Math 42: Fall 2015
Midterm 2
November 3, 2015

NAME: Solutions

Time: **180 minutes**

For each problem, you should write down all of your work carefully and legibly to receive full credit. When asked to justify your answer, you should use theorems and/or mathematical reasoning to support your answer, as appropriate.

Failure to follow these instructions will constitute a breach of the Stanford Honor Code:

- You may not use a calculator or any notes or book during the exam.
- You may not access your cell phone during the exam for any reason.
- You are bound by the Stanford Honor Code, which stipulates among other things that you may not communicate with anyone other than the instructor during the exam, or look at anyone else's solutions.

I understand and accept these instructions.

Signature: ____

Discussion Section: (Please circle)

9:30-10:20   10:30-11:20   11:30-12:20   12:30-1:20   ACE 1:30-3:20

There is a list of formulas and four pages of blank paper at the end of the exam.

Problem	**Value**	**Points**
1	12	
2	9	
3	16	
4	12	
5	7	
6	12	
7	9	
8	9	
9	7	
10	7	
Total	100	

1. (Short answer) You **do not** have to justify your answer to the following questions.

a. *(3 pts.)* (True or false) Because the radius of convergence of the Taylor series for $f(x) = e^x$ centered at $a = 0$ is ∞, the radius of convergence of the Taylor series for $g(x) = 1/e^x$ is 0.

False. $1/e^x = e^{-x}$, which has radius of convergence ∞.

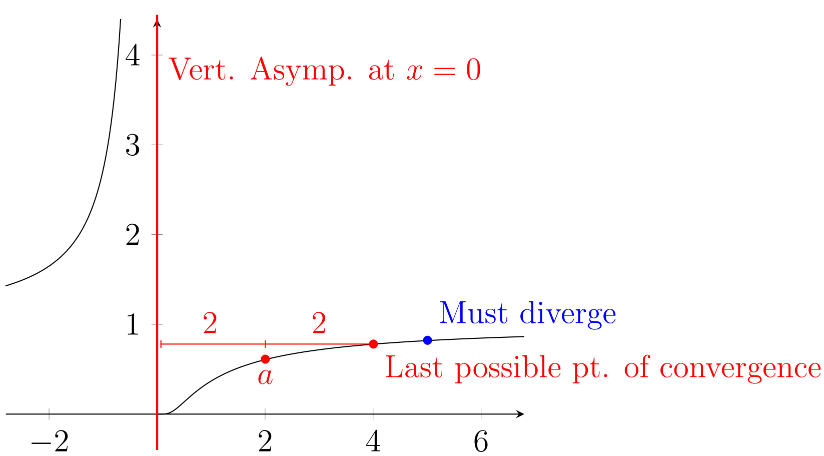
b. *(3 pts.)* Let $T_7(x)$ be the degree 7 Taylor polynomial for a function $f(x)$, centered at $a = 2$. If $f(x) = T_7(x)$ for all x, what is $f^{(9)}(3)$?

Since $f(x) = T_7(x)$ for all x, $f(x)$ is at most a degree 7 polynomial. Thus, its ninth derivative is exactly zero at any point. In particular, $f^{(9)}(3) = 0.$

c. *(3 pts.)* (True or false) Simpson's rule is always more accurate than the trapezoid rule.

False. Consider the approximations to $\int_{-1}^{1} |x|\, dx$ with $n = 2$. The trapezoid rule gives $\frac{1}{2}(|-1| + 2|0| + |1|) = 1$ and Simpson's rule gives $\frac{1}{3}(|-1| + 4|0| + |1|) = \frac{2}{3}$. The integral is equal to 1, so the trapezoid rule is exactly correct, whereas Simpson's rule is off by 1/3. For "most" integrals, Simpson's rule will be much more accurate, but it doesn't have to be.

d. *(3 pts.)* The graph of $f(x) = e^{-1/x}$ is given below. Consider its Taylor series centered at $a = 2$ (but please do not find it!). Does the series converge at $x = 5$?



Since $f(x)$ is not defined at $x = 0$, the radius of convergence of the series centered at $a = 2$ is at most $|2 - 0| = 2$. Thus, the interval of convergence is at most $(0, 4]$, so **the series does not converge at $x = 5$.**

2. a. *(6 pts.)* Find the Taylor series centered at $a = 0$ for $f(x) = x\ln(1+x)$. Express your answer in terms of sigma/summation notation.

