6. $h_{\text{ave}} = \frac{1}{b-a}\int_a^b h(u)\,du = \frac{1}{1-(-1)}\int_{-1}^1 (3-2u)^{-1}\,du = \frac{1}{2}\int_{-1}^1 \frac{1}{3-2u}\,du = \frac{1}{2}\int_5^1 \frac{1}{y}\left(-\frac{1}{2}\,dy\right)$ $[y = 3-2u,\ dy = -2\,du]$

$$= -\frac{1}{4}\Big[\ln|y|\Big]_5^1 = -\frac{1}{4}(\ln 1 - \ln 5) = \frac{1}{4}\ln 5$$

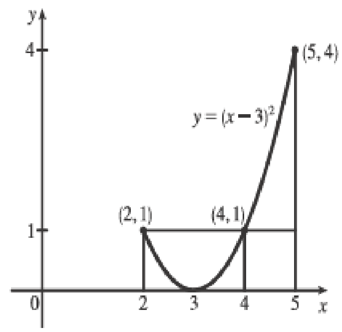
7. (a) $f_{\text{ave}} = \dfrac{1}{5-2}\displaystyle\int_2^5 (x-3)^2\,dx = \frac{1}{3}\left[\frac{1}{3}(x-3)^3\right]_2^5$

$$= \frac{1}{9}\left[2^3 - (-1)^3\right] = \frac{1}{9}(8+1) = 1$$

(b) $f(c) = f_{\text{ave}} \quad\Leftrightarrow\quad (c-3)^2 = 1 \quad\Leftrightarrow$

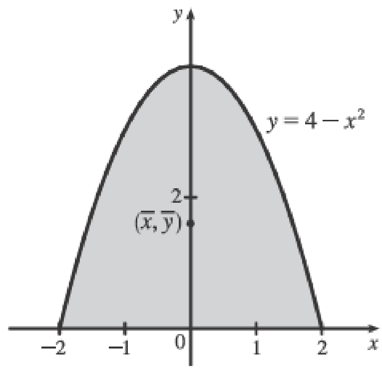
$c - 3 = \pm 1 \quad\Leftrightarrow\quad c = 2$ or 4

(c)



17. $\rho_{\text{ave}} = \dfrac{1}{8}\displaystyle\int_0^8 \frac{12}{\sqrt{x+1}}\,dx = \frac{3}{2}\int_0^8 (x+1)^{-1/2}\,dx = \left[3\sqrt{x+1}\right]_0^8 = 9 - 3 = 6$ kg/m

45. Since the region in the figure is symmetric about the y-axis, we know that $\overline{x} = 0$. The region is "bottom-heavy," so we know that $\overline{y} < 2$, and we might guess that $\overline{y} = 1.5$.



$A = \int_{-2}^2 (4-x^2)\,dx = 2\int_0^2 (4-x^2)\,dx = 2\left[4x - \frac{1}{3}x^3\right]_0^2$

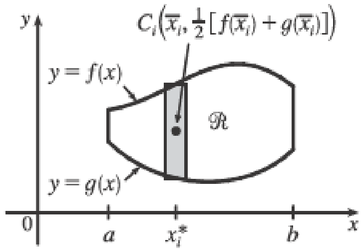
$\quad = 2\left(8 - \frac{8}{3}\right) = \frac{32}{3}$.

$\overline{x} = \frac{1}{A}\int_{-2}^2 x(4-x^2)\,dx = 0$ since $f(x) = x(4-x^2)$ is an odd function (or since the region is symmetric about the y-axis).

$$\overline{y} = \frac{1}{A}\int_{-2}^2 \frac{1}{2}(4-x^2)^2\,dx = \frac{3}{32}\cdot\frac{1}{2}\cdot 2\int_0^2 (16 - 8x^2 + x^4)\,dx = \frac{3}{32}\left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_0^2$$
$$= \frac{3}{32}\left(32 - \frac{64}{3} + \frac{32}{5}\right) = 3\left(1 - \frac{2}{3} + \frac{1}{5}\right) = 3\left(\frac{8}{15}\right) = \frac{8}{5}$$

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(0, \frac{8}{5}\right)$.

51. (a)



Suppose the region lies between two curves $y = f(x)$ and $y = g(x)$ where $f(x) \geq g(x)$, as illustrated in the figure. Use n subintervals determined by points x_i with $a = x_0 < x_1 < \cdots < x_n = b$ and choose $x_i^* = \overline{x}_i$ to be the midpoint of the ith subinterval; that is, $\overline{x}_i = \frac{1}{2}(x_{i-1} + x_i)$. Then the centroid of the ith approximating rectangle R_i is its center $C_i = \left(\overline{x}_i, \frac{1}{2}[f(\overline{x}_i) + g(\overline{x}_i)]\right)$.

Its area is $[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x$, so its mass is $\rho[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x$.

