Math 19: Fall 2014
Midterm 2
~~Green~~ versions
Both

NAME:

LECTURE: SOLUTIONS

Time: **75 minutes**

For each problem, you should write down all of your work carefully and legibly to receive full credit. When asked to justify your answer, you should use theorems and/or mathematical reasoning to support your answer, as appropriate.

Failure to follow these instructions will constitute a breach of the Stanford Honor Code:

- You may not use a calculator or any notes or book during the exam.
- You may not access your cell phone during the exam for any reason.
- You are required to sit in your assigned seat.
- You are bound by the Stanford Honor Code, which stipulates among other things that you may not communicate with anyone other than the instructor during the exam, or look at anyone else's solutions.

I understand and accept these instructions.

Signature: ______________________________

Problem	Value	Score
1	49	
2	10	
3	10	
4	15	
5	16	
TOTAL	100	

Problem 1 : (49 points) For each of the following, compute $\frac{dy}{dx}$.
There is no need to give the domain and codomain for the rule you obtain. You do not need to simplify your answer. Whenever necessary, solve for $\frac{dy}{dx}$.

a) $y = e^{x^2}$

$$\frac{dy}{dx} = e^{x^2} \cdot 2x$$

b) $y = \frac{x^3}{x^2 - 12}$

$$\frac{dy}{dx} = \frac{3x^2(x^2-12) - x^3(2x)}{(x^2-12)}$$

c) $y = e^{x^2} \sin(5x)$

$$\frac{dy}{dx} = e^{x^2} \cdot 2x \sin(5x) + e^{x^2} \cos(5x) \cdot 5$$

d) $y = \sqrt{1 + (x^2 + 4)^6}$

$$\frac{dy}{dx} = \frac{1}{2}\left(1 + (x^2 + 4)^6\right)^{-\frac{1}{2}} \left(6(x^2+4)^5\right) \cdot 2x$$

e) $y = \left(\dfrac{x+2}{1-x}\right)^{-3}$

$$\frac{dy}{dx} = -3\left(\frac{x+2}{1-x}\right)^{-4}\left(\frac{(1-x)-(x+2)(-1)}{(1-x)^2}\right)$$

f) $y = x^2y^3 + x^3y^2$

$$\frac{dy}{dx} = 2x\,y^3 + x^2\,3y^2\frac{dy}{dx} + 3x^2y^2 + x^3\,2y\,\frac{dy}{dx}$$

$$\left(1-3x^2y^2-2x^3y\right)\frac{dy}{dx} = 2xy^3 + 3x^2y^2$$

$$\frac{dy}{dx} = \frac{2xy^3+3x^2y^2}{1-3x^2y^2-2x^3y}$$

g) $(x-y)^2 = x+y-1$

$$2(x-y)\left(1-\frac{dy}{dx}\right) = 1+\frac{dy}{dx}$$

$$2(x-y) - 2(x-y)\frac{dy}{dx} = 1+\frac{dy}{dx}$$

$$2(x-y)-1 = \left(2(x-y)+1\right)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2(x-y)-1}{2(x-y)+1}$$

Problem 2 : (10 points) Consider the function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$. Use the limit definition of derivative to find the derivative of f with respect to x. You do not need to give a domain or a codomain.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{h\,(3x^2 + 3xh + h^2)}{h}$$

$$= \lim_{h \to 0} (3x^2 + 3xh + h^2) = 3x^2$$

Problem 3 : (10 points) Consider the function $f: \mathbb{R} \to \mathbb{R}$ given by the rule

$$f(x) = \begin{cases} 6x - 13 & \text{if } x \leq 3 \\ x^2 - 4 & \text{if } x > 3. \end{cases}$$

$$f(3) = 6 \cdot 3 - 13 = 18 - 13 = 5$$

a) (5 points) Is f differentiable at $x = 3$? Justify.

$$\lim_{h \to 0^+} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0^+} \frac{(3+h)^2 - 4 - 5}{h}$$

$$= \lim_{h \to 0^+} \frac{9 + 6h + h^2 - 9}{h} = \lim_{h \to 0^+} \frac{h(6+h)}{h}$$

$$= \lim_{h \to 0^+} (6+h) = 6$$

$$\lim_{h \to 0^-} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0^-} \frac{6(3+h) - 13 - 5}{h}$$

$$= \lim_{h \to 0^-} \frac{18 + 6h - 18}{h} = \lim_{h \to 0^-} \frac{6h}{h} = \lim_{h \to 0^-} 6 = 6$$