Notice that $\ln(1+x)$ is related to the geometric series

$$\begin{aligned}\frac{1}{1+x} &= 1 - x + x^2 - x^3 + x^4 - \dots \\ &= \sum_{n=0}^{\infty}(-1)^n x^n,\end{aligned}$$

provided $|x| < 1$. In particular, we have that

$$\begin{aligned}\ln(1+x) &= \int \frac{1}{1+x}\,dx \\ &= C + \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1}.\end{aligned}$$

We find C by plugging in the "easy point" $x = 0$, which yields $C = \ln(1) = 0$. Thus, we have that $\ln(1+x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1}$. Multiplying this by x yields

$$\boxed{x\ln(1+x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+2}}{n+1}.}$$

b. *(3 pts.)* What is the radius of convergence of the series in part **a.**?

First method: We use the ratio test, finding that

$$\begin{aligned}\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty}\frac{|x|^{n+3}/(n+2)}{|x|^{n+2}(n+1)|} \\ &= \lim_{n\to\infty}\frac{|x|^{n+3}}{|x|^{n+2}}\frac{n+1}{n+2} \\ &= |x|.\end{aligned}$$

Thus, the ratio test implies the series converges if $|x| < 1$ and not if $|x| > 1$, so the radius of convergence is $\boxed{R = 1.}$

Second method: To find the series in part **a.**, we used the formula for the sum of a geometric series, which converged if $|x| < 1$. That is, the radius of convergence for $\frac{1}{1+x}$ is $R = 1$. Integration doesn't change the radius of convergence, nor does multiplication by x, so the radius of convergence for $x\ln(1+x)$ must be $\boxed{R = 1.}$

3. a. *(6 pts.)* Find the Taylor series centered at $a = 2$ for $f(x) = x^{-2}$. Express your answer in terms of sigma/summation notation.

This isn't a series we know, nor can we easily relate it to a geometric series, so we take a bunch of derivatives and look for patterns. We find the following:

n	$f^{(n)}(x)$	$f^{(n)}(2)$	$c_n = f^{(n)}(2)/n!$
0	x^{-2}	2^{-2}	2^{-2}
1	$-2 \cdot x^{-3}$	$-2 \cdot 2^{-3}$	$-2 \cdot 2^{-3}$
2	$2 \cdot 3 \cdot x^{-4}$	$2 \cdot 3 \cdot 2^{-4}$	$2 \cdot 3 \cdot 2^{-4}/2! = 3 \cdot 2^{-4}$
3	$-2 \cdot 3 \cdot 4 \cdot x^{-5}$	$-2 \cdot 3 \cdot 4 \cdot 2^{-5}$	$-2 \cdot 3 \cdot 4 \cdot 2^{-5}/3! = -4 \cdot 2^{-5}$
4	$2 \cdot 3 \cdot 4 \cdot 5 \cdot x^{-6}$	$2 \cdot 3 \cdot 4 \cdot 5 \cdot 2^{-6}$	$2 \cdot 3 \cdot 4 \cdot 5 \cdot 2^{-6}/4! = 5 \cdot 2^{-6}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$ $\vdots$
n	$(-1)^n(n+1)! \cdot x^{-n-2}$	$(-1)^n(n+1)! \cdot 2^{-n-2}$	$(-1)^n(n+1)! \cdot 2^{-n-2}/n! = (-1)^n(n+1) \cdot 2^{-n-2}$

Thus, the Taylor series for $f(x)$ at $a = 2$ is given by

$$\sum_{n=0}^{\infty} c_n (x-a)^n = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n(n+1)}{2^{n+2}}(x-2)^n.}$$

b. *(4 pts.)* Find the interval of covergence of the series in part **a.**

We start by finding the radius of convergence via the ratio test. We compute that

$$\begin{aligned}\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{(n+2)|x-2|^{n+1}/2^{n+3}}{(n+1)|x-2|^n/2^{n+2}} \\ &= \lim_{n\to\infty} \frac{n+2}{n+1} \cdot \frac{|x-2|^{n+1}}{|x-2|^n} \cdot \frac{2^{n+2}}{2^{n+3}} \\ &= \frac{|x-2|}{2}.\end{aligned}$$

Thus, the series converges if $\frac{|x-2|}{2} < 1$, or $|x-2| < 2$, and the radius of convergence is 2. Since we're centered at $a = 2$, the interval of convergence will be of the form __0, 4__. We need to figure out what happens at the endpoints, $x = 0$ and $x = 4$.

