Midterm 2

Tuesday, 11/14/11

- Complete the following problems. You may use any result from class you like, but if you cite a theorem be sure to verify the hypotheses are satisfied.
- This is a closed-book, closed-notes exam. No calculators or other electronic aids permitted.
- In order to receive full credit, *you must show all of your work and justify your answers.* Your answer should be clearly labeled.
- If you need extra room, use the back sides of each page. Staple any scratch paper to your exam.
- The following formulas may be of use to you:

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B.$$

$$\sin(A \pm B) = \sin A \cos B \pm \sin B \cos A.$$

$$\sin(2A) = 2 \sin A \cos A.$$

$$\cos(2A) = \cos^2(A) - 1.$$

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1.$$

$$\lim_{\theta \to 0} \frac{\cos(\theta) - 1}{\theta} = 0.$$

- Please sign the following:

"On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.

Name: Solutions

Signature: ______________________

Please leave the following table blank for the grader.

1. ____________ (/25 points)
2. ____________ (/10 points)
3. ____________ (/10 points)
4. ____________ (/25 points)
5. ____________ (/12 points)
6. ____________ (/4 points)
7. ____________ (/4 points)

Total. ____________ (/90 points)

1. (25 points) Consider the function $y = f(x) = 2 + 2x^2 - x^4$.

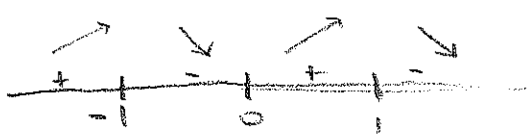
(a) (5 points) Compute $f'(x)$ and $f''(x)$.

$$f'(x) = 4x - 4x^3$$
$$f''(x) = 4 - 12x^2$$

(b) (5 points) On which intervals is f increasing? Decreasing? Where does f have a horizontal tangent line?

Incr/decr: $4x - 4x^3 = 0 \iff x - x^3 = 0 \iff x(1-x^2) = 0$
$\iff x = 0, x = \pm 1$

(sign of f')



on $(-\infty, -1)$, $f'(x) > 0$ f incr
$(-1, 0)$, $f'(x) < 0$ f decr
$(0, 1)$, $f'(x) > 0$ f incr
$(1, \infty)$ $f'(x) < 0$ f decr

So f is increasing on $(-\infty, -1) \cup (0, 1)$
decreasing on $(-1, 0) \cup (1, \infty)$

f has a horizontal tangent at $x = -1, 0, 1$

(c) (5 points) On which intervals if f concave up? Concave down? What are the inflection points of f?

$f''(x) = 4 - 12x^2$ $4 - 12x^2 = 0 \iff 1 - 3x^2 = 0$
$x = \pm \frac{1}{\sqrt{3}}$

(sign of f'') $-$ | $+$ | $-$ at $\frac{-1}{\sqrt{3}}$, $\frac{1}{\sqrt{3}}$

f is concave up on $(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$
concave dn on $(-\infty, \frac{-1}{\sqrt{3}}) \cup (\frac{1}{\sqrt{3}}, 1)$