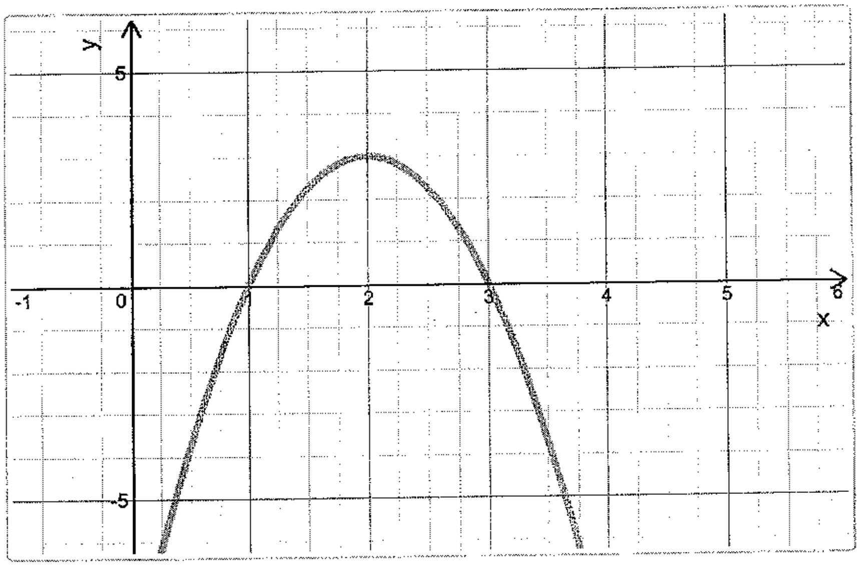
$f'(3) = \lim\limits_{h \to 0} \dfrac{f(3+h) - f(3)}{h}$ exists so f is differentiable at $x = 3$

b) (5 points) Is f continuous at $x = 3$? Justify.

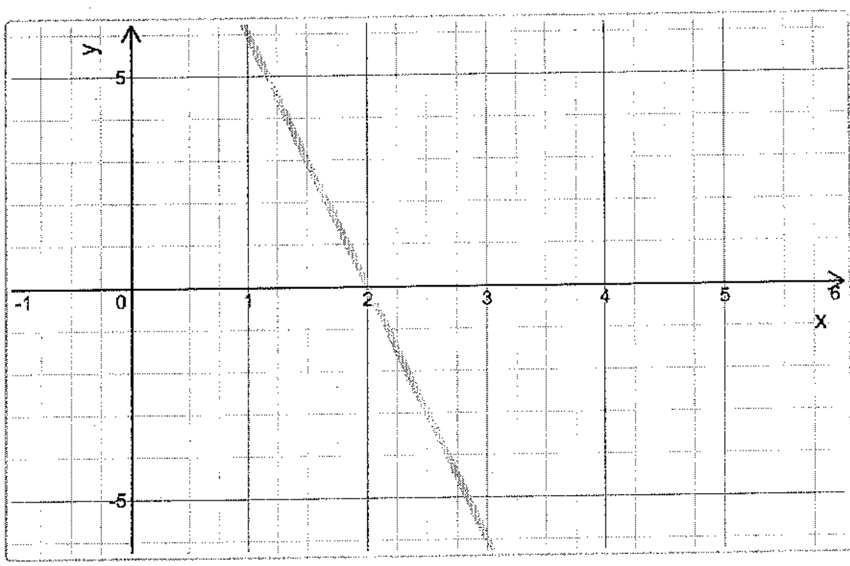
yes. since f is differentiable at $x=3$, it must be continuous at $X=3$.

Problem 4 : (15 points) Consider a function f whose domain is all real numbers.

This is the graph of f':



This is the graph of f'':



a) (5 points) What are the intervals of increase and decrease of f? Please write your answer using interval notation.

increase: $(1, 3)$

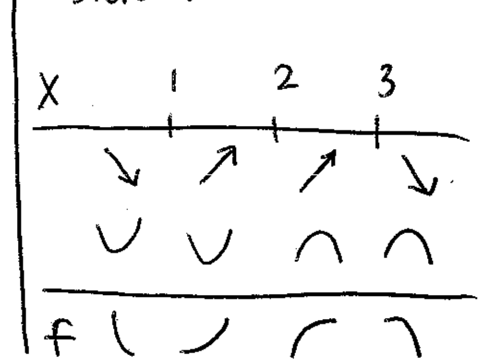
decrease: $(-\infty, 1) \cup (3, \infty)$

b) (4 points) What are the intervals on which f is concave up? What are the intervals on which f is concave down? Please write your answer in interval notation.

