Math 42: Fall 2015
Midterm 1
October 13, 2015

NAME: Solutions

Time: **180 minutes**

For each problem, you should write down all of your work carefully and legibly to receive full credit. When asked to justify your answer, you should use theorems and/or mathematical reasoning to support your answer, as appropriate.

Failure to follow these instructions will constitute a breach of the Stanford Honor Code:

- You may not use a calculator or any notes or book during the exam.
- You may not access your cell phone during the exam for any reason.
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I understand and accept these instructions.

Signature: ______________________________

Discussion Section: (Please circle)

9:30-10:20   10:30-11:20   11:30-12:20   12:30-1:20   ACE 1:30-3:20

There are four pages of blank paper at the end of the exam. Please use them for any scratch work and include them with your exam when you turn it in.

Problem	Value	Points
1	12	
2	10	
3	30	
4	10	
5	12	
6	10	
7	6	
8	10	
Total	100	

1. (Short answer) You **do not** have to justify your answer to the following questions.

a. *(2 pts.)* For what values of p does the *sequence* $\{1/n^p\}$ converge? $\underline{\mathbf{p \geq 0}}$
For $p > 0$, it converges to 0. For $p = 0$, it converges to 1.

b. *(2 pts.)* For what values of p does the *series* $\sum_{n=1}^{\infty} 1/n^p$ converge? $\underline{\mathbf{p > 1}}$

This is the p-series test

c. *(2 pts.)* (True or false) If $\int_1^\infty f(x)\,dx$ and $\int_1^\infty g(x)\,dx$ both converge, then $\int_1^\infty f(x) + g(x)\,dx$ must also converge.
True.

d. *(2 pts.)* (True or false) If $\int_1^\infty f(x)\,dx$ and $\int_1^\infty g(x)\,dx$ both diverge, then $\int_1^\infty f(x) + g(x)\,dx$ must also diverge.
False. This is tricky. For example, we could take $f(x) = 1$ and $g(x) = -1$. Both of these integrals diverge, but $f(x) + g(x) = 0$, whose integral definitely converges.

e. *(4 pts.)* Write out in terms of limits how to decompose the improper integral $\int_0^\infty \frac{\sqrt{x}}{(x-2)^2}\,dx$.
Do not evaluate the integral.
(To clarify: If I asked this question about $\int_1^\infty \frac{1}{x^2}\,dx$, I'd be looking for $\lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx$.
Your answer should be a sum of things like this.)

$$\lim_{t\to 2^-} \int_0^t \frac{\sqrt{x}}{(x-2)^2}\,dx + \lim_{t\to 2^+} \int_t^3 \frac{\sqrt{x}}{(x-2)^2}\,dx + \lim_{t\to\infty} \int_3^t \frac{\sqrt{x}}{(x-2)^2}\,dx$$

(In place of 3, you could use any number bigger than 2.)

2. Evaluate the following indefinite integrals.

a. *(5 pts.)* $\int \ln\sqrt{x}\,dx$

First way: Use integration by parts with

$$u = \ln\sqrt{x} \quad du = \frac{d}{dx}(\ln\sqrt{x})\,dx = \frac{1}{\sqrt{x}}\frac{1}{2\sqrt{x}}\,dx = \frac{1}{2x}\,dx$$
$$dv = dx \quad v = x$$

This gives

$$\int \ln\sqrt{x}\,dx = x\ln\sqrt{x} - \int \frac{x}{2x}\,dx = \boxed{x\ln\sqrt{x} - \frac{1}{2}x + C}.$$

Second way: Observe that $\ln\sqrt{x} = \frac{1}{2}\ln x$. Then use integration by parts with

$$u = \frac{1}{2}\ln x \qquad du = \frac{1}{2x}\,dx$$
$$dv = dx \qquad v = x,$$

to arrive at the same answer above. This just gives you a shortcut to the derivative.

