16. $\frac{dT}{dt} = k(T - 20)$. Let $y = T - 20$. Then $\frac{dy}{dt} = ky$, so $y(t) = y(0)e^{kt}$. $y(0) = T(0) - 20 = 95 - 20 = 75$, so $y(t) = 75e^{kt}$. When $T(t) = 70$, $\frac{dT}{dt} = -1°\mathrm{C/min}$. Equivalently, $\frac{dy}{dt} = -1$ when $y(t) = 50$. Thus, $-1 = \frac{dy}{dt} = ky(t) = 50k$ and $50 = y(t) = 75e^{kt}$. The first relation implies $k = -1/50$, so the second relation says $50 = 75e^{-t/50}$. Thus, $e^{-t/50} = \frac{2}{3} \Rightarrow -t/50 = \ln\left(\frac{2}{3}\right) \Rightarrow t = -50\ln\left(\frac{2}{3}\right) \approx 20.27$ min.

22. (a) $\frac{dy}{dt} = ky^{1+c} \Rightarrow y^{-1-c}\,dy = k\,dt \Rightarrow \frac{y^{-c}}{-c} = kt + C$. Since $y(0) = y_0$, we have $C = \frac{y_0^{-c}}{-c}$. Thus, $\frac{y^{-c}}{-c} = kt + \frac{y_0^{-c}}{-c}$, or $y^{-c} = y_0^{-c} - ckt$. So $y^c = \frac{1}{y_0^{-c} - ckt} = \frac{y_0^c}{1 - cy_0^c kt}$ and $y(t) = \frac{y_0}{(1 - cy_0^c kt)^{1/c}}$.

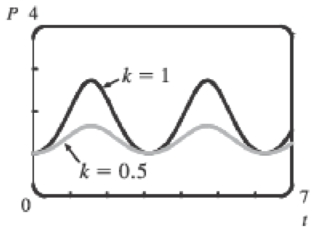
(b) $y(t) \to \infty$ as $1 - cy_0^c kt \to 0$, that is, as $t \to \frac{1}{cy_0^c k}$. Define $T = \frac{1}{cy_0^c k}$. Then $\lim_{t \to T^-} y(t) = \infty$.

(c) According to the data given, we have $c = 0.01$, $y(0) = 2$, and $y(3) = 16$, where the time t is given in months. Thus, $y_0 = 2$ and $16 = y(3) = \frac{y_0}{(1 - cy_0^c k \cdot 3)^{1/c}}$. Since $T = \frac{1}{cy_0^c k}$, we will solve for $cy_0^c k$. $16 = \frac{2}{(1 - 3cy_0^c k)^{100}} \Rightarrow$ $1 - 3cy_0^c k = \left(\frac{1}{8}\right)^{0.01} = 8^{-0.01} \Rightarrow cy_0^c k = \frac{1}{3}\left(1 - 8^{-0.01}\right)$. Thus, doomsday occurs when $t = T = \frac{1}{cy_0^c k} = \frac{3}{1 - 8^{-0.01}} \approx 145.77$ months or 12.15 years.

5. Using (4), $A = \frac{M - P_0}{P_0} = \frac{10{,}000 - 1000}{1000} = 9$, so $P(t) = \frac{10{,}000}{1 + 9e^{-kt}}$. $P(1) = 2500 \Rightarrow 2500 = \frac{10{,}000}{1 + 9e^{-k(1)}} \Rightarrow$ $1 + 9e^{-k} = 4 \Rightarrow 9e^{-k} = 3 \Rightarrow e^{-k} = \frac{1}{3} \Rightarrow -k = \ln\frac{1}{3} \Rightarrow k = \ln 3$. After another three years, $t = 4$, and $P(4) = \frac{10{,}000}{1 + 9e^{-(\ln 3)4}} = \frac{10{,}000}{1 + 9\left(e^{\ln 3}\right)^{-4}} = \frac{10{,}000}{1 + 9(3)^{-4}} = \frac{10{,}000}{1 + \frac{1}{9}} = \frac{10{,}000}{\frac{10}{9}} = 9000$.