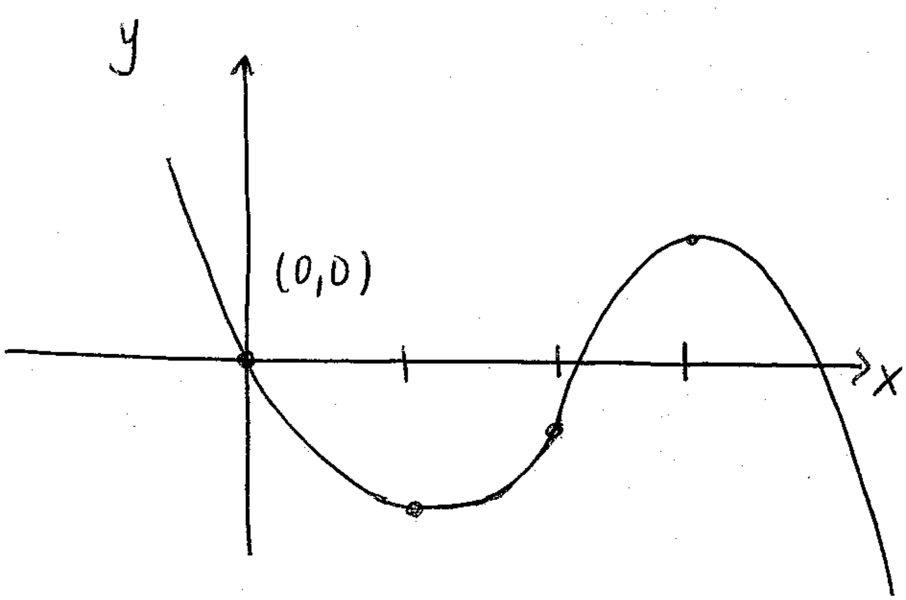
concave up: $(-\infty, 2)$

concave down: $(2, \infty)$

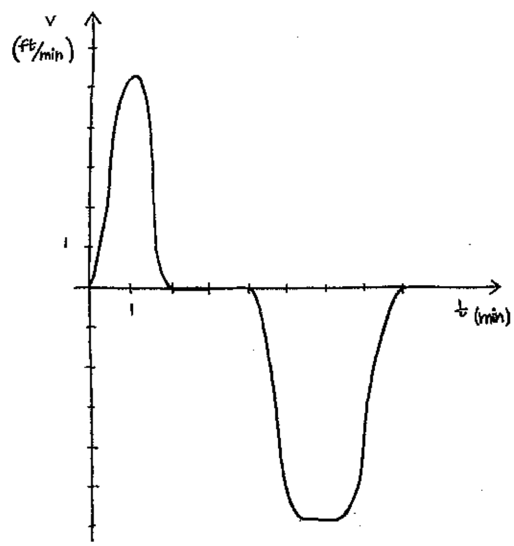
Side work:



c) (6 points) If $f(0) = 0$, sketch a plausible graph for f.

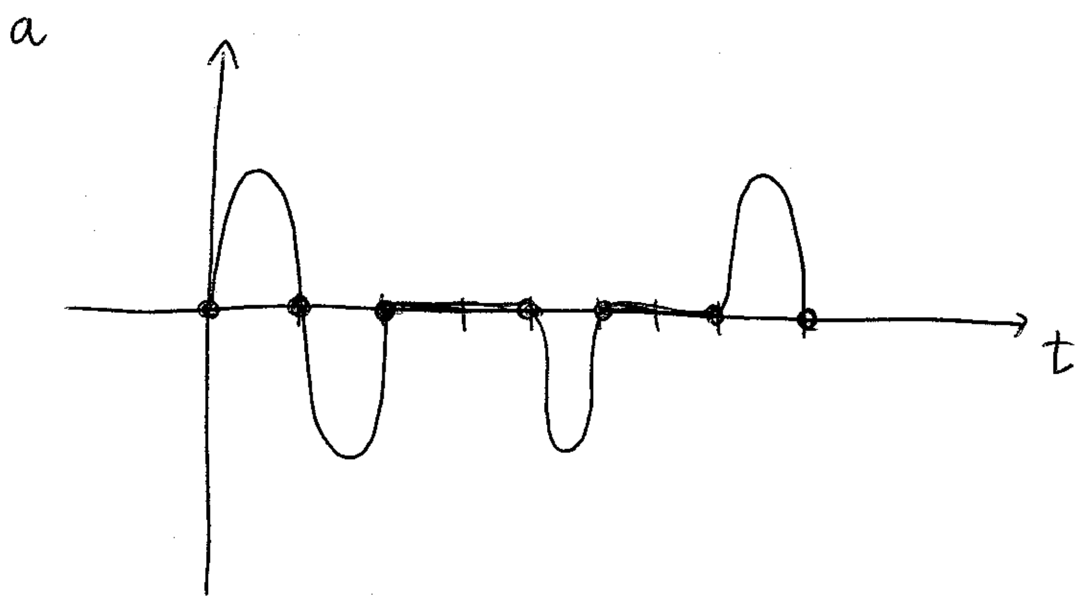


Problem 5 : (16 points) The graph below shows the velocity of an elevator as a function of time. This elevator is in a building with only three floors: the first floor, the second floor, and the third floor. The velocity is given in feet per minute and the time is given in minutes.