b. *(5 pts.)* $\int t^3 e^{-t^2}\,dt$

We make a u-substitution with $u = -t^2$, for which $du = -2t\,dt$. This yields

$$\begin{aligned}\int t^3 e^{-t^2}\,dt &= \frac{1}{2}\int ue^u\,du \\ \text{(Integration by parts)} &= \frac{1}{2}\left[ue^u - \int e^u\,du\right] \\ &= \frac{1}{2}\left[ue^u - e^u + C\right] \\ &= \boxed{-\frac{1}{2}t^2e^{-t^2} - \frac{1}{2}e^{-t^2} + C}.\end{aligned}$$

3. Determine whether each of the following improper integrals is convergent or divergent. If it's convergent, to what does it converge?

a. *(5 pts.)* $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx$

We make a substitution with $u = \ln x$, $du = \frac{1}{x}\,dx$, to find that

$$\begin{aligned}\int_1^\infty \frac{\ln x}{x}\,dx &= \lim_{t\to\infty}\int_1^t \frac{\ln x}{x}\,dx\\ &= \lim_{t\to\infty}\int_0^{\ln t} u\,du\\ &= \lim_{t\to\infty}\left.\frac{u^2}{2}\right|_0^{\ln t}\\ &= \lim_{t\to\infty}\frac{(\ln t)^2}{2}\\ &= +\infty.\end{aligned}$$

Thus, the integral $\boxed{\text{diverges}}$.

b. *(5 pts.)* $\displaystyle\int_0^2 \frac{x}{\sqrt{4-x^2}}\,dx$

We make a substitution with $u = 4 - x^2$, $du = -2x\,dx$, to find that

$$\begin{aligned}\int_0^2 \frac{x}{\sqrt{4-x^2}}\,dx &= \lim_{t\to 2^-}\int_0^t \frac{x}{\sqrt{4-x^2}}\,dx\\ &= \lim_{t\to 2^-} -\frac{1}{2}\int_4^{4-t^2}\frac{1}{\sqrt{u}}\,du\\ &= \lim_{t\to 2^-} -\frac{1}{2}\left.\frac{u^{1/2}}{1/2}\right|_4^{4-t^2}\\ &= \lim_{t\to 2^-} -\sqrt{4-t^2}+\sqrt{4}\\ &= 0.\end{aligned}$$

Thus, the integral $\boxed{\text{converges to 2}}$.

c. *(5 pts.)* $\displaystyle\int_{-\infty}^{\infty} \frac{x}{x^2+1}\,dx$

First way: We begin by splitting the integral up as

$$\int_{-\infty}^{\infty} \frac{x}{x^2+1}\,dx = \int_{-\infty}^{0} \frac{x}{x^2+1}\,dx + \int_{0}^{\infty} \frac{x}{x^2+1}\,dx.$$

We consider these halves *separately*. Let's work with the second one. Set $u = x^2+1$, $du = 2x\,dx$, so that

$$\begin{aligned}\int_{0}^{\infty} \frac{x}{x^2+1}\,dx &= \lim_{t\to\infty}\int_{0}^{t} \frac{x}{x^2+1}\,dx = \lim_{t\to\infty}\frac{1}{2}\int_{1}^{t^2+1}\frac{1}{u}\,du \\ &= \lim_{t\to\infty}\frac{1}{2}\ln|u|\Big|_{1}^{t^2+1} = \lim_{t\to\infty}\frac{1}{2}\ln(t^2+1) = +\infty.\end{aligned}$$

Thus, this half diverges, so the whole integral $\boxed{\text{diverges}}$.

Second way: As $x\to\infty$, $\dfrac{x}{x^2+1} \approx \dfrac{1}{x}$, so we might hope to do a comparison test. Notice that $1/x$ has a vertical asymptote at $x=0$, so if we want to do this, we must avoid $x=0$. Thus, we split our integral as

$$\int_{-\infty}^{\infty} \frac{x}{x^2+1}\,dx = \int_{-\infty}^{1} \frac{x}{x^2+1}\,dx + \int_{1}^{\infty} \frac{x}{x^2+1}\,dx.$$

For $x \geq 1$, we have that

$$\frac{x}{x^2+1} \geq \frac{x}{x^2+x^2} = \frac{1}{2x}.$$

Thus, we have that

$$\int_{1}^{\infty} \frac{x}{x^2+1}\,dx \geq \int_{1}^{\infty} \frac{1}{2x}\,dx,$$

which diverges ($p=1$). Thus, this half diverges by the direct comparison test, so the whole integral $\boxed{\text{diverges}}$.