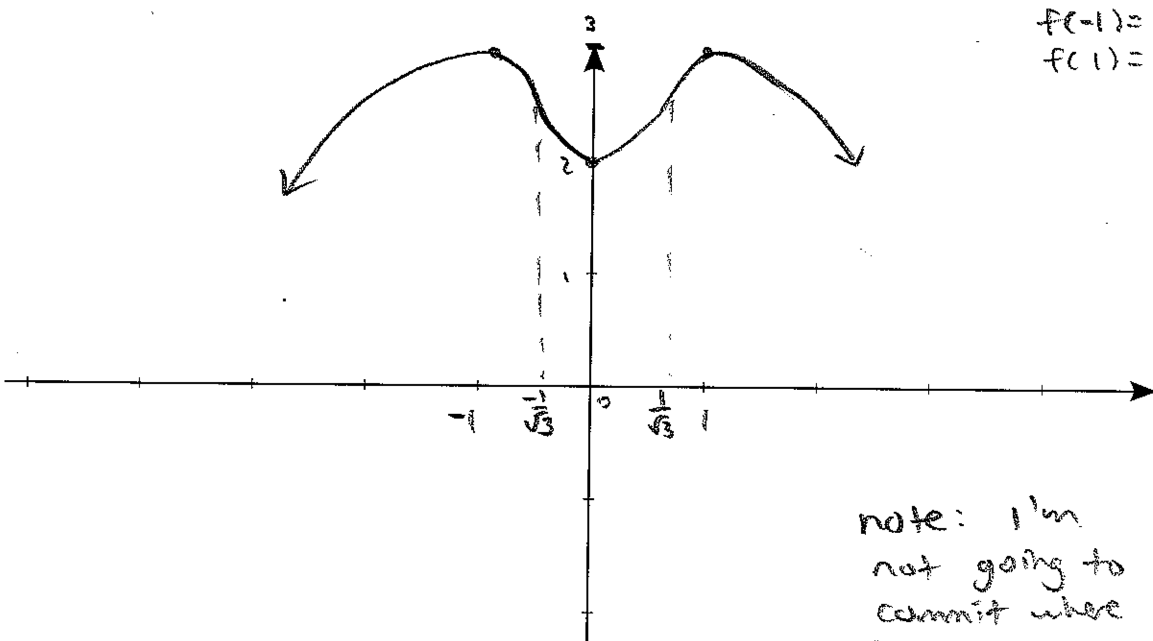
(d) (5 points) Determine the limits of f as x approaches ∞ and as x approaches $-\infty$.

$f(x) = 2 + 2x^2 - x^4$ is a polynomial, so behaves like the term with the highest degree as $x \to \pm\infty$.

So $\lim_{x \to \infty} 2 + 2x^2 - x^4 = -\infty$
$\lim_{x \to -\infty} 2 + 2x^2 - x^4 = -\infty$

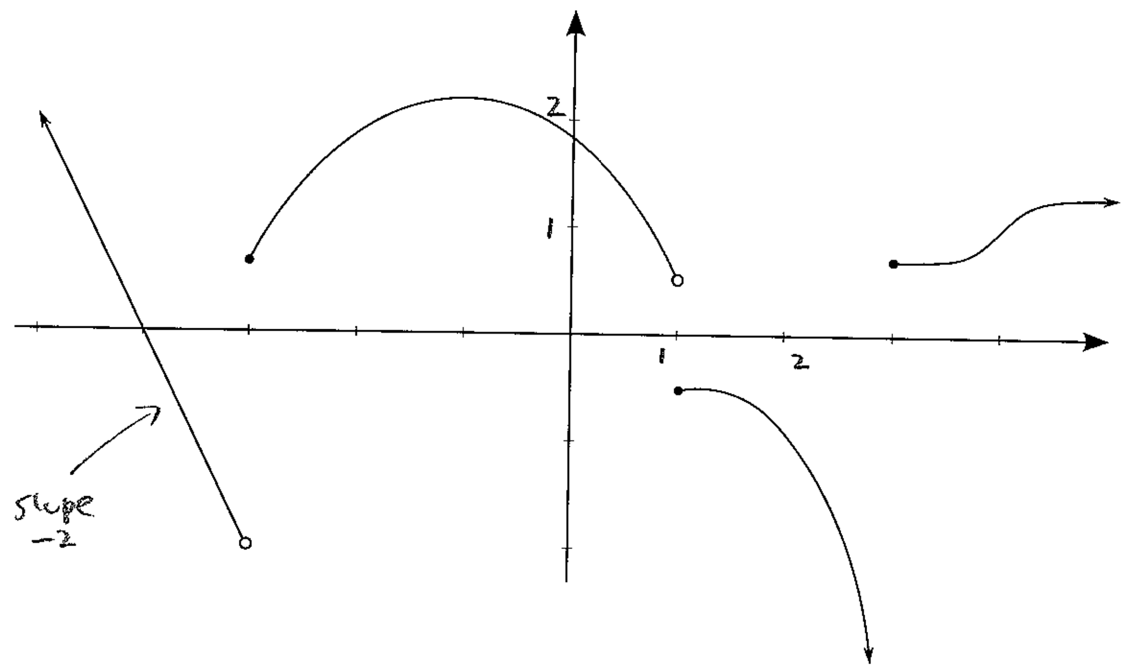
(e) (5 points) Sketch the graph of f as accurately as possible.

