Solutions:Homework 2

1 Chapter 4.5

Problem 25

First note that $\lim_{x\to 0}\cos(x) - 1 + \frac{1}{2}x^2 = 0$ and $\lim_{x\to 0} x^4 = 0$. Therefore we can apply l'hopital's rule and find that

$$\lim_{x\to 0}\frac{\cos(x) - 1 + \frac{1}{2}x^2}{x^4} = \lim_{x\to 0}\frac{-\sin(x) + x}{4x^3}.$$

Now we observe that once again the limit of the numerator and denominator is zero as $x \to 0$ so we can apply l'hopital's rule again to find that

$$\lim_{x\to 0}\frac{-\sin(x) + x}{4x^3} = \lim_{x\to 0}\frac{-\cos(x) + 1}{12x^2}.$$

Once again we find ourselves in a position to apply l'hopital's rule to find that

$$\lim_{x\to 0}\frac{-\cos(x) + 1}{12x^2} = \lim_{x\to 0}\frac{\sin(x)}{24x}.$$

The final l'hopital's rule yields the result of $\lim_{x\to 0}\frac{\cos(x)}{24} = \frac{1}{24}$ which is the final answer.

2 Chapter 5.10

Problem 31

Since $\frac{e^x}{e^x-1}$ has a discontinuity at $x = 0$ and is continuous everywhere else we classify our integral as an improper integral of type 2. Therefore, by the definition of an improper integral

$$\int_{-1}^{1}\frac{e^x}{e^x - 1}dx = \lim_{t\to 0^+}\int_{-1}^{t}\frac{e^x}{e^x - 1}dx + \lim_{t\to 0^-}\int_{t}^{1}\frac{e^x}{e^x - 1}dx.$$

In both integrals, we make the u-substitution $u = e^x - 1$ and $du = e^x dx$. So that

$$\int_{-1}^{1} \frac{e^x}{e^x - 1} dx = \lim_{t \to 0^+} \int_{e^{-1}-1}^{e^t - 1} \frac{1}{u} du + \lim_{t \to 0^-} \int_{e^t - 1}^{e-1} \frac{1}{u} du.$$

The antiderivative of $1/u$ is $\ln|u|$ so evaluating the first limit and integral in the above equation we have

$$\lim_{t \to 0^+} \int_{e^{-1}-1}^{e^t - 1} \frac{1}{u} du = \lim_{t \to 0^+} \ln|u| \Big|_{e^{-1}-1}^{e^t - 1} = \lim_{t \to 0^+} \ln|e^t - 1| - \ln|1/e - 1|$$

. Since $e^t \to 0$ as $t \to 0^+$, $\ln|e^t - 1| \to -\infty$ as $t \to 0^+$ so we conclude that the imroper integral **diverges**.

Problem 39

By the area interpretation of the integral, we are asked to compute the definite integral:

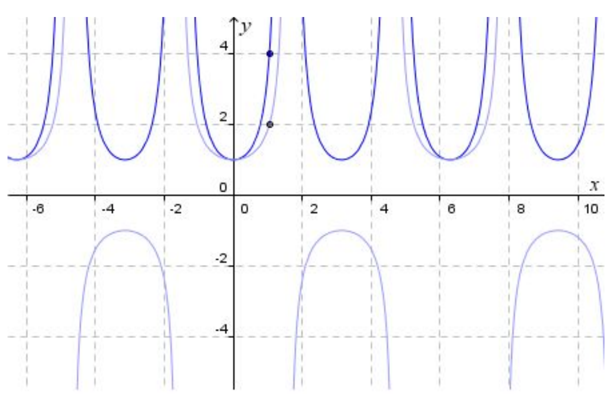
$$\int_0^{\pi/2} \sec^2(x) dx.$$

An antiderivative of $\sec^2(x)$ is $\tan(x)$ and $\sec^2(x)$ has a discontinuity at $\pi/2$ so we can evaluate this type 2 improper integral by

$$\int_0^{\pi/2} \sec^2(x) dx = \lim_{t \to \pi/2^-} \int_0^t \sec^2(x) dx = \lim_{t \to \pi/2^-} \tan(x)|_0^t = \lim_{t \to \pi/2^-} \tan(t) = \infty.$$

We conclude that this integral does not exist and the area is infinite.

A good sketch would look roughly like the one below with the area below the curve between $0 \leq x \leq \pi/2$.



Problem 44

We might guess that this integral is divergent. Therefore we pick the function $0 \leq 1/x \leq \frac{2+e^{-x}}{x}$ on the domain $x \geq 1$ to compare to. Since $\int_1^\infty \frac{1}{x}dx$ is divergent by the calculation done in section 5.10 of the book, we conclude by the comparison test that the integral in the problem also diverges.

Problem 46

Notice that $0 \leq \arctan(x) \leq \pi/2$ on the domain $x \geq 0$. The e^x in the denominator with a bounded numerator suggests that the improper integral is convergent. We therefore compare to the function $\frac{pi/2}{e^x} \geq \frac{\arctan(x)}{2+e^x}$ on $x \geq 0$. Now by the definition of a type 1 improper integral we have:

$$\int_{0\infty} \frac{pi/2}{e^x}dx = \lim_{t\to\infty}\int_0^t \frac{pi/2}{e^x}dx = \pi/2 \lim_{t\to\infty} -e^{-x}|_0^t = \pi/2 \lim_{t\to\infty} 1 - e^{-t} = \pi/2.$$

We conclude by the comparison test that the integral in our problem is also convergent.