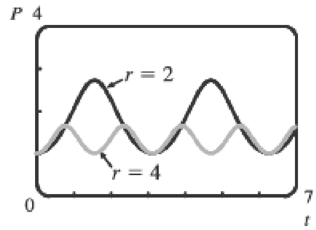
19. (a) $dP/dt = kP\cos(rt - \phi) \;\Rightarrow\; (dP)/P = k\cos(rt-\phi)\,dt \;\Rightarrow\; \int (dP)/P = k\int \cos(rt-\phi)\,dt \;\Rightarrow$ $\ln P = (k/r)\sin(rt-\phi) + C$. (Since this is a growth model, $P > 0$ and we can write $\ln P$ instead of $\ln|P|$.) Since $P(0) = P_0$, we obtain $\ln P_0 = (k/r)\sin(-\phi) + C = -(k/r)\sin\phi + C \;\Rightarrow\; C = \ln P_0 + (k/r)\sin\phi$. Thus, $\ln P = (k/r)\sin(rt-\phi) + \ln P_0 + (k/r)\sin\phi$, which we can rewrite as $\ln(P/P_0) = (k/r)[\sin(rt-\phi) + \sin\phi]$ or, after exponentiation, $P(t) = P_0 e^{(k/r)[\sin(rt-\phi)+\sin\phi]}$.

(b) As k increases, the amplitude increases, but the minimum value stays the same.



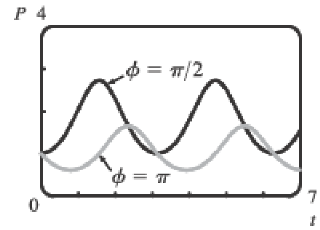
Comparing values of k with $P_0 = 1$, $r = 2$, and $\phi = \pi/2$

As r increases, the amplitude and the period decrease.



Comparing values of r with $P_0 = 1$, $k = 1$, and $\phi = \pi/2$

A change in ϕ produces slight adjustments in the phase shift and amplitude.



Comparing values of ϕ with $P_0 = 1$, $k = 1$, and $r = 2$

$P(t)$ oscillates between $P_0 e^{(k/r)(1+\sin\phi)}$ and $P_0 e^{(k/r)(-1+\sin\phi)}$ (the extreme values are attained when $rt - \phi$ is an odd multiple of $\frac{\pi}{2}$), so $\lim_{t\to\infty} P(t)$ does not exist.

20. (a) $dP/dt = kP\cos^2(rt-\phi) \;\Rightarrow\; (dP)/P = k\cos^2(rt-\phi)\,dt \;\Rightarrow\; \int(dP)/P = k\int\cos^2(rt-\phi)\,dt \;\Rightarrow$

$\ln P = k\displaystyle\int \frac{1+\cos(2(rt-\phi))}{2}\,dt = \frac{k}{2}t + \frac{k}{4r}\sin(2(rt-\phi)) + C$. From $P(0) = P_0$, we get

$\ln P_0 = \dfrac{k}{4r}\sin(-2\phi) + C = C - \dfrac{k}{4r}\sin 2\phi$, so $C = \ln P_0 + \dfrac{k}{4r}\sin 2\phi$ and

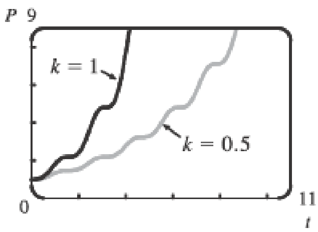
$\ln P = \dfrac{k}{2}t + \dfrac{k}{4r}\sin(2(rt-\phi)) + \ln P_0 + \dfrac{k}{4r}\sin 2\phi$. Simplifying, we get

$\ln\dfrac{P}{P_0} = \dfrac{k}{2}t + \dfrac{k}{4r}[\sin(2(rt-\phi)) + \sin 2\phi] = f(t)$, or $P(t) = P_0e^{f(t)}$.