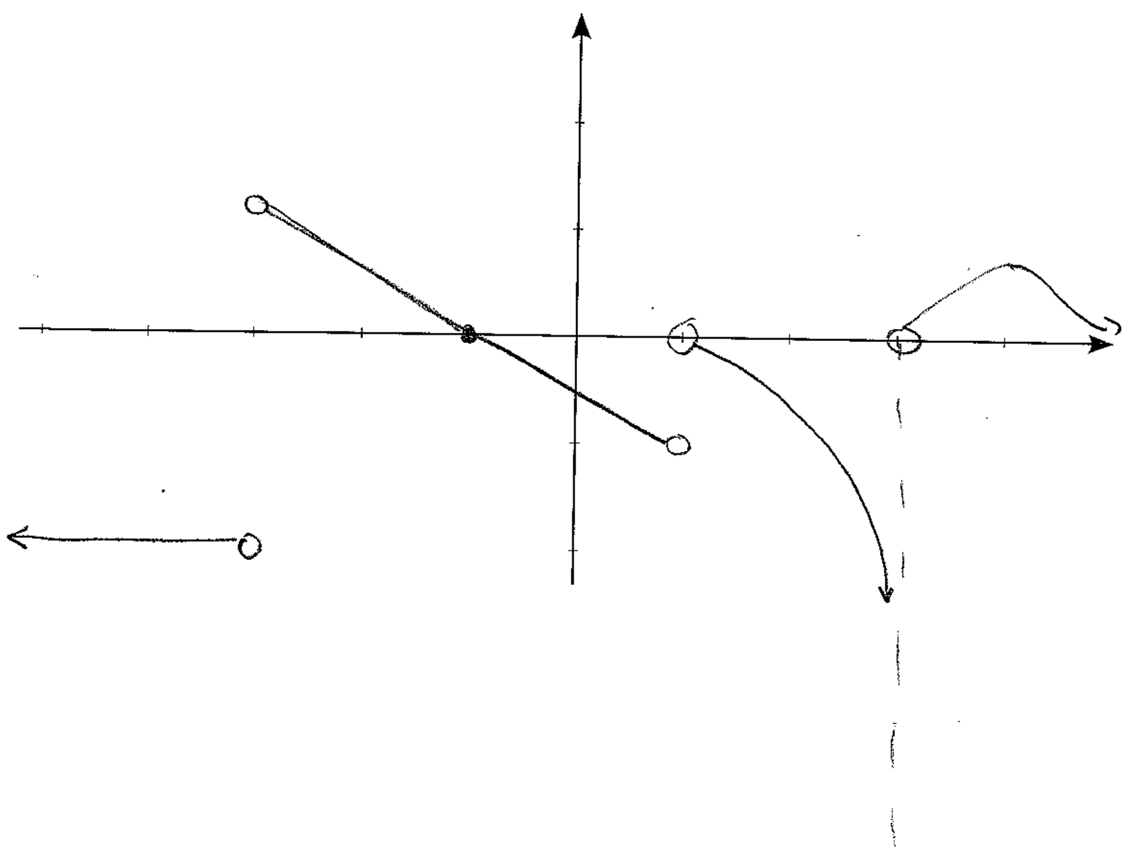
note
$f(0) = 2$
$f(-1) = 3$
$f(1) = 3$)



note: I'm not going to commit where $f(x) = 0$.

2. (10 points) Given the graph of $y = f(x)$ sketched below, sketch a plausible graph of $y = f'(x)$. Be as accurate as the information allows.





3. (a) (5 points) Find the equation of the tangent line to the curve $y = x^2$ at $x = a$ (for a constant real number a). Use this to give the equation of the tangent line to the point with $a = -1$.

