Math 42: Fall 2015
Midterm 2
November 3, 2015

NAME: ______________________________

Time: **180 minutes**

For each problem, you should write down all of your work carefully and legibly to receive full credit. When asked to justify your answer, you should use theorems and/or mathematical reasoning to support your answer, as appropriate.

Failure to follow these instructions will constitute a breach of the Stanford Honor Code:

- You may not use a calculator or any notes or book during the exam.
- You may not access your cell phone during the exam for any reason.
- You are bound by the Stanford Honor Code, which stipulates among other things that you may not communicate with anyone other than the instructor during the exam, or look at anyone else's solutions.

I understand and accept these instructions.

Signature: ______________________________

Discussion Section: (Please circle)

9:30-10:20 10:30-11:20 11:30-12:20 12:30-1:20 ACE 1:30-3:20

There is a list of formulas and four pages of blank paper at the end of the exam.

Problem	**Value**	**Points**
1	12	
2	9	
3	16	
4	12	
5	7	
6	12	
7	9	
8	9	
9	7	
10	7	
Total	100	

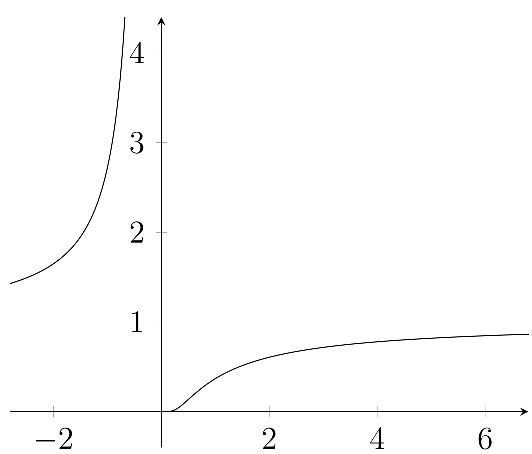
1. (Short answer) You **do not** have to justify your answer to the following questions.

a. *(3 pts.)* (True or false) Because the radius of convergence of the Taylor series for $f(x) = e^x$ centered at $a = 0$ is ∞, the radius of convergence of the Taylor series for $g(x) = 1/e^x$ is 0.

b. *(3 pts.)* Let $T_7(x)$ be the degree 7 Taylor polynomial for a function $f(x)$, centered at $a = 2$. If $f(x) = T_7(x)$ for all x, what is $f^{(9)}(3)$?

c. *(3 pts.)* (True or false) Simpson's rule is always more accurate than the trapezoid rule.

d. *(3 pts.)* The graph of $f(x) = e^{-1/x}$ is given below. Consider its Taylor series centered at $a = 2$ (but please do not find it!). Does the series converge at $x = 5$?



2. a. *(6 pts.)* Find the Taylor series centered at $a = 0$ for $f(x) = x\ln(1+x)$. Express your answer in terms of sigma/summation notation.

b. *(3 pts.)* What is the radius of convergence of the series in part **a.**?

3. a. *(6 pts.)* Find the Taylor series centered at $a = 2$ for $f(x) = x^{-2}$. Express your answer in terms of sigma/summation notation.

b. *(4 pts.)* Find the interval of covergence of the series in part **a.**

c. *(6 pts.)* How close is the approximation $f(x) \approx T_2(x)$ on the interval $[1.9, 2.1]$? (**Hint:** You do not need to have done parts **a.** or **b.** to do this part.)

4. Evaluate the following integrals.

a. *(6 pts.)* $\displaystyle\int_0^1 \frac{x}{(x+1)(x+2)}\,dx.$

b. *(6 pts.)* $\displaystyle\int \frac{6}{x(x+1)(x+3)}\,dx.$

5. *(7 pts.)* Consider the integral $\displaystyle\int_{-1}^{1} e^{x^2}\,dx$. How many subintervals are needed so that the midpoint approximation is within $0.0002 = 1/5000$?

6. a. *(7 pts.)* Find the area of the region bounded by the curves $x = 3y + 3$ and $x = y^2 + 3y + 2$.

b. *(5 pts.)* For what value of b does the the line $y = b$ divide the above region into two halves of equal area?

7. *(9 pts.)* The area between the curves $y = e^x$, $y = e^x\sqrt{x}$, and $x = 0$ is rotated around the x-axis. What is the volume of the resulting solid?

8. *(9 pts.)* Consider the region between the curves $y = (x-1)^2$ and $y = 5-(x-2)^2$. What is the volume of the solid formed by rotating this region around the line $x = -3$?

9. *(7 pts.)* A solid is to be formed in the following manner. Its base is to be given by the region bounded by the curves $y = x$ and $y = x^2$, and its cross section for each fixed x is to be a rectangle whose height is equal to x. What is the volume of this solid?

10. *(7 pts.)* Find the length of the curve governed by $y = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$ for $0 \leq x \leq 3$.

Formulas

The Binomial Theorem:

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n, \text{ where } \binom{k}{n} = \frac{k(k-1)(k-2)\ldots(k-n+1)}{n!}.$$

Radius of convergence = 1.

Alternating Series Estimation: $|S - S_N| \le b_{N+1}$.

Integral Test Estimation: $|S - S_N| \le \int_N^{\infty} f(x)\,dx$.

Taylor's Inequality:

$$|f(x) - T_N(x)| \le \frac{M}{(N+1)!} d^{N+1} \text{ for } |x-a| \le d,$$

where $M \ge |f^{(N+1)}(x)|$ on $[a-d, a+d]$.

Trapezoid and Midpoint Estimates: Suppose that $|f''(x)| \le K$ for x in $[a,b]$. Then

$$|\mathrm{Err}_{\mathrm{Trap}}| \le \frac{K(b-a)^3}{12n^2} \quad \text{and} \quad |\mathrm{Err}_{\mathrm{Midpt}}| \le \frac{K(b-a)^3}{24n^2}.$$

Simpson's Rule: Let n be even. Set $\Delta x = (b-a)/n$ and $x_i = a + i\Delta x$ for $0 \le i \le n$. Then

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right].$$

If $|f^{(4)}(x)| \le K_4$ on $[a,b]$, then the error in the above approximation is at most

$$\frac{K_4(b-a)^5}{180n^4}.$$

Arc Length: If a curve is given by $y = f(x)$ for $a \le x \le b$, then its length is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

Similarly, if it is given by $x = g(y)$ for $a \le y \le b$, then its length is

$$L = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy.$$