$x = 0$: We plug in $x = 0$ and find $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(n+1)}{2^{n+2}}(-2)^n = \sum_{n=0}^{\infty} \frac{n+1}{4}$, which diverges by the test for divergence (the terms don't go to 0).

$x = 4$: We plug in $x = 4$ and find $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(n+1)}{2^{n+2}}2^n = \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)}{4}$, which again diverges by the test for divergence.

Thus, $\boxed{\text{the interval of convergence is } (0,4).}$

c. *(6 pts.)* How close is the approximation $f(x) \approx T_2(x)$ on the interval $[1.9, 2.1]$? (**Hint:** You do not need to have done parts **a.** or **b.** to do this part.)

We use Taylor's inequality, which states that

$$|f(x) - T_N(x)| \leq \frac{M}{(N+1)!} d^{N+1}.$$

We apply this with $N = 2$ and $d = 0.1$. M is thus given by the max of $|f^{(3)}(x)|$ on $[1.9, 2.1]$. From part **a.** (or just computing it), we find that

$$f^{(3)}(x) = -24 \cdot x^{-5} = \frac{-24}{x^5}.$$

The max of $|f^{(3)}(x)|$ comes from minimizing the denominator, i.e. taking $x = 1.9$. Thus, $M = 24/(1.9)^5$. Putting this all together, we find that

$$|f(x) - T_2(x)| \leq \frac{24/(1.9)^5}{3!}(0.1)^3 = \boxed{\frac{4 \cdot (0.1)^3}{(1.9)^5}} \approx 0.00016 \text{ (if you were curious).}$$

4. Evaluate the following integrals.

a. *(6 pts.)* $\displaystyle\int_0^1 \frac{x}{(x+1)(x+2)}\, dx.$

We use partial fractions. Thus, we want to solve

$$\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2},$$

or, cross multiplying,

$$x = A(x+2) + B(x+1).$$

Plugging in $x = -2$ yields $-2 = -B$, or

$$B = 2,$$

and plugging in $x = -1$ yields

$$A = -1.$$

Thus,

$$\begin{aligned}
\int_0^1 \frac{x}{(x+1)(x+2)}\, dx &= \int_0^1 \left[\frac{-1}{x+1} + \frac{2}{x+2}\right] dx \\
&= \Big[-\ln|x+1| + 2\ln|x+2|\Big]\Big|_0^1 \\
&= \Big[-\ln(2) + 2\ln(3)\Big] - \Big[-\ln(1) + 2\ln(2)\Big] \\
&= \boxed{2\ln(3) - 3\ln(2).}
\end{aligned}$$

b. *(6 pts.)* $\displaystyle\int \frac{6}{x(x+1)(x+3)}\,dx.$

We use partial fractions once again. Thus, we're trying to solve

$$\frac{6}{x(x+1)(x+3)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+3},$$

or

$$6 = A(x+1)(x+3) + Bx(x+3) + Cx(x+1).$$

We plug in $x = 0$, $x = -1$, and $x = -3$ to find

$$A = 2, \quad B = -3, \quad \text{and } C = 1,$$

respectively. Thus,

$$\int \frac{6}{x(x+1)(x+3)}\,dx = \int \left[\frac{2}{x} - \frac{3}{x+1} + \frac{1}{x+3}\right] dx$$
$$= \boxed{2\ln|x| - 3\ln|x+1| + \ln|x+3| + C.}$$

5. *(7 pts.)* Consider the integral $\displaystyle\int_{-1}^{1} e^{x^2}\,dx$. How many subintervals are needed so that the midpoint approximation is within $0.0002 = 1/5000$?