d. *(5 pts.)* $\displaystyle\int_{0}^{\infty} \frac{1+e^{-x}}{\sqrt{x}}\,dx$

Notice that this is improper for two reasons: we have an infinite limit of integration and we have a vertical asymptote at $x=0$. Thus, we have to split our integral to consider these two features separately. Doing so, we find

$$\int_{0}^{\infty} \frac{1+e^{-x}}{\sqrt{x}}\,dx = \int_{0}^{1} \frac{1+e^{-x}}{\sqrt{x}}\,dx + \int_{1}^{\infty} \frac{1+e^{-x}}{\sqrt{x}}\,dx.$$

Working with the second integral, we find that

$$\int_{1}^{\infty} \frac{1+e^{-x}}{\sqrt{x}}\,dx \geq \int_{1}^{\infty} \frac{1}{\sqrt{x}}\,dx,$$

which diverges ($p=1/2$). Thus, this part diverges by the direct comparison test, so the whole integral $\boxed{\text{diverges}}$.

e. *(5 pts.)* $\displaystyle\int_0^\infty x^2 e^{-x}\,dx$

We're going to do integration by parts twice, so let's consider the indefinite integral first. For our first run, we set $u = x^2$ and $dv = e^{-x}\,dx$, so that $du = 2x\,dx$ and $v = -e^{-x}$. This gives

$$\int x^2 e^{-x}\,dx = -x^2 e^{-x} + \int 2xe^{-x}\,dx.$$

We now do it again, this time choosing $u = 2x$ and $dv = e^{-x}\,dx$, so that $du = 2\,dx$, $v = -e^{-x}$, and

$$\begin{aligned}\int x^2 e^{-x}\,dx &= -x^2 e^{-x} + \left[-2xe^{-x} + \int 2e^{-x}\,dx\right] \\ &= -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C.\end{aligned}$$

We now turn to the definite integral,

$$\begin{aligned}\int_0^\infty x^2 e^{-x}\,dx &= \lim_{t\to\infty} \int_0^t x^2 e^{-x}\,dx \\ &= \lim_{t\to\infty} \left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^t \\ &= \lim_{t\to\infty} \left[\left(-t^2e^{-t} - 2te^{-t} - 2e^{-t}\right) - (-2)\right] \\ &= \lim_{t\to\infty} \left[-\frac{t^2}{e^t} - \frac{2t}{e^t} - \frac{2}{e^t} + 2\right].\end{aligned}$$

We now use L'Hôpital's rule to see that

$$\lim_{t\to\infty} \frac{t^2}{e^t} = \lim_{t\to\infty} \frac{2t}{e^t} = \lim_{t\to\infty} \frac{2}{e^t} = 0.$$

Putting this all together, we find that the integral $\boxed{\text{converges to } 2}$.

f. *(5 pts.)* $\displaystyle\int_0^1 \ln x\,dx$

This is improper at $x = 0$, so we write

$$\begin{aligned}\int_0^1 \ln x\,dx &= \lim_{t\to 0^+} \int_t^1 \ln x\,dx \\ \text{(I.b.P.; } u = \ln x,\ dv = dx) &= \lim_{t\to 0^+} \left[x \ln x\Big|_t^1 - \int_t^1 1\,dx\right] \\ &= \lim_{t\to 0^+} \left[-t\ln t - (1-t)\right] \\ &= -1 - \lim_{t\to 0^+} t\ln t.\end{aligned}$$

We now use L'Hôpital's rule (with a bit of finagling) to see that

$$\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+} \frac{\ln t}{1/t} \stackrel{\text{L'Hôp}}{=} \lim_{t\to 0^+} \frac{1/t}{-1/t^2} = \lim_{t\to 0^+} -t = 0.$$

Thus, the integral $\boxed{\text{converges to } -1}$.

4. Suppose that $f(x)$ is a positive, continuous, and decreasing function on $[1, \infty)$ such that $\lim_{x\to\infty} f(x) = 3$. Define

$$a_n = (-1)^n f(n), \quad b_n = \frac{f(n)}{n}, \quad \text{and} \quad c_n = \frac{(-1)^n f(n)}{n}.$$

a. *(5 pts.)* Which of the sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ converge?

