Math 19: Fall 2013
Midterm 2

NAME: SOLUTIONS

LECTURE:

Time: **75 minutes**

This is a closed book and closed notes exam. Calculators and any electronic aid are not allowed.

For each problem, you should write down all of your work carefully and legibly to receive full credit. When asked to justify your answer, you should use theorems and/or mathematical reasoning to support your answer, as appropriate.

I understand and accept the provisions of the Stanford Honor Code.

Signature: ______________________________

Problem	Value	Score
1	20	
2	20	
3	7	
4	4	
5	8	
6	8	
7	7	
8	26	
TOTAL	100	

Limit Laws

Throughout, let a and c be real numbers, and suppose that both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist.

1. $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$
4. $\lim_{x\to a}(f(x)\cdot g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided that $\lim_{x\to a} g(x) \neq 0$
6. $\lim_{x\to a}(f(x))^n = (\lim_{x\to a} f(x))^n$, for n a positive integer
7. $\lim_{x\to a} c = c$
8. $\lim_{x\to a} x = a$
9. $\lim_{x\to a} x^n = a^n$, for n a positive integer
10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$, for n a positive integer, and provided that $a \geq 0$ if n is even
11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, for n a positive integer, and provided that $\lim_{x\to a} f(x) \geq 0$ if n is even

Problem 1 : (20 points) Evaluate the following limits, if they exist. Justify your work with limit rules or theorems.

a) **[Section 2.3 # 15]** $\lim_{h\to 0} \frac{(4+h)^2-16}{h}$ "$\frac{0}{0}$"

$$= \lim_{h\to 0} \frac{16+8h+h^2-16}{h}$$

$$= \lim_{h\to 0} \frac{h(8+h)}{h}$$

$$= \lim_{h\to 0} (8+h) = 8$$

by Rules 1, 7 and 8

b) $\lim_{x\to\infty} \frac{2+e^x}{6-e^x}$ "$\frac{\infty}{\infty}$"

$$= \lim_{x\to\infty} \frac{e^x\left(\frac{2}{e^x}+1\right)}{e^x\left(\frac{6}{e^x}-1\right)}$$

$$= \lim_{x\to\infty} \frac{\frac{2}{e^x}+1}{\frac{6}{e^x}-1} = \frac{0+1}{0-1} = -1$$

by Rules 5, 1, 7

c) $\lim_{x \to 0} x \cos\left(\frac{1}{x}\right)$

Squeeze Theorem:

$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1$

if $x > 0$, $-x \leq x\cos\left(\frac{1}{x}\right) \leq x$

if $x < 0$, $-x \geq x\cos\left(\frac{1}{x}\right) \geq x$

$\lim_{x \to 0} x = 0$ Rule 8

$\lim_{x \to 0} -x = 0$ Rules 3,8

Therefore by the Squeeze Theorem

$\lim_{x \to 0} x\cos\left(\frac{1}{x}\right) = 0$

d) $\lim_{x \to \infty} \left(\sqrt{9x^2 + x} - 3x\right)$ "$\infty - \infty$"

$$= \lim_{x \to \infty} \left(\sqrt{9x^2+x} - 3x\right) \frac{\left(\sqrt{9x^2+x} + 3x\right)}{\sqrt{9x^2+x} + 3x}$$

$$= \lim_{x \to \infty} \frac{9x^2 + x - 9x^2}{\sqrt{9x^2+x} + 3x}$$

$$= \lim_{x \to \infty} \frac{\cancel{x}}{\sqrt{x^2\left(9+\frac{1}{x}\right)} + 3x}$$

$$= \lim_{x \to \infty} \frac{x}{x\sqrt{9+\frac{1}{x}} + 3x} = \lim_{x \to \infty} \frac{1}{\sqrt{9+\frac{1}{x}} + 3} = \frac{1}{6}$$

by Rules 5,7,11,1

Problem 2 : (20 points) For each of the following expressions, compute the derivative with respect to x. There is no need to give the domain and codomain for the rule you obtain. You do not need to simplify your answer.