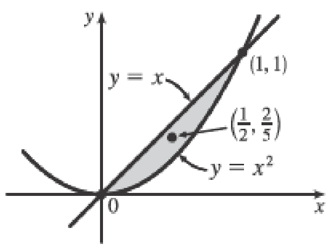
Thus, $M_y(R_i) = \rho[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x \cdot \overline{x}_i = \rho\overline{x}_i[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x$ and

$M_x(R_i) = \rho[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x \cdot \frac{1}{2}[f(\overline{x}_i) + g(\overline{x}_i)] = \rho \cdot \frac{1}{2}\left\{[f(\overline{x}_i)]^2 - [g(\overline{x}_i)]^2\right\}\Delta x$. Summing over i and taking the limit as $n \to \infty$, we get $M_y = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \rho\overline{x}_i[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x = \rho\int_a^b x[f(x) - g(x)]\,dx$ and

$M_x = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \rho \cdot \frac{1}{2}\left[f(\overline{x}_i)^2 - g(\overline{x}_i)^2\right]\Delta x = \rho\int_a^b \frac{1}{2}\left\{[f(x)]^2 - [g(x)]^2\right\}dx$. Thus,

$$\overline{x} = \frac{M_y}{m} = \frac{M_y}{\rho A} = \frac{1}{A}\int_a^b x[f(x) - g(x)]\,dx \text{ and } \overline{y} = \frac{M_x}{m} = \frac{M_x}{\rho A} = \frac{1}{A}\int_a^b \tfrac{1}{2}\left\{[f(x)]^2 - [g(x)]^2\right\}dx.$$

(b)



The region is sketched in the figure. We take $f(x) = x$, $g(x) = x^2$, $a = 0$, and $b = 1$ in the formulas in part (a). First we note that the area of the region is $A = \int_0^1 (x - x^2)\,dx = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 = \frac{1}{6}$.

Therefore, $\overline{x} = \frac{1}{A}\int_0^1 x[f(x) - g(x)]\,dx = \frac{1}{1/6}\int_0^1 x(x - x^2)\,dx = 6\int_0^1 (x^2 - x^3)\,dx = 6\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = \frac{1}{2}$

and $\overline{y} = \frac{1}{A}\int_0^1 \frac{1}{2}\left\{[f(x)]^2 - [g(x)]^2\right\}dx = \frac{1}{1/6}\int_0^1 \frac{1}{2}\left(x^2 - x^4\right)dx = 3\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1 = \frac{2}{5}$.

The centroid is $\left(\frac{1}{2}, \frac{2}{5}\right)$.

5. (a) In general, we must satisfy the two conditions that are mentioned before Example 1—namely, (1) $f(x) \geq 0$ for all x, and (2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$. If $c \geq 0$, then $f(x) \geq 0$, so condition (1) is satisfied. For condition (2), we see that

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{c}{1+x^2}\,dx \text{ and}$$

$$\int_0^{\infty} \frac{c}{1+x^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{c}{1+x^2}\,dx = c \lim_{t\to\infty} \left[\tan^{-1} x\right]_0^t = c \lim_{t\to\infty} \tan^{-1} t = c\left(\frac{\pi}{2}\right)$$

Similarly, $$\int_{-\infty}^{0} \frac{c}{1+x^2}\,dx = c\left(\frac{\pi}{2}\right), \text{ so } \int_{-\infty}^{\infty} \frac{c}{1+x^2}\,dx = 2c\left(\frac{\pi}{2}\right) = c\pi.$$

Since $c\pi$ must equal 1, we must have $c = 1/\pi$ so that f is a probability density function.

(b) $$P(-1 < X < 1) = \int_{-1}^{1} \frac{1/\pi}{1+x^2}\,dx = \frac{2}{\pi}\int_0^1 \frac{1}{1+x^2}\,dx = \frac{2}{\pi}\left[\tan^{-1} x\right]_0^1 = \frac{2}{\pi}\left(\frac{\pi}{4} - 0\right) = \frac{1}{2}$$

10. (a) $\mu = 1000 \quad\Rightarrow\quad f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{1000}e^{-t/1000} & \text{if } t \geq 0 \end{cases}$

(i) $P(0 \leq X \leq 200) = \int_0^{200} \frac{1}{1000}e^{-t/1000}\,dt = \left[-e^{-t/1000}\right]_0^{200} = -e^{-1/5} + 1 \approx 0.181$

(ii) $P(X > 800) = \int_{800}^{\infty} \frac{1}{1000}e^{-t/1000}\,dt = \lim_{x\to\infty} \left[-e^{-t/1000}\right]_{800}^{x} = 0 + e^{-4/5} \approx 0.449$

(b) We need to find m so that $\int_m^{\infty} f(t)\,dt = \frac{1}{2} \quad\Rightarrow\quad \lim_{x\to\infty} \int_m^x \frac{1}{1000}e^{-t/1000}\,dt = \frac{1}{2} \quad\Rightarrow\quad \lim_{x\to\infty} \left[-e^{-t/1000}\right]_m^x = \frac{1}{2} \quad\Rightarrow$

$0 + e^{-m/1000} = \frac{1}{2} \quad\Rightarrow\quad -m/1000 = \ln\frac{1}{2} \quad\Rightarrow\quad m = -1000\ln\frac{1}{2} = 1000\ln 2 \approx 693.1$ h.

15. (a) $P(0 \leq X \leq 100) = \int_0^{100} \frac{1}{8\sqrt{2\pi}} \exp\left(-\frac{(x-112)^2}{2\cdot 8^2}\right) dx \approx 0.0668$ (using a calculator or computer to estimate the integral), so there is about a 6.68% chance that a randomly chosen vehicle is traveling at a legal speed.

(b) $P(X \geq 125) = \int_{125}^{\infty} \frac{1}{8\sqrt{2\pi}} \exp\left(-\frac{(x-112)^2}{2\cdot 8^2}\right) dx = \int_{125}^{\infty} f(x)\,dx$. In this case, we could use a calculator or computer to estimate either $\int_{125}^{300} f(x)\,dx$ or $1 - \int_0^{125} f(x)\,dx$. Both are approximately 0.0521, so about 5.21% of the motorists are targeted.