$\{a_n\}$: Since $\lim_{x\to\infty} f(x) = 3$, a_n oscillates between something close to 3 and something close to -3. This must $\boxed{\text{diverge}}$.

$\{b_n\}$: We find that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{f(n)}{n} = \lim_{x\to\infty} \frac{f(x)}{x} \text{ “}=\text{” } \frac{3}{\infty} = 0$. This $\boxed{\text{converges}}$ to 0.

$\{c_n\}$: We have $c_n = (-1)^n b_n$. Since b_n converges to 0, the squeeze theorem implies that $\lim_{n\to\infty} c_n = 0$. Thus, $\{c_n\}$ $\boxed{\text{converges}}$ to 0.

b. *(5 pts.)* Which of the series $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$, and $\sum_{n=1}^{\infty} c_n$ converge?

$\sum_{n=1}^{\infty} a_n$ $\boxed{\text{diverges}}$ by the n^{th} term test (i.e. the test for divergence) and part a.

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{f(n)}{n} \geq \sum_{n=1}^{\infty} \frac{3}{n}$ because $f(x)$ is decreasing and $\lim_{x\to\infty} f(x) = 3$. This diverges $(p = 1)$, so $\sum_{n=1}^{\infty} b_n$ $\boxed{\text{diverges}}$ by the direct comparison test.

$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{(-1)^n f(n)}{n}$ is an alternating series. We found in part a. that $\lim_{n\to\infty} \frac{f(n)}{n} = 0$, and, because $f(x)$ is decreasing, so is $f(x)/x$. Thus, $\sum_{n=1}^{\infty} c_n$ $\boxed{\text{converges}}$ by the alternating series test.

5. For each of the following series, indicate whether it converges or diverges and what test you used.

a. *(3 pts.)* $\sum_{n=1}^{\infty} \frac{\pi^{n+4}}{4^{\pi+n}} = \sum_{n=1}^{\infty} \frac{\pi^4}{4^\pi} \left(\frac{\pi}{4}\right)^n$, which is geometric with $r = \pi/4$.

Convergent   Divergent   Test: Geometric series

b. *(3 pts.)* $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^3} \leq \sum_{n=1}^{\infty} \frac{1}{n^3}$, which converges ($p = 3$).

Convergent   Divergent   Test: Comparison w/ $p = 3$

c. *(3 pts.)* $\sum_{n=1}^{\infty} (2 + \sin n)^{1/n}$

$1 \leq (2 + \sin n)^{1/n} \leq 3^{1/n}$, and $\lim_{n\to\infty} 3^{1/n} = 1$. The squeeze theorem thus says that $\lim_{n\to\infty} (2 + \sin n)^{1/n} = 1$.

Convergent   Divergent   Test: Test for divergence

d. *(3 pts.)* $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n + \sqrt{n} + 1}$

Alternating series, terms are visibly decreasing and converging to 0.

Convergent   Divergent   Test: Alternating series

6. a. *(5 pts.)* For which values of p is the series

$$\sum_{n=1}^{\infty} \frac{4n}{(n^p+1)^3}$$

convergent? Justify your answer fully.

We think that $\frac{4n}{(n^p+1)^3} \approx \frac{4n}{n^{3p}} = \frac{4}{n^{3p-1}}$. We use the limit comparison test with $a_n = \frac{4n}{(n^p+1)^3}$ and $b_n = \frac{4}{n^{3p-1}}$, finding

$$\lim_{n\to\infty} \frac{4n/(n^p+1)^3}{4/n^{3p-1}} = \lim_{n\to\infty} \frac{n^{3p}}{(n^p+1)^3} = \lim_{n\to\infty} \frac{n^{3p}}{n^{3p} + \text{smaller terms}} = 1.$$

Thus, for any p, either $\sum_{n=1}^{\infty} \frac{4n}{(n^p+1)^3}$ and $\sum_{n=1}^{\infty} \frac{4}{n^{3p-1}}$ both converge or both diverge. We know that the latter converges exactly when $3p-1 > 1$, i.e. $\boxed{p > 2/3}$, so this is the case for the original series as well.