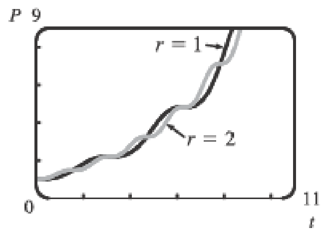
(b) An increase in k stretches the graph of P vertically while maintaining $P(0) = P_0$.

An increase in r compresses the graph of P horizontally—similar to changing the period in Exercise 19.

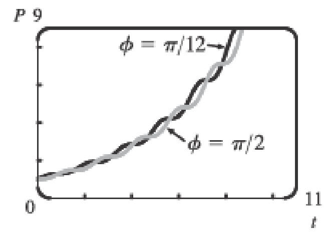
As in Exercise 19, a change in ϕ only makes slight adjustments in the growth of P, as shown in the figure.



Comparing values of k with $P_0 = 1$, $r = 2$, and $\phi = \pi/2$



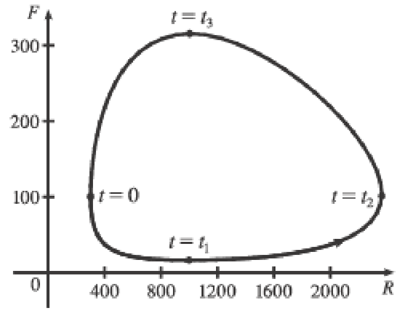
Comparing values of r with $P_0 = 1$, $k = 0.5$, and $\phi = \pi/2$



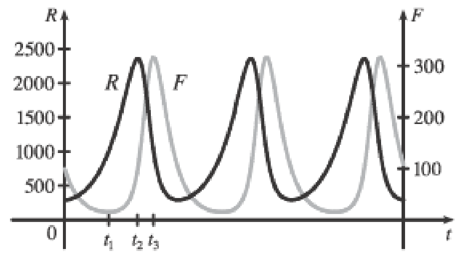
Comparing values of ϕ with $P_0 = 1$, $k = 0.5$, and $r = 2$

$f'(t) = k/2 + [k/(4r)][2r\cos(2(rt-\phi))] = (k/2)[1+\cos(2(rt-\phi))] \geq 0$. Since $P(t) = P_0e^{f(t)}$, we have $P'(t) = P_0f'(t)e^{f(t)} \geq 0$, with equality only when $\cos(2(rt-\phi)) = -1$; that is, when $rt-\phi$ is an odd multiple of $\frac{\pi}{2}$. Therefore, $P(t)$ is an increasing function on $(0,\infty)$. P can also be written as $P(t) = P_0e^{kt/2}e^{(k/4r)[\sin(2(rt-\phi))+\sin 2\phi]}$. The second exponential oscillates between $e^{(k/4r)(1+\sin 2\phi)}$ and $e^{(k/4r)(-1+\sin 2\phi)}$, while the first one, $e^{kt/2}$, grows without bound. So $\lim\limits_{t\to\infty} P(t) = \infty$.

5. (a) At $t = 0$, there are about 300 rabbits and 100 foxes. At $t = t_1$, the number of foxes reaches a minimum of about 20 while the number of rabbits is about 1000. At $t = t_2$, the number of rabbits reaches a maximum of about 2400, while the number of foxes rebounds to 100. At $t = t_3$, the number of rabbits decreases to about 1000 and the number of foxes reaches a maximum of about 315. As t increases, the number of foxes decreases greatly to 100, and the number of rabbits decreases to 300 (the initial populations), and the cycle starts again.



