac{1}{5 \cdot 2^5} - \cdots$$

9. Find the radius of convergence, and the interval of convergence, of the series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n^2+n}}$.

   Show all of your work.

10. Find the interval of convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n+1} x^n$.

11. Find the interval of convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-3x)^{n-1}}{3n+1}$.

12. Find the radius of convergence and interval of convergence of the power series $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{n^3}{2^{n+1}} x^n$.

13. (a) Find the Taylor series for $f(x) = \sqrt[3]{x}$ centered about $a = 27$, by explicitly finding the first four nonzero terms of the series. (You do not need to give a general formula for the coefficients of the power series.)

    (b) Use the *second* degree Taylor polynomial to approximate the value of $\sqrt[3]{28}$, and estimate the error in your approximation.

14. (a) Find the Taylor series for $f(x) = e^x$ centered at $a = 0$, either by giving a general formula or writing the first five nonzero terms of the series.

    (b) Use the *third* degree Taylor polynomial to approximate the value of e, and estimate the error in your approximation. (Your bound on the error should be in the form of some fraction of e.)