Need slope & pt. of tangency.

$y = x^2, \quad y' = 2x$

Slope at $x = a$, $m = y'(a) = 2a$

tangency: $x = a$, $y = a^2$, so (a, a^2)

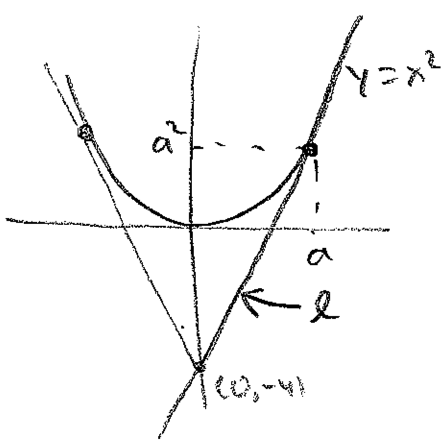
line: $y - a^2 = 2a(x - a)$

$y = 2ax - 2a^2 + a^2$

$\boxed{y = 2ax - a^2}$

when $a = -1$, $\boxed{y = -2x - 1}$

(b) (5 points) Draw a diagram to show that there are two tangent lines to the curve $y = x^2$ that pass through the point $(0, -4)$. Use part (a) to find the coordinates of these two points on the parabola.



let $x = a$ be the x-coord. of one of the points of tangency the line ℓ sketched has slope $m = 2a$ (part (a)) on the one hand, but also

$m = \frac{a^2 - (-4)}{a - 0} = \frac{a^2 + 4}{a}$

on the other.

So $2a = \frac{a^2+4}{a} \Rightarrow 2a^2 = a^2 + 4$

$a^2 = 4$

$a = \pm 2$.

the points are $(2, 4)$ and $(-2, 4)$.

4. (25 points) Compute the derivatives of the following functions. You do not need to simplify. It is okay to "do these in your head" instead of showing every step – BUT if you get it wrong we cannot give you much partial credit.

(a) (5 points)

$$y = \frac{x}{e^x}.$$

quotient rule

$$y' = \frac{e^x \frac{d}{dx}(x) - x \frac{d}{dx}(e^x)}{(e^x)^2}$$

$$= \frac{e^x \cdot 1 - xe^x}{(e^x)^2}$$

$$= \frac{e^x(1-x)}{(e^x)^2}$$

$$= \frac{1-x}{e^x}$$

(b) (5 points)

$$y = t^2 \sin(t).$$

product rule

$$\frac{dy}{dt} = \frac{d}{dt}(t^2) \cdot \sin(t) + t^2 \frac{d}{dt}(\sin(t))$$

$$= 2t \sin(t) + t^2 \cos(t)$$

(c) (5 points)

$$y = \sin(\tan(2\theta)).$$

chain rule. twice.

$$\frac{dy}{d\theta} = \cos(\tan(2\theta)) \cdot \frac{d}{d\theta}[\tan(2\theta)]$$

$$= \cos(\tan(2\theta)) \cdot \sec^2(2\theta) \cdot \frac{d}{d\theta}(2\theta)$$

$$= \cos(\tan(2\theta)) \cdot \sec^2(2\theta) \cdot 2$$

(d) (5 points)

$$y = (t^4 - 1)^3(t^3 + 1)^4.$$

product rule, then chain rule.

$$\frac{dy}{dt} = \frac{d}{dt}\left[(t^4-1)^3\right] \cdot (t^3+1)^4 + (t^4-1)^3 \frac{d}{dt}\left[(t^3+1)^4\right]$$

$$= 3(t^4-1)^2 \cdot 4t^3 \cdot (t^3+1)^4 + (t^4-1)^3 \cdot 4(t^3+1)^3(3t^2)$$

$$= (t^4-1)^2(t^3+1)^3\left[12t^3(t^3+1) + 12t^2(t^4-1)\right]$$

(e) (5 points)

$$y = e^{-x}\cos(x).$$

product rule, then chain rule

$$\frac{dy}{dx} = \frac{d}{dx}[e^{-x}] \cdot \cos(x) + e^{-x}\frac{d}{dx}[\cos(x)]$$