(b)



9. $\dfrac{dW}{dR} = \dfrac{-0.02W + 0.00002RW}{0.08R - 0.001RW} \quad \Leftrightarrow \quad (0.08 - 0.001W)R\,dW = (-0.02 + 0.00002R)W\,dR \quad \Leftrightarrow$

$$\frac{0.08 - 0.001W}{W}\,dW = \frac{-0.02 + 0.00002R}{R}\,dR \quad \Leftrightarrow \quad \int\left(\frac{0.08}{W} - 0.001\right)dW = \int\left(-\frac{0.02}{R} + 0.00002\right)dR \quad \Leftrightarrow$$

$$0.08\,\ln|W| - 0.001W = -0.02\,\ln|R| + 0.00002R + K \quad \Leftrightarrow \quad 0.08\,\ln W + 0.02\,\ln R = 0.001W + 0.00002R + K \quad \Leftrightarrow$$

$$\ln\!\left(W^{0.08}R^{0.02}\right) = 0.00002R + 0.001W + K \quad \Leftrightarrow \quad W^{0.08}R^{0.02} = e^{0.00002R + 0.001W + K} \quad \Leftrightarrow$$

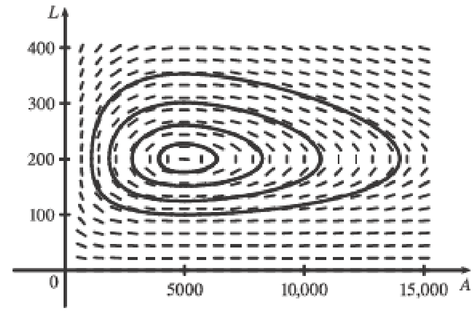
$R^{0.02}W^{0.08} = Ce^{0.00002R}e^{0.001W} \quad \Leftrightarrow \quad \dfrac{R^{0.02}W^{0.08}}{e^{0.00002R}e^{0.001W}} = C$. In general, if $\dfrac{dy}{dx} = \dfrac{-ry + bxy}{kx - axy}$, then $C = \dfrac{x^r y^k}{e^{bx}e^{ay}}$.

10. (a) A and L are constant $\Rightarrow$ $A' = 0$ and $L' = 0$ $\Rightarrow$ $\begin{Bmatrix} 0 = 2A - 0.01AL \\ 0 = -0.5L + 0.0001AL \end{Bmatrix}$ $\Rightarrow$ $\begin{cases} 0 = A(2 - 0.01L) \\ 0 = L(-0.5 + 0.0001A) \end{cases}$

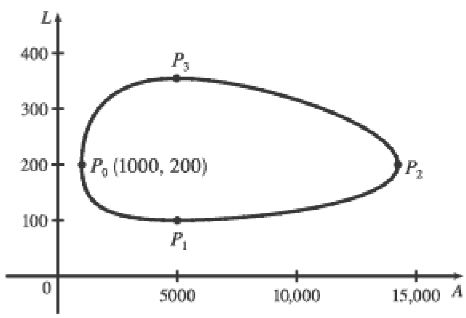
So either $A = L = 0$ or $L = \frac{2}{0.01} = 200$ and $A = \frac{0.5}{0.0001} = 5000$. The trivial solution $A = L = 0$ just says that if there aren't any aphids or ladybugs, then the populations will not change. The non-trivial solution, $L = 200$ and $A = 5000$, indicates the population sizes needed so that there are no changes in either the number of aphids or the number of ladybugs.

(b) $\dfrac{dL}{dA} = \dfrac{dL/dt}{dA/dt} = \dfrac{-0.5L + 0.0001AL}{2A - 0.01AL}$

(c) The solution curves (phase trajectories) are all closed curves that have the equilibrium point $(5000, 200)$ inside them.



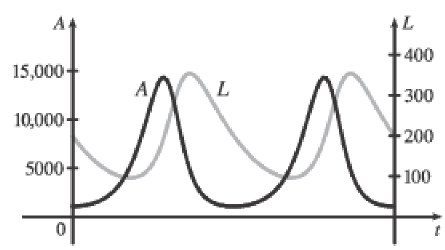
(d)



At $P_0(1000, 200)$, $dA/dt = 0$ and $dL/dt = -80 < 0$, so the number of ladybugs is decreasing and hence, we are proceeding in a counterclockwise direction. At P_0, there aren't enough aphids to support the ladybug population, so the number of ladybugs decreases and the number of aphids begins to increase. The ladybug population reaches a minimum at $P_1(5000, 100)$ while the aphid population increases in a dramatic way, reaching its maximum at $P_2(14, 250, 200)$.

