Solutions: Homework 8

1 Chapter 7.1

Problem 3

This is one of those problems where we substitute in solutions and just check to see if they work or not.

(a) Substituting $y = e^{rx}$ into the differential equation we obtain:

$$2r^2e^{rx} + re^{rx} - e^{rx} = 0 \iff 2r^2 + r - 1 = 0$$

We can use the quadratic formula and see that $r = \frac{-1\pm\sqrt{9}}{4} = \frac{1}{2}, -1$.

(b) Substituting $y = ae^{-x} + be^{x/2}$ into the differential equation we have

$$2(ae^{-x} + \frac{b}{4}e^{x/2}) - ae^{-x} + \frac{b}{2}e^{x/2} - ae^{-x} - be^{x/2}$$

which we see is equal to zero after collecting like terms.

Problem 9

The derivative $\frac{dP}{dt}$ represents the "change in P". So we simply have to understand the function $f(P) = 1.2P(1 - \frac{P}{4200})$ to answer the questions.

(a) When $P < 0$ or $P > 4200$, $f(P)$ is positive so that is when the population is increasing. However $P < 0$ is impossible for a real population so it is sufficient to mention only $P > 4200$.

(b) The population is decreasing when $0 < P < 4200$ because that is the range where $f(P)$ is negative.

(c) The population is not changin for $P = 0, 4200$ so those are the equilibrium solutions.

Problem 13

(a) This goes with graph III because of the two graphs with slope 1 at the origin, this is the one whose derivative y' is increasing as x, y both grow.

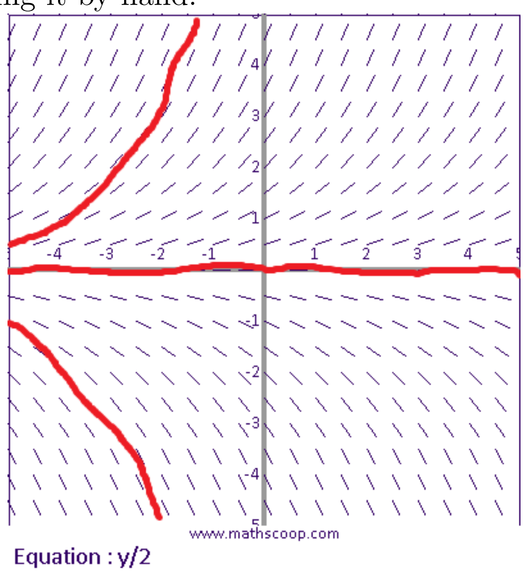
(b) This goes with graph I because this is the only differential equation with $y' = 0$ only at $x = 0$.

(c) This goes with graph IV because of the two graphs with slope 1 at the origin, this is the one whose derivative y' is going to zero as x, y both grow.

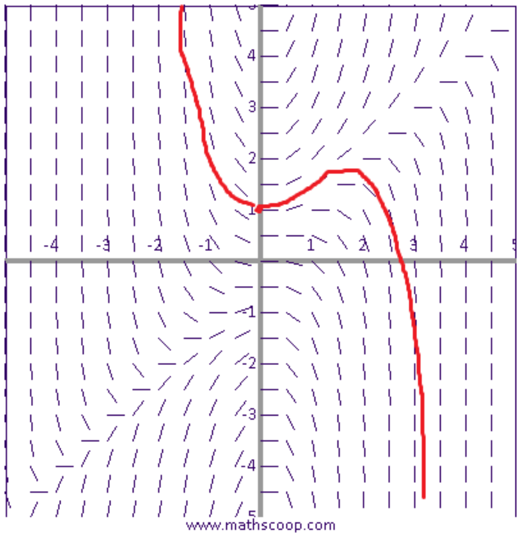
(d) This goes with graph II because the derivative $y' = 0$ when $y = 0$ in $y' = \sin(xy)\cos(xy)$.

2 Chapter 7.2

Problem 9 The slope field with three solutions in red. You should practice doing it by hand.

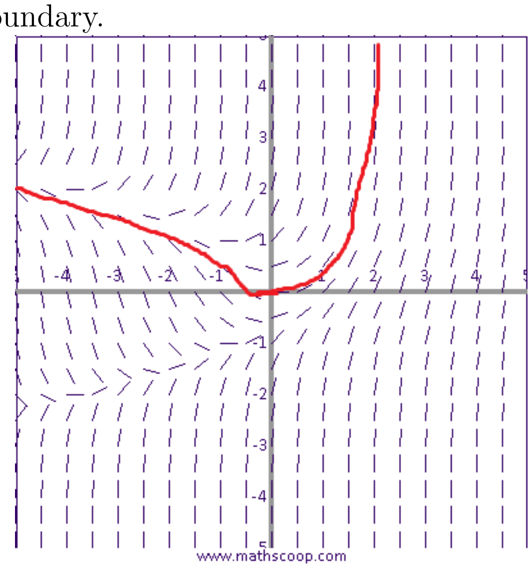


Problem 12 The slope field with the solution curve in red. You should practice doing it by hand.



Equation : x*y-x^2

Problem 14 The slope field with the solution curve in red. You should practice doing it by hand. The red curve is slightly tricky to think about for when $x < 0$ if at any point $y^2 + x > 0$ then the curve would have not passed through the origin. However if the curve goes to far below the $y^2 + x = 0$ curve then it would also not pass through the origin. Therefore, it was most accurate to draw the solution curve only slightly below the $y^2 + x = 0$ boundary.