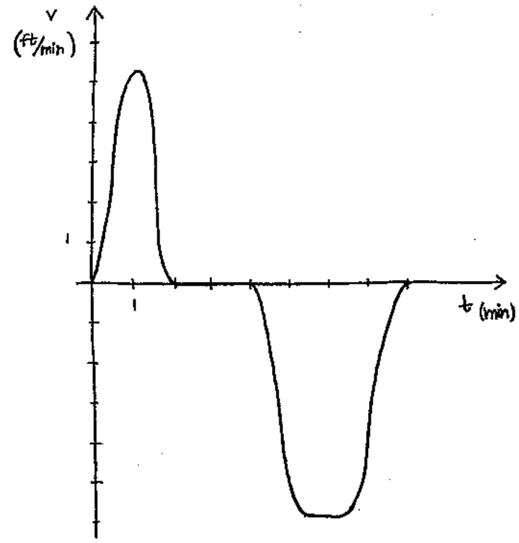


The graph begins at time $t = 0$ when the elevator doors closed on the second floor of the building, and ends at $t = 8$ when the doors opened at a different floor.

a) (5 points) Sketch a plausible graph for the acceleration of the elevator as a function of time.

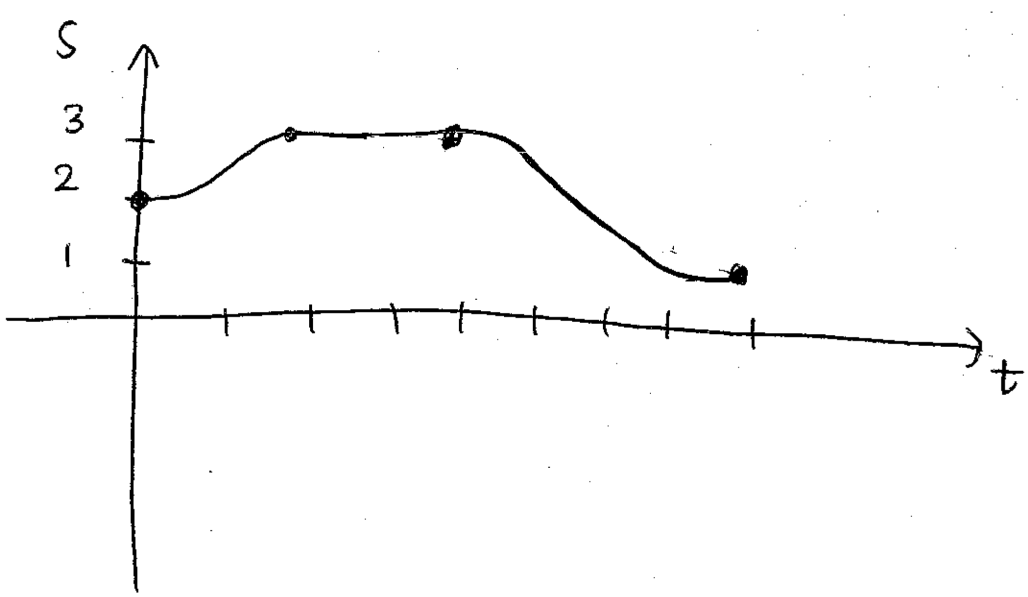


For your convenience, here is the graph of the velocity of the elevator again:



Recall that the graph begins at time $t = 0$ when the elevator doors closed on the second floor of the building, and ends at $t = 8$ when the doors opened at a different floor.

b) (5 points) Sketch a plausible graph for the position of the elevator as a function of time.



c) (2 points) What is the physical significance of a negative velocity in this problem? In other words, what is going on with the elevator when it has negative velocity?

The elevator is going down.

d) (2 points) What floor was the elevator at after 8 minutes?

floor 1

e) (2 points) What is the physical significance of a negative acceleration in this problem? In other words, what is going on with the elevator when it has negative acceleration?

It is either going up and slowly down
or going down and speeding up.