As written: We apply the midpoint approximation error estimate, which says that the error with n subintervals is at most

$$\frac{K(b-a)^3}{24n^2},$$

where K is anything $\geq |f''(x)|$ on $[a,b]$. We take $f(x) = e^{x^2}$ and compute that

$$f'(x) = 2x\,e^{x^2} \quad \text{and} \quad f''(x) = 2e^{x^2} + (2x)^2\,e^{x^2} = (2+4x^2)e^{x^2}.$$

Each constituent part of $f''(x)$ is maximized when $x = 1$ or -1, so we may take $K = 6e$. Thus, the error is at most

$$\frac{6e \cdot 2^3}{24n^2} = \frac{2e}{n^2}.$$

We set $\frac{2e}{n^2} \leq 1/5000$ and cross multiply to get

$$10000e \leq n^2,$$

which becomes

$$\boxed{n \geq \sqrt{10000e} = 100\sqrt{e}.}$$

As I intended, with e^{x^2-1} instead: Notice that $e^{x^2-1} = e^{x^2}/e$, so this function, and hence its derivatives, are $1/e$ times the above. Thus, we can take $K = 6$ here instead of $K = 6e$. When the dust settles, we find $\boxed{n \geq 100.}$

6. a. *(7 pts.)* Find the area of the region bounded by the curves $x = 3y + 3$ and $x = y^2 + 3y + 2$.

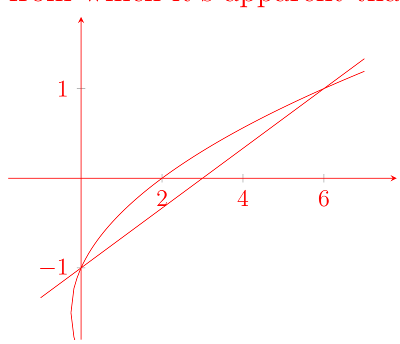
There's a graph below, but suppose we didn't know it. We start by finding the points of intersection. We set $3y + 3 = y^2 + 3y + 2$ and solve, finding $y^2 = 1$, so $y = \pm 1$. Our area is thus given by

$$A = \int_{-1}^{1} (\text{Right} - \text{Left})\ dy.$$

We need to find which function's on the right and which is on the left. Here's an easy way to do that: plug in $y = 0$ to each curve. The line has $x = 3$ and the parabola has $x = 2$, so the line has to be on the right. Thus,

$$\begin{aligned} A &= \int_{-1}^{1} \left[(3y+3) - (y^2 + 3y + 2)\right]\ dy \\ &= \int_{-1}^{1} \left[1 - y^2\right]\ dy \\ &= \left[y - \frac{y^3}{3}\right]\Big|_{-1}^{1} = \boxed{\frac{4}{3}.} \end{aligned}$$

Here's the graph of the curves, from which it's apparent that the line is on the right.



b. *(5 pts.)* For what value of b does the the line $y = b$ divide the above region into two halves of equal area?

We want the area above the line to be equal to the area below it, so we're solving

$$\int_{-1}^{b} \left[1 - y^2\right]\ dy = \int_{b}^{1} \left[1 - y^2\right]\ dy.$$

Equivalently, we can set one of these integrals equal to half of the total area, which we computed in part **a.** to be $4/3$. We do this for the first integral, which is

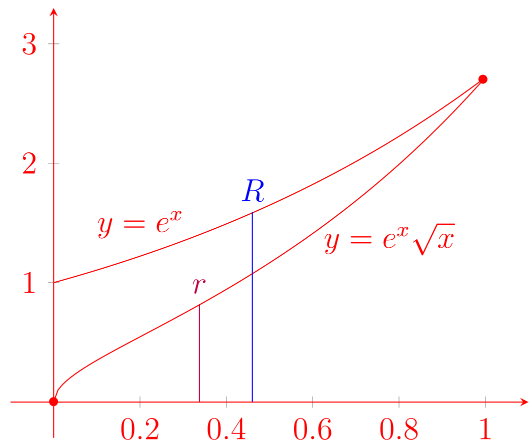
$$\int_{-1}^{b} \left[1 - y^2\right]\ dy = \left[y - \frac{y^3}{3}\right]\Big|_{-1}^{b} = \left[b - \frac{b^3}{3}\right] - \left[-\frac{2}{3}\right] = b - \frac{b^3}{3} + \frac{2}{3}.$$

Setting this equal to $2/3$, we find $b - \frac{b^3}{3} = 0$, so $b = 0$ or $\pm\sqrt{3}$. Both $\sqrt{3}$ and $-\sqrt{3}$ are outside our region, so $\boxed{b = 0.}$

Alternatively, notice that the integrand is an even function. Thus, by symmetry, $b = 0$ gives half the area.

7. *(9 pts.)* The area between the curves $y = e^x$, $y = e^x\sqrt{x}$, and $x = 0$ is rotated around the x-axis. What is the volume of the resulting solid?

