Solutions: Homework 4

1 Chapter 8.6

Problem 8.11

(a) Let $g(x) = \frac{1}{1+x}$. Then $-g'(x) = f(x)$. The power series for $g(x)$ is $\sum_{n=0}^{\infty}(-1)^n x^n$ with a radius of convergence equal to 1. Therefore, by differentiating the power series of $g(x)$ term-by-term and multiplying by -1, we see that the power series for $f(x)$ is

$$-\sum_{n=0}^{\infty}(-1)^n n x^{n-1} = \sum_{n=0}^{\infty}(-1)^n (n+1) x^n.$$

The power series has radius of convergence equal to 1 since the radius of convergence of a power series does not change when differentiating.

(b) Now notice that if $g(x) = \frac{1}{(1+x)^2}$ then $f(x) = -\frac{1}{2}g'(x)$. By differentiating the power series obtained in part (a) of this problem and then multiplying by $-\frac{1}{2}$ we see that the power series for $f(x)$ is

$$-\frac{1}{2}\sum_{n=0}(-1)^n (n+1) n x^{n-1} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n (n+2)(n+1) x^n.$$

(c) All we need to do is multiply the power series obtained in part (b) by x^2. Therefore, the power series is

$$\frac{x^2}{(1+x)^3} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n (n+2)(n+1) x^{n+2} = \frac{1}{2}\sum_{n=2}^{\infty}(-1)^n (n)(n-1) x^n.$$

Problem 8.13

The function $f(x) = \ln(5-x)$ is an integral of $g(x) = \frac{-1}{5-x}$. Our strategy is to find the power series representation of $g(x)$ and its radius of convergence. Then, we will integrate the power series, compute its constant term for $\ln(5-x)$, and note that the radius of convergence does not change under integration.

Note that $g(x) = \frac{-1}{5}\frac{1}{1-\frac{x}{5}}$. This small algebraic manipulation allows us to substitute $\frac{x}{5}$ into the standard geometric series and see that

$$g(x) = \frac{-1}{5}\sum_{n=0}^{\infty}\frac{x^n}{5^n}$$

with radius of convergence 5. Integrating this power series term by term we see that

$$\ln(5-x) = C + \frac{-1}{5}\sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)5^n}.$$

By substituting $x = 0$, we see that $C = \ln(5)$. The radius of convergence is still 5.

Problem 8.38

(a) Note that the power series $\sum_{n=1}^{\infty} nx^{n-1}$ is the derivative of the power series for $\frac{1}{1-x}$ and is therefore a power series convergent to $\frac{1}{(1-x)^2}$ for $|x| < 1$.

(b) (i) The relation between this power series and the one for $\frac{1}{(1-x)^2}$ is that we need to multiply by one factor of x. Therefore, this power series represents the function $\frac{x}{(1-x)^2}$.

(ii) Since this series is just the specialization of the power series in part (i) by subsituting $x = 1/2$ and because is in the interval of convergence of the power series in part (i), we conclude that it is converges to $\frac{1/2}{(1-1/2)^2} = 2$.

(c) (i) From part (a), we know that $\sum_{n=0}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$ for $|x| < 1$. By differentiating we see that

$$\frac{2}{(1-x)^3} = \sum_{n=0}^{\infty} n(n-1)x^{n-2} = \sum_{n=2}^{\infty} n(n-1)x^{n-2}$$

for $|x| < 1$. After multiplying by $|x|^2$ we see that

$$\frac{2x^2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^n$$

for $|x| < 1$.

(ii) Note that the series $\sum_{n=2}^{\infty} \frac{n^2-n}{2^n}$ is the same as the power series of part (i) evaluated at $x = \frac{1}{2}$. Since $1/2$ is in the interval of convergence of that power series, we conclude that it converges to $\frac{2(1/2)^2}{(1-1/2)^3} = 4$.

(iii) The key idea is to split the given series into a sum of two series as follows (why is this a good idea?):

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \sum_{n=1}^{\infty} \frac{n}{2^n} + \sum_{n=2}^{\infty} \frac{n^2-n}{2^n}.$$

The first series on the right hand side is equal to 2 by the calculation from part (b)[ii] of this problem. The second series on the right hand side is equal to 4 by part (c)[ii] of this problem. Therefore the given series converges to $2 + 4 = 6$.

2 Chapter 8.7

Problem 9

We compute the derivatives of $f(x) = e^{5x}$ using the chain rule and see that $f^{(n)}(x) = 5^n e^{5x}$. Therefore the taylor coefficients expanded around 0 are

$$c_n = \frac{f^{(n)}(0)}{n!} = \frac{5^n}{n!}.$$

The radius of convergence can be computed by the taking the ratio test:

$$\lim_{n\to\infty} \left| \frac{c_{n+1}x^{n+1}}{c_n x^n} \right| = \lim_{n\to\infty} |x| \left| \frac{\frac{5^{n+1}}{(n+1)!}}{\frac{5^n}{n!}} \right| = \lim_{n\to\infty} |x| \left| \frac{5}{n+1} \right| = 0 < 1.$$

Therefore, the radius of convergence of the taylor series is ∞.