a) $f(x) = \dfrac{x-2}{3-x^5}$

$$\frac{df}{dx} = \frac{1 \cdot (3-x^5) - (x-2)(-5x^4)}{(3-x^5)^2}$$

b) [Section 3.1 # 22] $f(x) = ae^x + \dfrac{b}{x} + \dfrac{c}{x^2} = ae^x + bx^{-1} + cx^{-2}$

$$\frac{df}{dx} = ae^x + b(-1)x^{-2} + c(-2)x^{-3}$$

c) $f(x) = \sqrt{4 + \sqrt{x^2 + 9}}$

$$\frac{df}{dx} = \frac{1}{2}\left(4 + \sqrt{x^2+9}\right)^{-\frac{1}{2}} \left(\tfrac{1}{2}(x^2+9)^{-\frac{1}{2}}\right) \cdot 2x$$

d) $f(x) = \frac{\sec x}{1 + \sec x}$ $= \dfrac{\frac{1}{\cos x}}{1 + \frac{1}{\cos x}} \cdot \dfrac{\cos x}{\cos x} = \dfrac{1}{\cos x + 1} = (\cos x + 1)^{-1}$

$$\frac{df}{dx} = -1\,(\cos x + 1)^{-2} \cdot (-\sin x)$$

Alternatively, $\frac{d}{dx} \sec x = -1(\cos x)^{-2}(-\sin x) = \tan x \sec x$

$$\frac{df}{dx} = \frac{\tan x \sec x\,(1+\sec x) - \sec x\,(\tan x \sec x)}{(1+\sec x)^2}$$

Problem 3 : (7 points)

a) (3 points) Write the three-step definition for the following statement: The function f is continuous at $x = a$.

1- $f(a)$ exists

2- $\lim_{x \to a} f(x)$ exists

3- $f(a) = \lim_{x \to a} f(x)$

b) (4 points) Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by the rule

$$f(x) = \begin{cases} 2 & \text{if } x \leq 0, \\ x^2 + 3x - 1 & \text{if } 0 < x < 1, \\ 3 & \text{if } x = 1, \\ \sqrt{x+8} & \text{if } x > 1. \end{cases}$$

Use the definition in part a) to determine if f is continuous at $x = 0$.

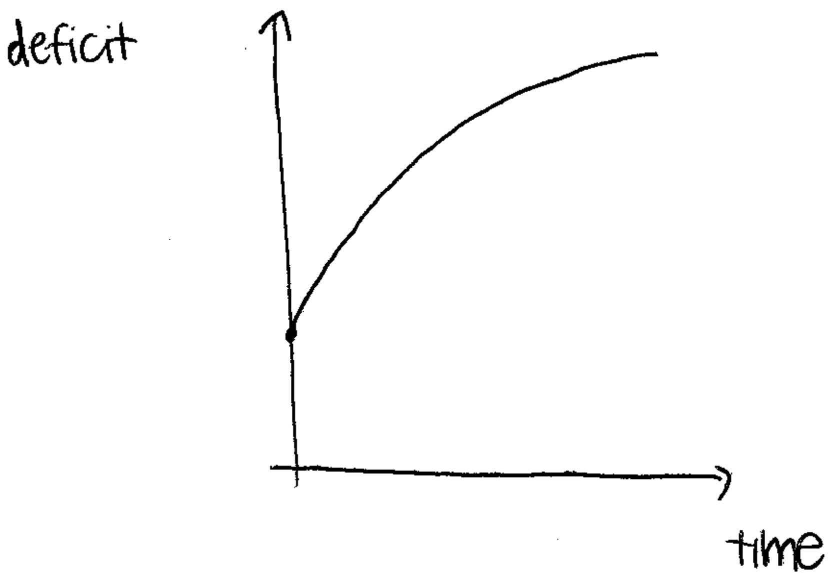
$f(0) = 2$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 2 = 2$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x^2 + 3x - 1) = -1$

$\lim_{x \to 0} f(x)$ does not exist and f is not continuous at $x=0$.