We need at least a rough picture to decide which method to use. We know what $y = e^x$ looks like, so it's $y = e^x\sqrt{x}$ that's confusing. Notice that at $x = 0$, we have $y = 0$. We find the point(s) of intersection by setting $e^x = e^x\sqrt{x}$. This gives $\sqrt{x} = 1$, so $x = 1$ is the only point of intersection. These two bits of data inform our graph.



We want to do this problem dx, so we're going to use the washer method. R and r are marked in the graph. We thus find that

$$\begin{aligned} V &= \int_0^1 \left[\pi R^2 - \pi r^2\right]\,dx \\ &= \pi \int_0^1 \left[(e^x)^2 - (e^x\sqrt{x})^2\right]\,dx \\ &= \pi \int_0^1 \left[e^{2x} - xe^{2x}\right]\,dx \\ &= \pi \int_0^1 e^{2x}\,dx - \pi \int_0^1 xe^{2x}\,dx. \end{aligned}$$

We do the first integral with a u-substitution, taking $u = 2x$ and $du = 2\,dx$, to find

$$\pi \int_0^1 e^{2x}\,dx = \frac{\pi}{2}\int_0^2 e^u\,du = \frac{\pi}{2}\left[e^u\right]\Big|_0^2 = \frac{\pi}{2}e^2 - \frac{\pi}{2}.$$

We use integration by parts in the second integral, and we find

$$\begin{aligned} \pi \int_0^1 xe^{2x}\,dx &= \frac{\pi}{2}xe^{2x}\Big|_0^1 - \frac{\pi}{2}\int_0^1 e^{2x}\,dx \\ &= \frac{\pi}{2}e^2 - \frac{\pi}{4}\left[e^{2x}\right]\Big|_0^1 = \frac{\pi}{4}e^2 + \frac{\pi}{4}. \end{aligned}$$

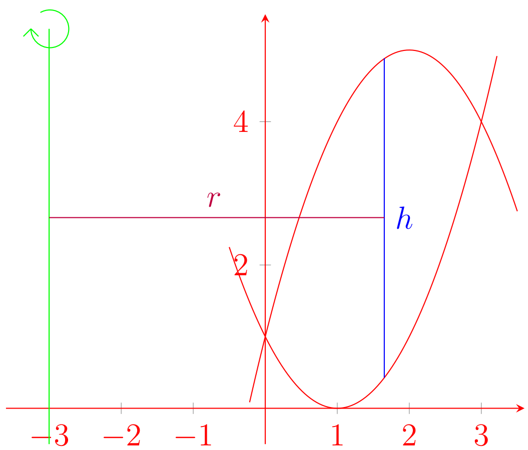
Putting this all together, we find that $\boxed{V = \frac{\pi}{4}e^2 - \frac{3\pi}{4}.}$

8. *(9 pts.)* Consider the region between the curves $y = (x-1)^2$ and $y = 5-(x-2)^2$. What is the volume of the solid formed by rotating this region around the line $x = -3$?

One of these curves is an upward-facing parabola starting at $(1, 0)$, the other is a downward-facing parabola starting at $(2, 5)$. Thus, we should already have a loose picture of what this region will be (roughly speaking, it should be the region between the two "cups"). We find the points of intersection:

$$\begin{aligned} (x-1)^2 &= 5-(x-2)^2, \quad \text{so} \\ x^2-2x+1 &= 5-(x^2-4x+4), \quad \text{so} \\ 2x^2-6x &= 0, \quad \text{so} \\ 2x(x-3) &= 0. \end{aligned}$$

Thus, $x = 0$ and $x = 3$ give the points of intersection. Our graph looks like the following.



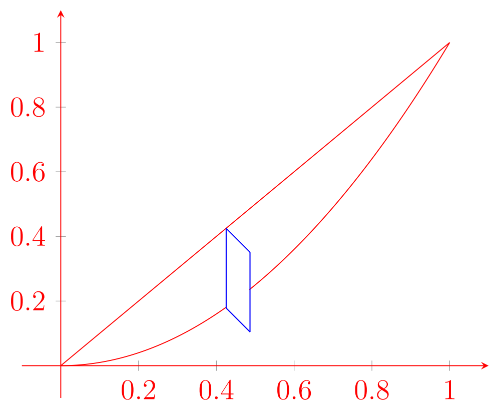
We want to do this problem dx, so we are forced to use the method of cylindrical shells; r and h have been marked in the graph. Explicitly, we have