Equation : x+y^2

3 Chapter 7.3

Problem 2

We use the method of solution for separable equations to obtain the general solution:

$$\begin{aligned}
\frac{dy}{dx} &= xe^{-y}, \\
e^y dy &= xdx, \\
\int e^y dy &= \int xdx, \\
e^y + C_1 &= x^2/2 + C_2, \\
e^y &= x^2/2 + C_3, \\
y &= \ln(x^2/2 + C_3).
\end{aligned}$$

Problem 7

We use the method of solution for separable equations to obtain the general solution:

$$\begin{aligned}
\frac{dy}{dt} &= \frac{te^t}{y\sqrt{1+y^2}}, \\
y\sqrt{1+y^2}dy &= te^t dt, \\
\int y\sqrt{1+y^2}dy &= \int te^t dt, \\
\frac{1}{3}(1+y^2)^{3/2} + C_1 &= te^t - e^t + C_2, \\
\frac{1}{3}(1+y^2)^{3/2} &= te^t - e^t + C_3, \\
y &= \pm\sqrt{[3(te^t - e^t + C_3)]^{2/3} - 1}.
\end{aligned}$$

On the left hand side we used u substitution to integrate. On the right hand side we used integration by parts.

Problem 12

We use the method of solution for separable equations to obtain the general solution:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\ln(x)}{xy}, \\
ydy &= \frac{\ln(x)}{x}dx, \\
\int ydy &= \int \frac{\ln(x)}{x}dx, \\
y^2/2 &= \frac{\ln(x)^2}{2} + C.
\end{aligned}$$

The integral on the right hand side used a u substitution $u = \ln(x)$. To solve for C, we use $y(1) = 2$;

$$2^2/2 = 0 + C \to C = 2.$$

Therefore, $y^2 = \ln(x)^2 + 4 \Rightarrow y = +\sqrt{\ln(x)^2 + 4}$.

Problem 47

The solution to the problem involves first setting up the differential equation corresponding to this "mixing problem", setting up the initial conditions, solving the initial value problem, and then using that solution to find the alcohol proporitions after an hour.

To set things up let $y(t)$ be the number of gallons of alcohol in the vat after t minutes. Then $y(0) = .04(500) = 20$. The mixing equation says that

$$\frac{dy}{dt} = (rate\ in) - (rate\ out) = .06 * 5 - \frac{y(t)}{500} * 5 = .3 - \frac{y}{100}$$

To solve this differentiable equation we notice that it is separable so

$$\begin{aligned}
\frac{dy}{dt} &= \frac{30 - y}{100}, \\
\frac{100dy}{30 - y} &= dt, \\
\int \frac{100dy}{30 - y} &= \int dt, \\
-100 \ln|30 - y| &= t + C.
\end{aligned}$$

Since, $y(0) = 20$, we see that $C = -100 \ln|30 - 20| = -100 \ln(10)$. To find out the percentage of alcohol at $t = 60$, we substitute:

$$
\begin{aligned}
-100 \ln|30 - y| &= 60 - 100 \ln(10), \\
\ln|30 - y| &= -.6 + \ln(10), \\
|30 - y| &= 10e^{-.6}
\end{aligned}
$$

Notice that $y < 30$ because y is continuous, $y(0) = 20$ and there would be a discontinuity if $y = 30$. Therefore $30 - y = 10e^{-6} \Rightarrow y = 30 - 10e^{-.6}$. The percentage of alcohol is therefore $(y(60)/500) * 100 = 6 - 2^{e^{-.6}}$.

4 Chapter 7.4

Problem 3

(a) With this setup we know that $\frac{dP}{dt} = kP$ for some growth rate k and with $P(0) = 100$ and $P(1) = 420$. The general solution is $P(t) = Ae^{kt}$. Since $P(0) = 100$, $A = 100$. Since $P(1) = 420$, we know that $420 = 100e^k \Rightarrow k = \ln(4.2)$. So the solution is $P(t) = 100 * 4.2^t$.

(b) After three hours the population is $P(3) = 100 * 4.2^3 \approx 7408$.

(c) The growth rate after three hours is $\frac{dP}{dt} = \ln(4.2)P(3) = \ln(4.2)*7408 \approx 10631\ bacteria/hour$.

(d) Solving for $P(t) = 10000 = 100*4.2^t \Rightarrow 4.2^t = 100 \Rightarrow t = \log_{4.2}(100) \approx 3.2089\ hours$.

Problem 9

(a) With this setup we know that $\frac{dP}{dt} = kP$ for some growth rate k and with $P(0) = 100$ and $P(30) = 50$. The general solution is $P(t) = Ae^{kt}$. Since $P(0) = 100$, $A = 100$. Since $P(30) = 50$, we know that $50 = 100e^{30k} \Rightarrow 30k = \ln(1/2)$. So the solution is $P(t) = 100/2^{t/30}$.

(b) After 100 years $P(100) = 100/2^{100/30} \approx 9.92\ mg$.

(c) Solving for $P(t) = 1 = 100/2^{t/30} \Rightarrow t = 30 \log_2(100) \approx 199.31\ years$.