Problem 21

We have to apply the binomial series theorem for $k = 1/2$. In other words:

$$\sqrt(1+x) = (1+x)^{1/2} = 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \binom{1/2}{n} x^n$$

for $|x| < 1$. The coefficients $\binom{1/2}{n}$ can, for $n \geq 2$, be written as

$$\frac{(1/2)(-1/2)(-3/2)\cdots(-2n-3/2)}{n!} = (-1)^{n-1}\frac{(1)(3)(5)\cdots(2n-3)}{2^n n!}.$$

Problem 27

The Maclaurin series for e^x is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. The Maclaurin series for e^{2x}, computed by substituting $2x$ in for x, is $\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$. Both power series are convergent everywhere. Therefore,

$$e^x + e^{2x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{2^n x^n}{n!} = \sum_{n=0}^{\infty} \frac{(1+2^n)x^n}{n!}.$$

Problem 64

We recognize that $1 - \ln 2 + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} + \cdots$ is the power series representation of e^x evaluated at $x = -\ln(2)$. The power series for e^x is convergent everywhere to e^x so the series converges to $e^{-\ln(2)} = 1/2$.

3 Chapter 8.8

Problem 14ab

(a) To compute the taylor series centered at $x = \pi/6$ up to 4th order we find the derivatives of $f(x) = \sin(x)$:

$f'(x) = \cos(x),\ f^{(2)}(x) = -\sin(x),\ f^{(3)}(x) = -\cos(x),\ f^{(4)}(x) = \sin(x).$

We can evalute these derivates at $x = \pi/6$ to obtain:

$$f'(\pi/6) = \frac{\sqrt{3}}{2},\ f^{(2)}(x) = -\frac{1}{2},\ f^{(3)}(x) = -\frac{\sqrt{3}}{2},\ f^{(4)}(x) = \frac{1}{2}.$$

The taylor series for $\sin(x)$ centered at $\pi/6$ is therefore:

$$\begin{aligned} T_4(x) &= \frac{1}{2} + \frac{\sqrt{3}}{2}(x-\pi/6) + \frac{-1/2}{2}(x-\pi/6)^2 + \frac{-\frac{\sqrt{3}}{2}}{6}(x-\pi/6)^3 + \frac{1/2}{24}(x-\pi/6)^4 \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2}(x-\pi/6) - \frac{1}{4}(x-\pi/6)^2 - \frac{\sqrt{3}}{12}(x-\pi/6)^3 + \frac{1}{48}(x-\pi/6)^4. \end{aligned}$$

(b) By applying Taylor's theorem with $n = 4$, $a = \pi/6$ and $d = \pi/6$, we know that $|R_4(x)| \leq \frac{M}{5!}|x - \pi/6|^5$ for $0 \leq x \leq \pi/3$ and an appropriate choice of M. The choice of M must satisfy $M \geq |f^{(5)}(x)| = |\cos(x)|$ on the interval $0 \leq x \leq \pi/6$. By choosing $M = 1$, we see that for $0 \leq x \leq \pi/3$,

$$|R_4(x)| \leq \frac{M}{5!}|x - \pi/6|^5 \leq \frac{1}{5!}(\pi/6)^5 \approx 0.000328.$$

Problem 23

The series expansion for $\sin(x)$ starts with $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$. This series is a convergent alternating series when $x > 0$ with decreasing coefficients so we know, by the Alternating Series Estimation Theorem, that the remainder term $|\sin(x) - (x - \frac{x^3}{3!})| \leq \frac{x^5}{5!}$. We use the following string of equivalent inequalities to find the range x for which the error term is less than .01:

$$\left|\frac{x^5}{5!}\right| \leq .01 \iff |x|^5 \leq 1.2 \iff |x| \leq 1.043\ldots$$

Therefore, we can say that the error is less than .01 for $|x| \leq 1.043$. This can be confirmed graphically through any graphing utility.

Problem 26

The taylor series centered at 4 for f is $\sum_{n=0}^{\infty} c_n(x-4)^n$ for $c_n = \frac{f^{(n)}(4)}{n!} = \frac{(-1)^n}{3^n(n+1)}$. Since we are asked to estimate the remainder term $R_5(5)$ we can either use Taylor's inequality or the Alternating Series Remainder term. Since we don't know the derivative of f on the entire interval of convergence we have to use the Alternating Series Estimation Theorem and luckily it is clear that the c_n are alternating. First however, we must compute the radius of convergence for the series:

$$\lim_{n\to\infty}\left|\frac{c_{n+1}(x-4)^{n+1}}{c_n(x-4)^n}\right| = |x-4|\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}}{3^{n+1}(n+2)}}{\frac{(-1)^n}{3^n(n+1)}}\right| = |x-4|\frac{1}{3}.$$

Therefore the radius of convergence is 3 centered at 4 so $x = 5$ is within the radius of convergence. At $x = 5$, the power series is alternating and the coefficients are decreasing in absolute value. Therefore, by the Alternating Series Estimation Theorem, the error $|R_5(5)| \leq |5-4|^6 \frac{1}{3^6 \times 7} = \frac{1}{5103} \approx .000196 < .0002$.