$$r = x+3 \quad \text{and} \quad h = [5-(x-2)^2]-[(x-1)^2] = -2x^2+6x.$$

Thus, our formula for the volume gives

$$\begin{aligned} V &= \int_0^3 2\pi r h \, dx \\ &= \int_0^3 2\pi(x+3)(-2x^2+6x)\, dx \\ &= 2\pi \int_0^3 (-2x^3+18x)\, dx \\ &= 2\pi \left[-\frac{x^4}{2}+9x^2\right]\Big|_0^3 \\ &= 2\pi\left[-\frac{81}{2}+81\right] = \boxed{81\pi.} \end{aligned}$$

9. *(7 pts.)* A solid is to be formed in the following manner. Its base is to be given by the region bounded by the curves $y = x$ and $y = x^2$, and its cross section for each fixed x is to be a rectangle whose height is equal to x. What is the volume of this solid?



(The blue figure is a rectangle coming out of the page.)

The area of the blue rectangle formed by taking the cross section at x is

$$\begin{aligned} A(x) &= (\text{Base})(\text{Height}) \\ &= (x - x^2)(x) \\ &= x^2 - x^3. \end{aligned}$$

The volume is given by integrating this expression, i.e.

$$\begin{aligned} V &= \int_0^1 A(x)\,dx \\ &= \int_0^1 x^2 - x^3\,dx \\ &= \left[\frac{x^3}{3} - \frac{x^4}{4}\right]\Big|_0^1 = \boxed{\frac{1}{12}.} \end{aligned}$$

10. *(7 pts.)* Find the length of the curve governed by $y = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$ for $0 \le x \le 3$.

The formula for arc length is

$$L = \int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

We gradually manipulate the derivative, finding

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2}e^x - \frac{1}{2}e^{-x} \\ \left(\frac{dy}{dx}\right)^2 &= \left(\frac{1}{2}e^x - \frac{1}{2}e^{-x}\right)^2 \\ &= \frac{1}{4}e^{2x} - \frac{1}{2} + \frac{1}{4}e^{-2x} \\ 1 + \left(\frac{dy}{dx}\right)^2 &= \frac{1}{4}e^{2x} + \frac{1}{2} + \frac{1}{4}e^{-2x} \\ &= \left(\frac{1}{2}e^x + \frac{1}{2}e^{-x}\right)^2. \end{aligned}$$

Thus,

$$L = \int_0^3 \left(\frac{1}{2}e^x + \frac{1}{2}e^{-x}\right)dx = \left[\frac{1}{2}e^x - \frac{1}{2}e^{-x}\right]\Big|_0^3 = \boxed{\frac{1}{2}e^3 - \frac{1}{2}e^{-3}.}$$

Formulas

The Binomial Theorem:

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n, \text{ where } \binom{k}{n} = \frac{k(k-1)(k-2)\ldots(k-n+1)}{n!}.$$

Radius of convergence = 1.

Alternating Series Estimation: $|S - S_N| \leq b_{N+1}$.

Integral Test Estimation: $|S - S_N| \leq \int_N^\infty f(x)\,dx$.

Taylor's Inequality:

$$|f(x) - T_N(x)| \leq \frac{M}{(N+1)!} d^{N+1} \text{ for } |x-a| \leq d,$$

where $M \geq |f^{(N+1)}(x)|$ on $[a-d, a+d]$.

Trapezoid and Midpoint Estimates: Suppose that $|f''(x)| \leq K$ for x in $[a,b]$. Then

$$|\text{Err}_{\text{Trap}}| \leq \frac{K(b-a)^3}{12n^2} \quad \text{and} \quad |\text{Err}_{\text{Midpt}}| \leq \frac{K(b-a)^3}{24n^2}.$$

Simpson's Rule: Let n be even. Set $\Delta x = (b-a)/n$ and $x_i = a + i\Delta x$ for $0 \leq i \leq n$. Then

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right].$$

If $|f^{(4)}(x)| \leq K_4$ on $[a,b]$, then the error in the above approximation is at most

$$\frac{K_4(b-a)^5}{180n^4}.$$

Arc Length: If a curve is given by $y = f(x)$ for $a \leq x \leq b$, then its length is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

Similarly, if it is given by $x = g(y)$ for $a \leq y \leq b$, then its length is

$$L = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy.$$