$$= -e^{-x} \cdot \cos(x) + e^{-x}(-\sin(x))$$

$$= -e^{-x}(\cos(x) + \sin(x))$$

5. (a) (8 points) Using the limit definition of the derivative, compute the derivative of $f(x) = x^2 - 4x + 4$. You MUST use the limit definition or you will get zero points. (You may use the derivative rules to double check your answer.)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - 4(x+h) + 4 - (x^2 - 4x + 4)}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 4x - 4h + 4 - x^2 + 4x - 4}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 - 4h}{h}$$

$$= \lim_{h \to 0} 2x + h - 4$$

$$= 2x - 4.$$

(b) (4 points) Find the equation of the tangent line to $y = f(x) = x^2 - 4x + 4$ at $x = 2$.

Need slope & point of tangency.

Slope $m = f'(2) = 2(2) - 4 = 0$

point, $x = 2$, $y = f(2) = 2^2 - 4(2) + 4 = 4 - 8 + 4 = 0$

line: $y - 0 = 0(x - 2)$

$\boxed{y = 0}$ (i.e. the x-axis)

6. (4 points) Suppose the tangent line to $y = g(x)$ at $x = 2$ is $y = -3x + 4$. Determine the values of $g(2)$ and $g'(2)$ or write IMPOSSIBLE if it is impossible to tell from only this information. (*hint*: draw a picture).

$g'(2)$ = slope of the tangent line to $y = g(x)$ at $x = 2$

= slope of $y = -3x+4$

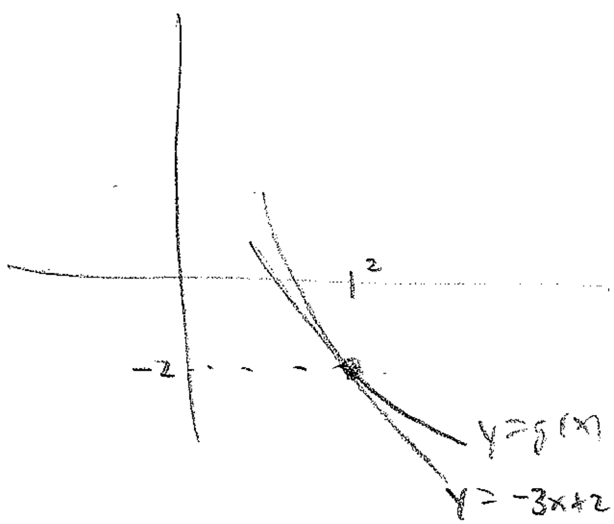
= -3

Since a tangent line passes through the pt. of tangency, $g(2)$ = value of $y = -3x+4$ at $x=2$

= $-3(2)+4$

= $-6+4$

= -2.



7. (4 points) Consider the following table of values of the functions f, g, h, and their derivatives. The leftmost column lists the possible x values, and each other column represents the value of the different functions at that x value.

x	$f(x)$	$g(x)$	$h(x)$	$f'(x)$	$g'(x)$	$h'(x)$
-4	2	5	2	3	3	0
0	3	3	1	-1	-4	2
1	-1	5	0	2	-3	0
2	7	-2	3	0	5	-1
3	4	1	6	5	0	3
5	3	6	-4	3	1	5

Using this information:

(a) Compute $(g \circ h)'(0)$.

(b) Compute the derivative of $f(g(h(x)))$ evaluated at $x = 0$.

a) $(g \circ h)'(0) = g'(h(0)) \cdot h'(0)$
$= g'(1) \cdot 2$
$= -3 \cdot 2 = -6$

b) $[f(g(h(x)))]'$
$= f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$
at $x=0$,
$f(g(h(0))) \cdot g'(h(0)) \cdot h'(0)$
$= f(g(1)) \cdot g'(1) \cdot h'(0)$
$= f(5) \cdot (-3)(2)$
$= 3 \cdot (-3) \cdot 2 = -18$