Meanwhile, the ladybug population is increasing from P_1 to $P_3(5000, 355)$, and as we pass through P_2, the increasing number of ladybugs starts to deplete the aphid population. At P_3 the ladybugs reach a maximum population, and start to decrease due to the reduced aphid population. Both populations then decrease until P_0, where the cycle starts over again.

(e) Both graphs have the same period and the graph of L peaks about a quarter of a cycle after the graph of A.



11. (a) Letting $W = 0$ gives us $dR/dt = 0.08R(1 - 0.0002R)$. $dR/dt = 0 \Leftrightarrow R = 0$ or 5000. Since $dR/dt > 0$ for $0 < R < 5000$, we would expect the rabbit population to *increase* to 5000 for these values of R. Since $dR/dt < 0$ for $R > 5000$, we would expect the rabbit population to *decrease* to 5000 for these values of R. Hence, in the absence of wolves, we would expect the rabbit population to stabilize at 5000.

(b) R and W are constant $\Rightarrow R' = 0$ and $W' = 0 \Rightarrow$

$$\begin{cases} 0 = 0.08R(1 - 0.0002R) - 0.001RW \\ 0 = -0.02W + 0.00002RW \end{cases} \Rightarrow \begin{cases} 0 = R[0.08(1 - 0.0002R) - 0.001W] \\ 0 = W(-0.02 + 0.00002R) \end{cases}$$

The second equation is true if $W = 0$ or $R = \frac{0.02}{0.00002} = 1000$. If $W = 0$ in the first equation, then either $R = 0$ or $R = \frac{1}{0.0002} = 5000$ [as in part (a)]. If $R = 1000$, then $0 = 1000[0.08(1 - 0.0002 \cdot 1000) - 0.001W] \Leftrightarrow 0 = 80(1 - 0.2) - W \Leftrightarrow W = 64$.

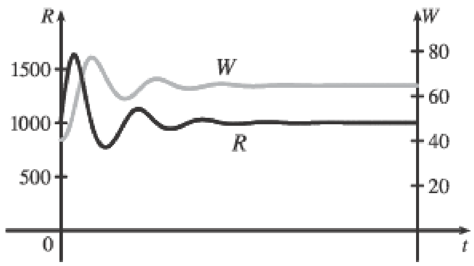
Case (i): $W = 0$, $R = 0$: both populations are zero

Case (ii): $W = 0$, $R = 5000$: see part (a)

Case (iii): $R = 1000$, $W = 64$: the predator/prey interaction balances and the populations are stable.

(c) The populations of wolves and rabbits fluctuate around 64 and 1000, respectively, and eventually stabilize at those values.

(d)



24. First note that, in this question, "weighs" is used in the informal sense, so what we really require is Barbara's mass m in kg as a function of t. Barbara's net intake of calories per day at time t (measured in days) is $c(t) = 1600 - 850 - 15m(t) = 750 - 15m(t)$, where $m(t)$ is her mass at time t. We are given that $m(0) = 60$ kg and $\frac{dm}{dt} = \frac{c(t)}{10{,}000}$, so $\frac{dm}{dt} = \frac{750 - 15m}{10{,}000} = \frac{150 - 3m}{2000} = \frac{-3(m - 50)}{2000}$ with $m(0) = 60$. From $\int \frac{dm}{m - 50} = \int \frac{-3\,dt}{2000}$, we get $\ln|m - 50| = -\frac{3}{2000}t + C$. Since $m(0) = 60$, $C = \ln 10$. Now $\ln \frac{|m - 50|}{10} = -\frac{3t}{2000}$, so $|m - 50| = 10e^{-3t/2000}$. The quantity $m - 50$ is continuous, initially positive, and the right-hand side is never zero. Thus, $m - 50$ is positive for all t, and $m(t) = 50 + 10e^{-3t/2000}$ kg. As $t \to \infty$, $m(t) \to 50$ kg. Thus, Barbara's mass gradually settles down to 50 kg.