b. *(5 pts.)* For $p = 2$, how many terms are needed to compute the series in part a. to within $1/100$? That is, find N so that $|S - S_N| \le 1/100$, where

$$S = \sum_{n=1}^{\infty} \frac{4n}{(n^2+1)^3} \qquad \text{and} \qquad S_N = \sum_{n=1}^{N} \frac{4n}{(n^2+1)^3}.$$

We use the remainder estimate for the integral test, which says that if $R_N = S - S_N$, then

$$\begin{aligned} R_N &\le \int_N^{\infty} \frac{4x}{(x^2+1)^3}\,dx \\ &= \lim_{t\to\infty} \int_N^t \frac{4x}{(x^2+1)^3}\,dx \\ (u = x^2+1) \quad &= \lim_{t\to\infty} \int_{N^2+1}^{t^2+1} \frac{2}{u^3}\,du \\ &= \lim_{t\to\infty} -u^{-2}\Big|_{N^2+1}^{t^2+1} \\ &= \lim_{t\to\infty} \left[\frac{1}{(N^2+1)^2} - \frac{1}{(t^2+1)^2}\right] = \frac{1}{(N^2+1)^2}. \end{aligned}$$

We thus set $\frac{1}{(N^2+1)^2} \le \frac{1}{100}$ and solve: $(N^2+1)^2 \ge 100$, so $N^2+1 \ge 10$, or $\boxed{N \ge 3}$.

7. *(6 pts.)* Determine whether the series

$$\sum_{n=1}^{\infty} \frac{2^{n^2}}{n!}$$

converges or diverges, showing all necessary work.

We use the ratio test. We compute that

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{2^{(n+1)^2}/(n+1)!}{2^{n^2}/n!} \right| \\
&= \lim_{n\to\infty} \frac{2^{(n+1)^2}}{2^{n^2}} \frac{n!}{(n+1)!} \\
&= \lim_{n\to\infty} \frac{2^{n^2+2n+1}}{2^{n^2}} \frac{1}{n+1} \\
&= \lim_{n\to\infty} \frac{2^{2n+1}}{n+1} \\
&= \lim_{n\to\infty} \frac{2 \cdot 4^n}{n+1}.
\end{aligned}$$

We now use L'Hôpital's rule to check that

$$\lim_{n\to\infty} \frac{4^n}{n+1} = \lim_{x\to\infty} \frac{4^x}{x+1} = \lim_{x\to\infty} \frac{4^x \ln 4}{1} = +\infty.$$

Thus, the ratio test implies that the series $\boxed{\text{diverges}}$.

8. Consider the power series
$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^{n+2}\sqrt{n+1}}.$$

a. *(5 pts.)* What is its radius of convergence?

We use the ratio test, finding

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}/3^{n+3}\sqrt{n+2}}{(-1)^n x^n/3^{n+2}\sqrt{n+1}}\right| \\ &= \lim_{n\to\infty}\frac{|x|^{n+1}}{|x|^n}\frac{3^{n+2}}{3^{n+3}}\frac{\sqrt{n+1}}{\sqrt{n+2}} \\ &= \lim_{n\to\infty}\frac{|x|}{3}\sqrt{\frac{n+1}{n+2}} \\ &= \frac{|x|}{3}.\end{aligned}$$

Thus, the ratio test implies the power series converges if $|x|/3 < 1$, or $|x| < 3$. Thus, the radius of convergence is $\boxed{R = 3}$.

b. *(5 pts.)* What is its interval of convergence?

We have to check $x = 3$ and $x = -3$. For $x = 3$, we find
$$\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{3^{n+2}\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{9\sqrt{n+1}},$$
which is an alternating series. Its terms are visibly decreasing to 0, so the series converges at $x = 3$.

For $x = -3$, we find
$$\sum_{n=1}^{\infty} \frac{(-1)^n(-3)^n}{3^{n+2}\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{1}{9\sqrt{n+1}},$$
which diverges ($p = 1/2$). Thus, the interval of convergence is $\boxed{(-3, 3]}$.