# Solutions:Homework 3

## 1 Chapter 8.3

**Problem 8** Consider the function $f(x) = \frac{1}{\sqrt{x+4}}$. It is a positive, decreasing function, so we may apply the integral test to it. The integral $\int_1^\infty \frac{dx}{\sqrt{x+4}}$ equals $\lim_{b\to\infty} \frac{1}{2}(\sqrt{b+4} - \sqrt{1+4})$, which diverges. Therefore the series $\sum_{n=1} \frac{1}{\sqrt{n+4}}$ diverges.

**Problem 19**

To see that $\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^2+1}$ converges, we will use the comparison test for convergence. Since $0 \leq cos^2(n) \leq 1$ and $n^2+1 \geq n^2$, it follws that $0 \leq \frac{cos^2(n)}{n^2+1} \leq 1/n^2$. Now, since $\sum_{n=1}^{\infty} 1/n^2$ converges by the p-series test for $p = 2$, the original series converges by the comparison test.

**Problem 26**

Again, we will use the comparison test. We have the inequality $0 \leq 1/\sqrt{n^3+1} \leq n^{-3/2}$, for $n \geq 1$. Since $n = 1^\infty 1/n^{3/2}$ converges by the p-series test for $p = 3/2$, the original series converges by the comparison test.

**Problem 39** We want to bound the n-th partial sum $s_n$ of $\sum_{n=1}^{\infty} \frac{1}{n}$ from above. We place a rectangle of height $1/n$ between the points $x = n-1$ and $x = n$. This is the "n-th rectangle". This rectangle lies below the graph of the function $f(x) = \frac{1}{x}$. The area of the 1st through n-th rectangles equals $s_n$, so the area of the 2nd through n-th rectangles equals $s_n - 1$, and is less than the area under the graph of $f(x)$ between $x = 1$ and $x = n$. Therefore, $s_n - 1 \leq \int_1^n \frac{1}{x} dx = \ln(n) - \ln(1) = \ln(n)$.

For part b), we compute $\ln(10)$ to be approximately 2.30, so $1+\ln(10^6) \approx 1 + 6 \times 2.30 = 14.8$, and $1 + \ln(10^9) \approx 1 + 9 \times 2.30 = 21.7$.

## 2 Chapter 8.4

**Problem 9**

We check that $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^2+9}$ satisfies the conditions of the alternating series test for convergence, with $b_n = \frac{n}{n^2+9}$. The sequence $b_n$ is positive, since its numerator and denominator are. The limit $\lim_{n\to\infty}\frac{n}{n^2+9} = 0$, by looking at leading terms. It is decreasing for $n \geq 3$, since the inequality $b_n \geq b_{n+1}$ is equivalent to $n^3 + 2n^2 + 10n \geq n^3 + n^2 + 9n + 9$ (cross-multiply and expand), which reduces to the inequality $n^2 + n \geq 9$, which is true for $n \geq 3$. Since we may rewrite our series as $-1/10 + 2/13 + \sum_{n=3}(-1)^{n+1}\frac{n}{n^2+9}$, the series in this expression converges by the alternating series test, hence our original series converges.

**Problem 13**

If $p \leq 0$, the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^p}$ diverges by the limit test for divergence. Otherwise, if $p > 0$, we may apply the alternating series test with $b_n = 1/n^p$, which is positive, decreasing, and satisfying $\lim_{n\to\infty} 1/n^p = 0$, to see that the series converges.

**Problem 27**

As in problem 13, with $p = 1/2$, the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$ converges. However, the sequence $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges, by the p-series test for $p = 1/2 \leq 1$. Therefore the original series converges, but not absolutely.

**Problem 32**

Using the ratio test, $\lim_{n\to\infty}|a_{n+1}/a_n| = \lim \frac{2^{n+1}(n+1)!}{(2(n+1))!}\cdot\frac{(2n)!}{2^n n!} = \lim \frac{2(n+1)}{(2n+2)(2n+1)} = \lim_{n\to\infty}\frac{1}{2n+1} = 0$ is less than 1, so the series is absolutely convergent.

# 3 Chapter 8.5

**Problem 6**

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{\sqrt{n+1}|x|^{n+1}}{\sqrt{n}|x|^n} = |x| \cdot \lim(1+1/n)^{1/2} = |x|$, so by the ratio test, the series converges for $|x| < 1$, diverges for $|x| > 1$, and could either converge or diverge for $|x| = 1$.

However, by the limit test for divergence, the series diverges at $x = 1$ and $x = -1$, since the limits $\lim_{n\to\infty}\sqrt{n}(1)^n$ and $\lim_{n\to\infty}\sqrt{n}(-1)^n$ are both divergent.

**Problem 7**

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{n!|x|^{n+1}}{(n+1)!|x|^n} = |x| \cdot \lim \frac{1}{n+1} = 0$. By the ratio test, the series converges for all values of $x$.

**Problem 11**

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{2^{n+1}|x|^{n+1}}{(n+1)^{1/4}} \cdot \frac{n^{1/4}}{2^n|x|^n} = 2|x|\lim(\frac{n}{n+1})^{1/4} = 2|x|$. The ratio test says that the series diverges if $2|x| > 1$, converges if $2|x| < 1$, and is inconclusive for $2|x| = 1$.

When $x = 1/2$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/4}}$, which converges by the alternating series test, as in problem 13 of section 8.5.

When $x = -1/2$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{n^{1/4}}$, which diverges by the p-series test with $p = 1/4$.

Therefore the interval of convergence is the half-open interval $(-1/2, 1/2]$.

**Problem 23**

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{1\cdot 3 \ldots (2n-1)|x|^{n+1}}{1\cdot 3 \ldots (2n-1)(2(n+1)-1)|x|^n} = |x| \lim \frac{1}{2n+1} = 0$, so by the ratio test, the series converges for every value of $x$.