Problem 4 : (4 points) **[Section 2.8 # 9]** The President announces that the national deficit is increasing, but at a decreasing rate. Sketch the graph of a possible function f which could give the national deficit as a function of time.



Recall that
$$\lim_{h\to 0}\frac{\sin h}{h}=1 \qquad \text{and} \qquad \lim_{h\to 0}\frac{\cos h-1}{h}=0.$$

Problem 5 : (8 points) Consider the function $f\colon \mathbb{R}\to\mathbb{R}$, $f(x)=\cos(2x)$.

a) (5 points) Use the limit definition of derivative to find the derivative of f with respect to x.

$$\frac{d}{dx}\cos 2x = \lim_{h\to 0}\frac{\cos(2(x+h))-\cos 2x}{h}$$

$$=\lim_{h\to 0}\frac{\cos 2x\cos 2h-\sin 2x\sin 2h-\cos 2x}{h}$$

$$=\lim_{h\to 0}\cos 2x\left(\frac{\cos 2h-1}{h}\right)-\lim_{h\to 0}\frac{\sin 2x\sin 2h}{h}$$

$$=\cos 2x\lim_{h\to 0}\frac{2(\cos 2h-1)}{2h}$$

$$-\sin 2x\lim_{h\to 0}\frac{2\sin 2h}{2h}$$

$$=\cos 2x\cdot 2\cdot 0-\sin 2x\cdot 2\cdot 1$$

$$=-2\sin 2x$$

b) (3 points) Use the derivative rules to find the derivative of f with respect to x.

$$\frac{df}{dx}=-\sin 2x\cdot 2=-2\sin 2x$$

Problem 6 : (8 points) Doctor Campisi and Doctor Vincent are having a race! Doctor Campisi's position function is

$$s_1\colon [0,30] \to \mathbb{R} \qquad s_1(t) = 528t,$$

and Doctor Vincent's position function is

$$s_2\colon [0,30] \to \mathbb{R} \qquad s_2(t) = \frac{t^3}{3} - 8t^2 + 264t.$$

In this problem, time is given in units of minutes, and distance is given in feet, so that velocity is given in feet per minute.

a) (2 points) What is Doctor Campisi's velocity **function**?

$v_1 : [0,30] \to \mathbb{R}$

$v_1(t) = 528$

b) (2 points) What is Doctor Vincent's velocity **function**?

$v_2 : [0,30] \to \mathbb{R}$

$v_2(t) = t^2 - 16t + 264$

c) (4 points) Show that there is a time at which Doctor Campisi and Doctor Vincent have exactly the same velocity.

Show $t^2 - 16t + 264 = 528$ has a solution between 0 and 30:

- v_2 is continuous
- $v_2(0) = 264$

  $v_2(30) = (30)^2 - 16(30) + 264 = 900 - 480 + 264 = 684$
- By the Intermediate Value Theorem there is t between 0 and 30 such that $v_2(t) = 528$.

Problem 7 : (7 points)

a) (3 points) Write down the definition for the following statement: The function f is differentiable at $x = a$.

The limit $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

exists.

b) (4 points) Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by the rule

$$f(x) = \begin{cases} x^2 + 2x + 3 & \text{if } x \leq 1, \\ 4x + 2 & \text{if } x > 1. \end{cases}$$

Is f differentiable at $x = 1$?

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{(1+h)^2 + 2(1+h) + 3 - 6}{h}$$

$$= \lim_{h \to 0^-} \frac{1 + 2h + h^2 + 2 + 2h + 3 - 6}{h}$$

$$= \lim_{h \to 0^-} \frac{h(4+h)}{h} = \lim_{h \to 0^-} (4+h) = 4$$

$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{4(1+h) + 2 - 6}{h}$$

$$= \lim_{h \to 0^+} \frac{4 + 4h + 2 - 6}{h} = \lim_{h \to 0^+} 4 = 4$$

$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$ exists so f is differentiable at $x=1$.