Problem 55

1. For part (a) the problem asks us to make a rough sketch of $F(t)$. Any increasing function starting $F(0) = 0$, with a horizontal asymptote at $y = 1$, and centered roughly around $x = 700$ would be acceptable.

2. For part (b), the derivative $r(t)$ of the fraction $F(t)$ is the probability density of the chance that a lightbulb would burn out at time t.

3. For part (c), the integral of $r(t)$ from time 0 to ∞ must be exactly 1 since
$$\int_0^\infty r(t)dt = \lim_{T\to\infty}\int_0^T r(t)dt = \lim_{T\to\infty} F(T) - F(0).$$
We know that $F(0) = 0$ since at time 0 no light bulbs have burned out. However as $T \to \infty$, $F(T) = 1$ since eventually all lightbulbs burn out.

3 Chapter 8.1

Problem 16

We should be able to recognize that this sequence $a_n = 9 \times \left(\frac{3}{5}\right)^n$ is a geometric sequence. Notice that the function $f(x) = 9 \times \left(\frac{3}{5}\right)^x$ approaches zero as $x \to \infty$. Since $a_n = f(n)$, Theorem 2 of section 8.1 of our textbook implies that $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = 0$.

Problem 18

By the last limit law for sequences on page 557 of our textbook applied with $p = .5$ we know that

$$\lim_{n\to\infty}\sqrt{\frac{n+1}{9n+1}} = \sqrt{\lim_{n\to\infty}\frac{n+1}{9n+1}}.$$

By a bit of algebra, we see that $\frac{n+1}{9n+1} = \frac{1}{9}\frac{9n+9}{9n+1} = \frac{1}{9}(1+\frac{8}{9n+1})$. Therefore by the limit laws of sequences on page 557 of our textbook the sequence converges to

$$\sqrt{\lim_{n\to\infty}\frac{n+1}{9n+1}} = \sqrt{\lim_{n\to\infty}\frac{1}{9}(1+\frac{8}{9n+1}))} = \sqrt{1/9} = 1/3.$$

Problem 29

Note that $\frac{(2n-1)!}{(2n+1)!} = \frac{(2n-1)(2n-2)\cdots 1}{(2n+1)(2n)(2n-1)(2n-1)\cdots 1} = \frac{1}{(2n+1)(2n)}$. Clearly, $\lim_{n\to\infty}\frac{1}{(2n)(2n+1)} = 0$ so the sequence converges to zero.

Problem 34

It suffices to show $\left|\frac{(-3)^n}{n!}\right| \to 0$ by the squeeze theorem. Notice that

$$\frac{3^n}{n!} = \frac{3}{n}\frac{3}{n-1}\frac{3}{n-2}\cdots\frac{3}{3}\frac{3}{2}\frac{3}{1} \leq \frac{3}{n} * \frac{9}{2}.$$

Clearly, $\frac{3}{n} * \frac{9}{2} \to 0$ so the sequence converges to zero by the squeeze theorem.

Problem 49

The sequence $a_n = \frac{1}{2n+3}$ is decreasing. To show this we have to check that $a_{n+1} < a_n$ or equivalently $\frac{1}{2n+5} < \frac{1}{2n+3}$. This follows because by cross multiplying we can verify that $2n + 5 > 2n + 3$.

In addition, we know that the sequence $a_n > 0$ is positive and it is bounded above by the first term in the sequence since it is decreasing. Therefore we conldue that a_n is decreasing and bounded.

4 Chapter 8.2

Problem 9

1. Part a: We compute $\lim_{n\to\infty}\frac{2n}{3n+1} = \lim_{n\to\infty}\frac{2}{3+1/n} = 2/3$ so the sequence a_n converges.

2. Part b: By the divergence test we see that the series $\sum a_n$ must diverge.

Problem 12

Notice that this series is just a geometric series $a + ar + ar^2 + \ldots$ for $a = 4$ and $r = 3/4$. Since $|r| < 1$, the series converges to $\frac{a}{1-r} = 16$.

Problem 20

We apply the divergence test and note that $\lim_{k\to\infty} \frac{k^2+2k}{k^2+6k+9} = 1$ so the series diverges.

Problem 23 Notice that $\sum_{n=1}^{\infty} \frac{1}{3^n} = 1/2$ because it is a geometric series with $a = 1/3$ and $r = 1/3$. Also, $\sum_{n=1}^{\infty} \frac{2^n}{3^n} = 2$ because it is a geometric series with $a = 2/3$ and $r = 2/3$. The series in the problem is the sum of these two convergent series so it is also convergent to $2 + 1/2 = 5/2$.

Problem 41

The series $\sum_{n=1}^{\infty} \frac{x^n}{3^n}$ is a geometric series with starting value $a = x/3$ and common ratio $r = x/3$. Therefore, it converges if and only if $|x| < 3$. In that case, it converges, by the formula for the sum of convergent geometric series, to $\frac{x/3}{1-x/3} = \frac{x}{3-x}$.