Problem 8 : (26 points) [HW 5 # 1 b)]
Consider the function given by the rule

$$f(x) = \frac{x^2 + x - 6}{(x+1)(x-2)}.$$

a) (2 points) What is the domain of f?

all real numbers except $x = -1, x = 2$

b) (2 points) List all x-intercepts of f, if any.

$0 = \frac{x^2+x-6}{(x+1)(x-2)}$ when $x^2+x-6=0$ & x is in the domain.

$x^2+x-6=(x+3)(x-2)=0$

we reject $x=2$ (not in domain)

and the x-intercept is $(-3,0)$

c) (2 points) List all y-intercepts of f, if any.

$f(0) = \frac{-6}{1\cdot(-2)} = 3$ $(0,3)$

d) (3 points) Find all vertical asymptotes of f, if any. For each vertical asymptotes, use limits to prove that you have truly found a vertical asymptote, and to investigate the behavior of f on either side of the asymptote.

There is a vertical asymptote with equation $x=-1$:

$$\lim_{x\to -1^-} \frac{x^2+x-6}{(x+1)(x-2)} = \frac{-4}{0^-(-3)} = -\infty$$

$$\lim_{x\to -1^+} \frac{x^2+x-6}{(x+1)(x-2)} = \frac{-4}{0^+(-3)} = +\infty$$

e) (3 points) Find all horizontal asymptotes of f, if any. Use limits to show your work.

$$\lim_{x\to\infty} \frac{x^2+x-6}{(x+1)(x-2)} = \lim_{x\to\infty} \frac{x^2(1+\frac{1}{x}-\frac{6}{x^2})}{x(1+\frac{1}{x})x(1-\frac{2}{x})}$$

$$= \lim_{x\to\infty} \frac{1+\frac{1}{x}-\frac{6}{x^2}}{(1+\frac{1}{x})(1-\frac{2}{x})} = 1$$

$$\lim_{x\to -\infty} \frac{x^2+x-6}{(x+1)(x-2)} = \lim_{x\to -\infty} \frac{1+\frac{1}{x}-\frac{6}{x^2}}{(1+\frac{1}{x})(1-\frac{2}{x})} = 1$$

horizontal asymptote $y=1$ on both sides.

f) (3 points) List all removable discontinuities of f, if any. Use limits to compute the location of the hole in the graph.

Removable discontinuity when $x=2$

$$\lim_{x\to 2} \frac{x^2+x-6}{(x+1)(x-2)} = \lim_{x\to 2} \frac{(x+3)(x-2)}{(x+1)(x-2)}$$

$$= \lim_{x\to 2} \frac{x+3}{x+1} = \frac{5}{3}$$

location of hole: $\left(2, \frac{5}{3}\right)$

g) (3 points) List all jump discontinuities of f, if any. Use limits to investigate the behavior of f at these jumps.

None

h) Copy the information you have found about f here. Please write "none" if you did not find any.

- Domain:

  $x \neq -1, 2$

- x-intercept(s):

  $(-3, 0)$

- y-intercept(s):

  $(0, 3)$

- Equations of the vertical asymptotes:

  $x = -1$

- Equations of the horizontal asymptotes:

  $y = 1$

- Location of removable discontinuities:

  $(2, \frac{5}{3})$

- Location of jump discontinuities:

  none

In addition, Doctor Campisi and Doctor Vincent have found the following information for you:

- f is decreasing everywhere on its domain.
- On its domain, f is concave down when $x < -1$, and f is concave up when $-1 < x$.

i) (8 points) With the information listed on the previous page, graph f. You will be graded on how well your graph satisfies all of the conditions you have written on the previous page. In particular, if you did not copy your information you will lose points.

Label all asymptotes with their equation, and label all special points (intercepts, holes, maxima, minima, points of inflection) with their coordinates.